A LYNDON-HOCHSCHILD-SERRE-TYPE SPECTRAL SEQUENCE FOR DISCRETE G-SPECTRA

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Abstract. These are some notes for my talk in the bell show at the conference *Structured Ring Spectra – TNG*, on August 4th, 2011, in Hamburg, Germany.

1. Notation for the talk

- G is a profinite group with finite vcd (that is, finite virtual cohomological dimension): there exists some $U <_o G$ and some natural number l such that
$$H^s_c(U; M) = 0, \text{ for all } s > l \text{ and all discrete } U\text{–modules } M.$$
- Note: in practice, the above is not too restrictive a hypothesis because it is satisfied by many of the profinite groups that one cares about, such as any compact p-adic analytic Lie group.
- H and K are closed subgroups of G, with $H \lhd K$. This implies that K/H is a profinite group.
- Spt_G is the model category of discrete G–spectra.

2. Motivation for our theorem

Let
$$\begin{aligned} G &= G_n \\ &= S_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \\ &= \text{the extended Morava stabilizer group} \end{aligned}$$
and let Z be any finite spectrum.

Work of Ethan Devinatz, Mike Hopkins, Mark Behrens, and myself shows that there is a strongly convergent descent spectral sequence that has the form
$$E_2^{s,t} = H^s_c\big(K/H; \pi_t\big((E_n \wedge Z)^{hH}\big)\big) \Longrightarrow \pi_{t-s}\big((E_n \wedge Z)^{hK}\big).$$
This is referred to as a "Lyndon-Hochschild-Serre spectral sequence" because the abutment is the total right derived functor of K–fixed points and the E_2–term is the K/H–continuous cohomology of the total right derived functor of H–fixed points.

Now let $X \in \mathrm{Spt}_G$. The above spectral sequence leads one to ask if there is a descent spectral sequence that has the form
$$H^s_c\big(K/H; \pi_t\big(X^{hH}\big)\big) \Longrightarrow \pi_{t-s}(X^{hK}).$$

3. Dog-gone-it, back to reality

When X is a totally hyperfibrant discrete G–spectrum and K/H has finite vcd, then it is known that the desired descent spectral sequence exists. But, in general, we are not able to say that this spectral sequence exists.

For example:

- there are cases where X^{hH} is not a discrete K/H–spectrum;
- in general, it is not known how to view $\pi_t(X^{hH})$ as a topological K/H–module; and
- in general, it is not known how to define
$$\big(X^{hH}\big)^{hK/H},$$
which is the way one expects to build the above spectral sequence if it exists.

Nevertheless, it is still possible, in general, by using the speaker's framework of delta-discrete K/H–spectra, to build a descent spectral sequence for

$$\big(X^{hH}\big)^{h_\delta K/H} \xleftarrow{\simeq} X^{hK}$$

that is a Lyndon-Hochschild-Serre-type spectral sequence.

4. Two definitions to help state our result

Definition 4.1. Let $P = \lim_\alpha P_\alpha$ be a profinite set, so that each P_α is a finite set. Then let

$$\mathrm{Map}_c(P, X) := \operatorname*{colim}_\alpha \textstyle\prod_\alpha X,$$

where the colimit and product are formed in Spt_G (and hence, in this case, in the category of spectra). Thus, $\mathrm{Map}_c(P, X) \in \mathrm{Spt}_G$.

Definition 4.2. Let $\widehat{X} = \operatorname*{colim}_{N \lhd_o G} (X^N)_f$, where $(-)_f$ denotes fibrant replacement in the category of spectra. Then $\widehat{X} \in \mathrm{Spt}_G$ and there is a map $X \to \widehat{X}$ that is a weak equivalence in Spt_G.

5. Our result

Theorem 5.1. *There is a conditionally convergent descent spectral sequence*

$$E_2^{s,t} = H^s\Big[\pi_t\big(\mathrm{Map}_c(K/H^*, \widehat{X})^{hH}\big)\Big] \Longrightarrow \pi_{t-s}(X^{hK}),$$

with

$$E_2^{s,t} = H^s\Big[\pi_t(X^{hH}) \to \pi_t\big(\mathrm{Map}_c(K/H, \widehat{X})^{hH}\big) \to \cdots\Big].$$

This spectral sequence has the desired abutment and the E_2–term involves the total right derived functor of H–fixed points and an expression that is related to the continuous cochains for K/H of continuous cohomology, so this spectral sequence is of Lyndon-Hochschild-Serre-type.

This is the nicest way I know to write the general spectral sequence, but its form is not meant to imply that the cochain complex above comes from the usual simplicial object $K/H^\bullet$, because it does not, and the definition of the cochain complex does (as it should) involve the K/H–action on the discrete K/H–spectra $\mathrm{Map}_c(G^*, \widehat{X})^H$.

Remark 5.2. One can ask: if X^{hH} is a discrete K/H–spectrum and K/H has finite vcd, which is what happens in the cases of interest in chromatic stable homotopy theory, then is the above spectral sequence isomorphic to the usual descent spectral sequence

$$H_c^s\big(K/H; \pi_t\big(X^{hH}\big)\big) \Longrightarrow \pi_{t-s}\Big(\big(X^{hH}\big)^{hK/H}\Big)\ ?$$

I expect to be able to show that in the main case when this happens (that is, when X is a totally hyperfibrant discrete G–spectrum), this is indeed correct, but this is work in progress.