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CYCLIC GAMMA HOMOLOGY AND GAMMA HOMOLOGY FOR ASSOCIATIVE ALGEBRAS

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For Mamuka Jibladze
on the occasion of his 50th birthday

Abstract. The stabilization of Hochschild homology of commutative algebras is Gamma homology. We describe a cyclic variant of Gamma homology and prove that the associated analogue of Connes' periodicity sequence becomes almost trivial, because the cyclic version coincides with the ordinary version from homological degree two on. We show that a possible desymmetrized definition of Gamma homology coincides with a shifted version of Hochschild homology and its associated cyclic theory does the same.

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1. Introduction

Given a commutative algebra A, there are several homology theories available that can help to understand A. Hochschild homology and cyclic homology of A are related by Connes' periodicity sequence

$$\ldots \to HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \to \ldots$$

which is a good means for comparing Hochschild homology with its cyclic variant.

Using the commutativity of A we could consider André–Quillen homology as well. Viewing A as an E_∞-algebra with trivial homotopies for commutativity allows us to consider André–Quillen homology in the category of differential graded E_∞-algebras as defined by Mike Mandell [7]. This homology theory coincides with Alan Robinson's Gamma homology [15, 1] which in turn can be interpreted as stabilization of Hochschild homology of A by [11, Theorem 1]. Gamma homology has the feature that it agrees with André–Quillen homology for $\mathbb{Q}$-algebras [17, Theorem 6.4].

The Hodge decomposition for Hochschild homology for flat commutative algebras, with the base ring containing rationals, splits André–Quillen homology off as the first summand $HH_*^{(1)} \cong AQ_{*-1}$ in the decomposition (see [3, 6, 9]). Cyclic homology splits similarly and from degree three on the first summand $HC_*^{(1)}$ of that decomposition is again André–Quillen homology, AQ_{*-1}. It is known that the periodicity sequence passes to a sequence for the decomposition

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summands [6, 9],

$$\ldots \to HH_n^{(i)}(A) \xrightarrow{I} HC_n^{(i)}(A) \xrightarrow{S} HC_{n-2}^{(i-1)}(A) \xrightarrow{B} HH_{n-1}^{(i)}(A) \to \ldots$$

Therefore rationally the periodicity sequence collapses in higher degrees for the first decomposition summand, because there the map I becomes an isomorphism. One could guess that this is a defect of working over the rationals, but we will show in the course of this paper that this is not the case.

Robinson and Whitehouse proposed a cyclic variant of Gamma homology of differential graded E_∞-algebras over a cyclic E_∞-operad in [17].

We construct a cyclic variant of Gamma homology which arises naturally from the interpretation of Gamma homology as stable homotopy of certain Γ-modules.

It turns out, however, (compare Corollary 4.4) that this definition of cyclic Gamma homology coincides with usual Gamma homology from homological degree two on; hence this sequence collapses. We explicitly describe (see Propositions 5.4 and 5.5) cyclic Gamma homology in small degrees in terms of ordinary cyclic homology and deRham cohomology.

An alternative approach for a cyclic version of Gamma homology is to first extend the definition of Gamma homology to associative algebras and then to build a cyclic version of the resulting theory. In Section 6 we show that such a desymmetrized definition of Gamma homology coincides with a shifted version of Hochschild homology. Gamma homology of commutative algebras views a commutative algebra as an E_∞-algebra and computes André–Quillen homology in the category of E_∞-algebra. If the characteristic of the base ring is not zero, then the non-trivial homology groups of symmetric groups lead to the fact that Gamma homology is not isomorphic to classical André–Quillen homology of commutative algebras.

Gamma homology of associative algebras should be viewed as André–Quillen homology of A_∞-algebras, so one interpretation of our result is that the A_∞-homology of associative algebras is Hochschild homology. Robinson and Whitehouse proved an analogous result using a different model [17, Corollary 4.2].

The associated cyclic theory of Gamma homology of associative algebras coincides with shifted Hochschild homology as well, so the two approaches to producing cyclic Gamma homology that we present in this paper fail to give something different.

Lars Hesselholt proved an analogous phenomenon in the setting of topological Hochschild homology. Fix an arbitrary prime p. In [4] he showed that the equivalence between stable K-theory and topological Hochschild homology is reflected in an equivalence between the p-completions of the stabilization of topological cyclic homology and p-completed topological Hochschild homology.

Our methods of proof use the extension of the definitions of cyclic, Hochschild and Gamma homology to functor categories. We recall the necessary prerequisites from [5, 10, 11, 12].

2. The Category $\mathcal{F}$ and $\mathcal{F}$-Modules

We recall the definition of cyclic homology from [5, §6] (see also [10, §3]). Let $\mathcal{F}$ denote the skeleton of the category of finite unpointed sets and let $\underline{n}$ be the object $\{0, \ldots, n\}$ in $\mathcal{F}$. We call functors from $\mathcal{F}$ to the category of k-modules $\mathcal{F}$-modules. Here k is an arbitrary commutative ring with unit. For a set S we denote by $k[S]$ the free k-module generated by S.

The projective generators for the category of $\mathcal{F}$-modules are the functors $\mathcal{F}^n$ given by

$$\mathcal{F}^n(\underline{m}) := k[\mathcal{F}(\underline{n}, \underline{m})];$$

whereas the category of contravariant functors from $\mathcal{F}$ to k-modules has the family $\mathcal{F}_n$ with

$$\mathcal{F}_n(\underline{m}) := k[\mathcal{F}(\underline{m}, \underline{n})]$$

as generators.

For two $\mathcal{F}$-modules F and F' let $F \otimes F'$ be the pointwise tensor product of F and F', *i.e.*, $F \otimes F'(\underline{n}) = F(\underline{n}) \otimes F'(\underline{n})$. As a map in $\mathcal{F}$ from the object $\underline{0}$ to an object $\underline{m}$ just picks an arbitrary element, one obtains that $(\mathcal{F}^0)^{\otimes n} \cong \mathcal{F}^{n-1}$. The functor $\mathcal{F}^0$ is an analog of the functor L from [11] in the unpointed setting and the tensor powers $(\mathcal{F}^0)^{\otimes n} \cong \mathcal{F}^{n-1}$ for $n > 1$ correspond to $L^{\otimes n}$.

Given a unital commutative k-algebra A, the $\mathcal{F}$-module which gives rise to cyclic homology of A is the functor $\mathcal{L}(A)$ that sends $\underline{n}$ to $A^{\otimes n+1}$. A map $f\colon \underline{n} \to \underline{m}$ induces $f_*\colon \mathcal{L}(A)(\underline{n}) \to \mathcal{L}(A)(\underline{m})$ via

$$f_*(a_0 \otimes \ldots \otimes a_n) = b_0 \otimes \ldots \otimes b_m, \text{ with } b_i = \prod_{j \in f^{-1}(i)} a_j.$$

Here we set $b_i = 1$ if the preimage of i is empty.

For any $\mathcal{F}$-module F, cyclic homology of F, $HC_*(F)$, can be defined [10, 3.4] as the homology of the total complex associated to the bicomplex

$$\begin{tikzcd}
\vdots \arrow[d, "b"] & \vdots \arrow[d, "b"] & \vdots \arrow[d, "b"] \\
F(\underline{2}) \arrow[d, "b"] & F(\underline{1}) \arrow[l, "B"'] \arrow[d, "b"] & F(\underline{0}) \arrow[l, "B"'] \\
F(\underline{1}) \arrow[d, "b"] & F(\underline{0}) \arrow[l, "B"'] & \\
F(\underline{0}) & &
\end{tikzcd}$$

In particular, cyclic homology of A, $HC_*(A)$, is the homology of this total complex applied to the functor $\mathcal{L}(A)$. We recall the definition of b in (1) and the one of B in Definition 5.1.

3. The Relationship to the Category Γ

Let Γ be the skeleton of the category of pointed finite sets and let $[n]$ be the object $[n] = \{0, \dots, n\}$ with 0 as basepoint. The projective generators of the category of Γ-modules are the functors Γ^n given by

$$\Gamma^n[m] = k[\Gamma([n], [m])].$$

There is a natural forgetful functor $\mu\colon \Gamma \to \mathcal{F}$ and the left adjoint to μ, $\nu\colon \mathcal{F} \to \Gamma$, which adds an extra basepoint $\nu(\underline{m}) = [m+1]$. Pulling back with these functors transforms Γ-modules into $\mathcal{F}$-modules and vice versa:

$$\nu^*\colon \Gamma\text{-modules} \rightleftarrows \mathcal{F}\text{-modules} : \mu^*$$

In [10, Proposition 3.3] Pirashvili shows that

$$\mathsf{Tor}_*^{\mathcal{F}}(\nu^* F, G) \cong \mathsf{Tor}_*^{\Gamma}(F, \mu^* G).$$

Lemma 3.1. *The functor $\mathcal{F}^n$ pulled back along μ is isomorphic to Γ^{n+1}.*

Proof. We first show that the Γ-module $\mu^*(\mathcal{F}^0)$ is isomorphic to Γ^1: for every object $[n]$ we obtain that

$$\mu^*(\mathcal{F}^0)[n] = \mathcal{F}^0(\underline{n}) = k[\mathcal{F}(\underline{0}, \underline{n})] \cong k^{n+1}$$

because the value of a function $f \in \mathcal{F}(\underline{0}, \underline{n})$ on 0 can be an arbitrary element $i \in \underline{n}$. The Γ-module Γ^1 has the same value on $[n]$, because a function $g \in \Gamma([1], [n])$ has an arbitrary value on 1 but sends zero to zero. As we just allow pointed maps, the two functors are isomorphic.

The general case easily follows by direct considerations or by using the decompositions of $\mathcal{F}^n$ and Γ^{n+1} as $(n+1)$-fold tensor products $\mathcal{F}^n \cong (\mathcal{F}^0)^{\otimes n+1}$ and $\Gamma^{n+1} \cong (\Gamma^1)^{\otimes n+1}$. $\square$

Recall, that Hochschild homology of a Γ-module G, $HH_*(G)$, can be defined as the homology of the complex

$$G[0] \xleftarrow{b} G[1] \xleftarrow{b} \cdots \tag{1}$$

where $b = \sum_{i=0}^n (-1)^i G(d_i)$ and d_i is the map of pointed sets that for $i < n$ sends i and $i+1$ to i and is bijective and order preserving on the other values in $[n]$. The last map, d_n, maps 0 and n to 0 and is the identity for all other elements of $[n]$.

Later, we will need the following auxiliary result.

Lemma 3.2. *Hochschild homology of a Gamma module G is isomorphic to the homology of the normalized complex which consist of $G[n]/D_n$ in chain degree n where $D_n \subset G[n]$ consists of all elements of the form $(s_i)_* F[n-1]$ where s_i is the order preserving injection from $[n-1]$ to $[n]$ which misses i.*

Proof. This result just uses the standard fact that the Hochschild complex is the chain complex associated to a simplicial k-module and the elements in D_n correspond to the degenerate elements; therefore the complex D_* is acyclic. $\square$

A similar result applies to cyclic homology of $\mathcal{F}$-modules.

Let $\mathbb{S}^1 = \Delta^1/\partial\Delta^1$ denote the standard model of the simplicial 1-sphere. Recall from [5, 10] that Hochschild homology of a commutative unital k-algebra A, $HH_*(A)$, coincides with the homotopy groups of the simplicial k-module $\mu^*\mathcal{L}(A)(\mathbb{S}^1)$. Here, we evaluate $\mu^*\mathcal{L}(A)$ degreewise. Note that more generally, Hochschild homology of any Γ-module G as defined above coincides with $\pi_*G(\mathbb{S}^1)$.

4. Gamma Homology and Its Cyclic Version

Let t be the contravariant functor from Γ to k-modules which is defined as

$$t[n] = \mathrm{Hom}_{\mathsf{Sets}_*}([n], k)$$

where Sets_* denotes the category of pointed sets. Pirashvili and the author proved in [11] that Gamma homology of any Γ-module G, $H\Gamma_*(G)$, is isomorphic to $\mathsf{Tor}_*^\Gamma(t, G)$. In particular, Gamma homology of the algebra A, $H\Gamma_*(A)$ is isomorphic to $\mathsf{Tor}_*^\Gamma(t, \mu^*\mathcal{L}(A))$.

For a cyclic variant of Gamma homology, we have to transform the functor t into a contravariant $\mathcal{F}$-module. Choosing ν^*t does this, but it inserts an extra basepoint. Killing the value on an additional point amounts to defining the $\mathcal{F}$-module $\bar{t}$ by the following exact sequence:

$$0 \longrightarrow \mathcal{F}_0 \longrightarrow \nu^*(t) \longrightarrow \bar{t} \longrightarrow 0. \tag{2}$$

The transformation from $\mathcal{F}_0$ to ν^*t is given by sending a scalar multiple λf of a map $f\colon \underline{n} \to \underline{0}$ to the function in $\mathrm{Hom}_{\mathsf{Sets}_*}([n+1], k)$ which sends the points $1, \ldots, n+1$ to λ.

Proposition 4.1. *On the family of projective generators $(\mathcal{F}^n)_{n\geq 0}$ the torsion groups with respect to $\bar{t}$ are as follows:*

$$\mathsf{Tor}_*^{\mathcal{F}}(\bar{t}, \mathcal{F}^n) \cong \begin{cases} 0 & \text{for} \quad * > 0, \\ k^n & \text{for} \quad * = 0. \end{cases}$$

Proof. It is clear that the torsion groups vanish in positive degrees because the functors $\mathcal{F}^n$ are projective. We have to prove the claim in degree zero, but the tensor products in question are easy to calculate:

$$\bar{t} \otimes_{\mathcal{F}} \mathcal{F}^n \cong \bar{t}(\underline{n}) \cong k^n. \qquad \square$$

Definition 4.2. We call the group $\mathsf{Tor}_n^{\mathcal{F}}(\bar{t}, F)$ the *nth cyclic Gamma homology group of the $\mathcal{F}$-module F* and denote it by $H\Gamma C_n(F)$.

Remark 4.3. We will see in 5.4 that cyclic Gamma homology of an algebra A in degree zero behaves analogously to usual Gamma homology whose value in homological degree zero gives Hochschild homology of degree one.

Robinson and Whitehouse proposed a cyclic version of Gamma homology for algebras over cyclic differential graded E_∞-operads [17, Definition 3.10]. Alan Robinson [16] assured the author that their definition agrees with ours in the case of commutative algebras.

As the functor $\mathcal{F}_0$ is projective, the calculation in proposition 4.1 allows us to draw the following conclusion.

Corollary 4.4. *Cyclic Gamma homology of any $\mathcal{F}$-module F coincides with Gamma homology of the induced Gamma module $\mu^*(F)$ in degrees higher than* 1, i.e.,

$$H\Gamma C_*(F) = \mathsf{Tor}_*^{\mathcal{F}}(\bar{t}, F) \cong \mathsf{Tor}_*^{\Gamma}(t, \mu^* F) \cong H\Gamma_*(\mu^*(F)) \quad \forall * > 1.$$

In low degrees the difference between cyclic and ordinary Gamma homologies is measured by the following exact sequence:

$$0 \to \mathsf{Tor}_1^{\mathcal{F}}(\nu^* t, F) \to \mathsf{Tor}_1^{\mathcal{F}}(\bar{t}, F) \xrightarrow{\delta} F(\underline{0}) \to \nu^* t \otimes_{\mathcal{F}} F \to \bar{t} \otimes_{\mathcal{F}} F \to 0$$

which is nothing but

$$0 \to H\Gamma_1(\mu^* F) \to H\Gamma C_1(F) \to \mu^* F(\underline{0}) \xrightarrow{\delta} H\Gamma_0(\mu^* F) \to H\Gamma C_0(F) \to 0.$$

We will obtain more explicit descriptions in the algebraic case in the next section.

5. The B Operator

In an unstable situation there is a map B which connects cyclic homology and Hochschild homology and which gives rise to Connes' important periodicity sequence

$$\cdots \longrightarrow HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \longrightarrow \cdots$$

In low degrees the map B sends the zeroth cyclic homology of a k-algebra A which is nothing but A again to the first Hochschild homology group of A which consists of the module of Kähler differentials $\Omega^1_{A|k}$ and the map is given by $B(a) = da$. If we consider the first nontrivial parts in the long exact sequence of Tor-groups as above, arising from the short exact sequence $0 \to \mathcal{F}_0 \to \nu^* t \to \bar{t} \to 0$ then, for the functor $\mathcal{L}(A)$, we obtain

$$\cdots \to A \to \nu^* t \otimes_{\mathcal{F}} \mathcal{L}(A) \to \bar{t} \otimes_{\mathcal{F}} \mathcal{L}(A) \to 0$$

and $\nu^* t \otimes_{\mathcal{F}} \mathcal{L}(A)$ is isomorphic to the zeroth Gamma homology group of A which is the module of Kähler differentials. The map is induced by the natural transformation from $\mathcal{F}_0$ to $\nu^* t$. The aim of this section is to prove that this map is given by the B-map.

Let us recall the general definition of the B-map for cyclic and Hochschild homology of functors. If T is the generator of the cyclic group on $n+1$ elements viewed as the permutation $T : \underline{n} \to \underline{n}$, $T(i) = i + 1 \bmod n + 1$. Thus T acts on $F(\underline{n})$ for every $\mathcal{F}$-module F. We define τ to be $(-1)^n T$.

The B-map from cyclic homology to Hochschild homology can be viewed as a map from the nth generator $\mathcal{F}_n$ to the $(n + 1)$st in the following manner:

Definition 5.1. Let s be the map of finite sets which sends i to $i + 1$. Then the B-map is defined as a map $B \colon \mathcal{F}_n \to \mathcal{F}_{n+1}$. On a generator $f \colon \underline{m} \to \underline{n}$ it is $B(f) := (1 - \tau) \circ s \circ N \circ f$ where N is the norm map $N = \sum_{i=1}^{n+1} \tau^i$.

On the part $F(\underline{n}) \cong \mathcal{F}_n \otimes_{\mathcal{F}} F$ of the complex for cyclic homology of F this induces the usual B-map known from the algebraic case $F = \mathcal{L}(A)$, for a commutative algebra A. By the very definition of the map it is clear that it is well-defined on the tensor product.

In our situation we apply the B-map to the first column of the double complex for cyclic homology of F

$$\begin{tikzcd}
\vdots \arrow[d, "b"] & \vdots \arrow[d, "b"] & \vdots \arrow[d, "b"] \\
F(\underline{2}) \arrow[d, "b"] & F(\underline{1}) \arrow[l, "B"'] \arrow[d, "b"] & F(\underline{0}) \arrow[l, "B"'] \\
F(\underline{1}) \arrow[d, "b"] & F(\underline{0}) \arrow[l, "B"'] & \\
F(\underline{0}) & &
\end{tikzcd}$$

and send all other columns to zero. In [10, 3.2] it is shown that $\nu^*\Gamma_n \cong \mathcal{F}_n$. Using this we obtain an isomorphism $F(\underline{n}) \cong \mathcal{F}_n \otimes_{\mathcal{F}} F \cong \nu^*\Gamma_n \otimes_{\mathcal{F}} F \cong \Gamma_n \otimes_\Gamma \mu^* F$ and see that B gives rise to a map from the total complex for cyclic homology of F to the complex for Hochschild homology of $\mu^*(F)$.

A verbatim translation of the proof for ([5, 2.5.10, 2.1]) in the case of a cyclic module to our setting gives the following result:

Lemma 5.2. *The map B is a map of chain complexes and therefore induces a map from $HC_*(F)$ to $HH_{*+1}(\mu^* F)$.*

Remark 5.3. In degree zero, the B-map from $\mathcal{F}_0$ to $\mathcal{F}_1$ applied to an $f \in \mathcal{F}_0(\underline{n})$ is given by $(1-\tau) \circ s \circ f$.

We should first make sure that cyclic Gamma homology has the right value in homological dimension zero.

Proposition 5.4. *Cyclic Gamma homology in degree zero is isomorphic to cyclic homology in degree one. In particular, $H\Gamma C_0(A) \cong HC_1(A)$.*

Proof. The cokernel of the map $F(\underline{0}) \to \nu^*(t) \otimes_{\mathcal{F}} F$ can be determined by a map from $F(\underline{0})$ to $F(\underline{1})$: we use the beginning of the resolution of t

$$\ldots \to \Gamma_2 \xrightarrow{\alpha} \Gamma_1 \xrightarrow{\beta} t \to 0$$

(see [10, §1.4]). Here, α is the sum of maps $d_0 - d_1 + d_2$ whereas β sends a map g from $[n]$ to $[1]$ to $\sum_{g(i)=1} \chi_i$ where χ_i is the characteristic function of i. The exactness of ν^* turns this into an exact sequence $\ldots \to \mathcal{F}_2 \xrightarrow{\nu^*\alpha} \mathcal{F}_1 \xrightarrow{\nu^*\beta} \nu^* t \to 0$. We can choose a lift of the inclusion map i from $\mathcal{F}_0$ to $\nu^* t$ to $\mathcal{F}_1$ by sending a generator $f \in \mathcal{F}_0(\underline{n})$ to $s \circ f$. We call this lift j.

$$\begin{tikzcd} & & \mathcal{F}_2 \arrow[d, "\nu^*\alpha"] & & \\ & & \mathcal{F}_1 \arrow[d, "\nu^*\beta"] & & \\ 0 \arrow[r] & \mathcal{F}_0 \arrow[ur, "j"] \arrow[r, "i"] & \nu^* t \arrow[r, "\pi"] & \bar{t} \arrow[r] & 0 \end{tikzcd}$$

It is easy to see that $\nu^*\beta \circ j = i$. The snake lemma allows us to calculate the cokernel of i via the quotient of $\mathcal{F}_1$ by the image of j and the kernel of $\nu^*\beta$ which is the image of $\nu^*\alpha$. The map j corresponds to the reduced part of the B-map and $\nu^*\alpha$ is the Hochschild boundary.

As Hochschild and cyclic homology coincide with their normalized version (see Lemma 3.2), we obtain that the cokernel of i is isomorphic to the first cyclic homology group. $\square$

Cyclic Gamma homology in dimension one can be explicitly described as well. In small degrees our Tor-exact sequence looks as follows:

$$0 \to H\Gamma_1(\mu^* F) \to H\Gamma C_1(F) \xrightarrow{\delta} F(\underline{0}) \xrightarrow{B} HH_1(F).$$

Therefore we obtain the following.

Proposition 5.5. *The difference between cyclic Gamma homology and ordinary Gamma homology in degree one is measured by the kernel of the B-map.*

In the case of the functor $\mathcal{L}(A)$ the exact sequence is

$$0 \to H\Gamma_1(A) \to H\Gamma C_1(A) \xrightarrow{\delta} A \xrightarrow{d} \Omega^1_{A|k}.$$

Thus in degree one the difference between Gamma homology and its cyclic version is measured by the zeroth deRham cohomology of A. For instance, if A is étale, then $H\Gamma_1(A) = 0 = \Omega^1_{A|k}$ and therefore $H\Gamma C_1(A) \cong A$.

The above calculations in small dimensions suggest that one should view the sequence of Tor-groups coming from the sequence $0 \to \mathcal{F}_0 \to \nu^* t \to \bar{t} \to 0$ as the stable version of the periodicity sequence. In the algebraic case the two sequences are nicely related in the following way.

$$\begin{tikzcd} HC_1(A) \arrow[r, "B"] & HH_2(A) \arrow[r, "I"] \arrow[d, "\mathsf{stab}"] & HC_2(A) \arrow[r, "S"] & HC_0(A) \arrow[r, "B"] \arrow[d, equal] & HH_1(A) \arrow[r] \arrow[d, "\cong"', "\mathsf{stab}"] & HC_1(A) \arrow[r] \arrow[d, "\cong"'] & 0 \\ 0 \arrow[r] & H\Gamma_1(A) \arrow[r] & H\Gamma C_1(A) \arrow[r, "\delta"] & A \arrow[r, "B"] & H\Gamma_0(A) \arrow[r] & H\Gamma C_0(A) \arrow[r] & 0 \end{tikzcd}$$

But in higher dimensions the transformation I from the periodicity sequence becomes an isomorphism. The term $\mathcal{F}_0 \otimes_{\mathcal{F}} F \cong F(\underline{0})$ plays the role of cyclic Gamma homology in dimension -1.

6. Gamma Homology for Associative Algebras

6.1. **Gamma homology for algebras over operads.** An alternative natural extension of the definition of Gamma homology is to construct a version that allows associative algebras as input. One could hope that an alternative version of cyclic Gamma homology could be gained as a cyclic version of Gamma homology of associative algebras.

More generally, if $\widetilde{\mathcal{O}}$ is an operad in the category of sets and $\mathcal{O}$ is the corresponding operad in the category of k-modules for some commutative ring with unit k which is defined as $\mathcal{O}(n) := k[\widetilde{\mathcal{O}}(n)]$ with $k[-]$ denoting the free k-module, then there is an associated category $\Gamma_{\mathcal{O}}$. In [8] this category was called *category of operators* and refer to the precise definition of $\Gamma_{\mathcal{O}}$ to their paper. The category $\Gamma_{\mathcal{O}}$ has the same objects as the category Γ. Roughly speaking the morphisms are maps f of finite pointed sets together with operad elements $w_i \in \widetilde{\mathcal{O}}(|f^{-1}(i)|)$.

The functor t is defined as the cokernel of two representable functors and we will define an analogous contravariant functor on $\Gamma_{\mathcal{O}}$. Let $\Gamma_n^{\mathcal{O}}$ be the representable functor $\Gamma_n^{\mathcal{O}}([m]) = k[\Gamma^{\mathcal{O}}([m],[n])]$. Fix an element $w \in \widetilde{\mathcal{O}(2)}$.

Definition 6.1. For a functor F from the category $\Gamma_{\mathcal{O}}$ to the category of k-modules the *Gamma-$\mathcal{O}$-homology of F* is defined as

$$H\Gamma\mathcal{O}_*(F) := \mathrm{Tor}_*(t_{\mathcal{O}}, F)$$

where $t_{\mathcal{O}}$ is defined as the cokernel of the map

$$\Gamma_2^{\mathcal{O}} \xrightarrow{\alpha} \Gamma_1^{\mathcal{O}}.$$

Here, α is the same sum of maps of pointed sets $d_0 - d_1 + d_2$ as before, but d_2 carries $\tau.w$ as operad element for the the preimage of zero, whereas d_0, d_1 carry w as their operad element for the preimage of zero resp. one.

This definition depends of course on the chosen element w. But we think of w as some canonical element in $\widetilde{\mathcal{O}(2)}$ that parametrizes a multiplication on $\mathcal{O}$-algebras and therefore we refrain from decorating $H\Gamma\mathcal{O}_*(F)$ with w.

As the operad of commutative monoids $\mathcal{O} = \mathcal{C}om$ comes from the operad whose degree n part consists of the one-elementian set, the category $\Gamma_{\mathcal{C}om}$ is identical with the category Γ and therefore Gamma-$\mathcal{C}om$-homology is ordinary Gamma homology. In the following we will identify $H\Gamma\mathcal{A}s_*(F)$ explicitly and thus we will focus on the operad $\mathcal{O} = \mathcal{A}s$ for associative monoids from now on. We will denote $H\Gamma\mathcal{A}s_*(F)$ by $H\Gamma(as)_*(F)$ and $t_{\mathcal{A}s}$ by $t(as)$. Here we choose w to be the identity in the symmetric group on two elements.

The category $\Gamma(as)$ can be described explicitly as in [12] as follows. Objects of $\Gamma(as)$ are finite pointed sets

$$[n] := \{0, 1, \ldots, n\}, \quad n \geqslant 0,$$

and a morphism $[n] \to [m]$ is a map $f : [n] \to [m]$ of finite pointed sets together with a total ordering on the preimages $f^{-1}(j)$ for all $j \in [m]$. In order to define the composition in $\Gamma(as)$ we recall the definition of the ordered union of ordered sets. If S is a totally ordered set and if X_s is a totally ordered set for each $s \in S$ the the disjoint union $X = \coprod_{s \in S} X_s$ will be ordered as follows: For $x \in X_r$ and $y \in X_s$ we declare $x \leqslant y$ in X if and only if $r < s$ or $r = s$ and $x \leqslant y$ in X_s.

If $f : [n] \to [m]$ and $g : [m] \to [k]$ are morphisms in $\Gamma(as)$, then the composite of g and f as a map is $g \circ f$, while the total ordering in $(g \circ f)^{-1}(i)$, $i \in [n]$ is given by the identification $(g \circ f)^{-1}(i) = \coprod_{j \in g^{-1}(i)} f^{-1}(j)$.

6.2. **Hochschild and cyclic homology.** In order to identify Gamma-$\mathcal{A}s$-homology, we recall the main results from [12]. Loday observed that the simplicial circle $C\colon \Delta^{op} \to \Gamma$ has a lift to $\Gamma(as)$

$$\hat{C} : \Delta^{op} \to \Gamma(as)$$

with $\hat{C}_n = [n]$ and face maps $d_i\colon [n] \to [n-1]$

$$d_i(j) = \begin{cases} j, & j \leqslant i, \\ j-1, & j > i, \end{cases} \quad \text{for} \quad i \neq n \quad \text{and} \quad d_n(j) = \begin{cases} j, & j \neq n, \\ 0, & j = n. \end{cases}$$

Here, the ordering on the preimages $d_i^{-1}\{j\}$ with more than one element is $i < i+1$ for $i \neq n$ and $n < 0$ for $i = n$. Note that the homology of the simplicial module $\mathcal{L}(A; M)(\hat{C})$ is Hochschild homology of an associative algebra A with coefficients in an A-bimodule M. In [12, Theorem 1.3] we identify Hochschild homology of any functor F form $\Gamma(as)$ to k-modules with $\mathrm{Tor}_*^{\Gamma(as)}(\bar{b}, F)$ and cyclic homology of any functor G from $\mathcal{F}(as)$ to k-modules with $\mathrm{Tor}_*^{\mathcal{F}(as)}(b, G)$. Here, b is the cokernel of $d_0 - d_1\colon \mathcal{F}(as)_1 \to \mathcal{F}(as)_0$ and $\bar{b}$ is the cokernel of $d_0 - d_1\colon \Gamma(as)_1 \to \Gamma(as)_0$.

6.3. **Gamma-$\mathcal{A}s$-homology.** With these prerequisites at hand it is straightforward to prove the following result.

Theorem 6.2. *For any functor F from $\Gamma(as)$ to the category of k-modules Gamma-$\mathcal{A}s$-homology of F is isomorphic to a shifted version of Hochschild homology of F,* i.e.,

$$H\Gamma(as)_*(F) \cong \begin{cases} HH_{*+1}(F) & * > 0, \\ t(as) \otimes_{\Gamma(as)} F & * = 0. \end{cases}$$

Proof. From [12, Proposition 2.2] we know that for every $[n]$ in $\Gamma(as)$ the simplicial module

$$\Delta^{op} \xrightarrow{\hat{C}} \Gamma(as) \xrightarrow{k[\Gamma(as)([n],-)]} k\text{-modules}$$

has homology concentrated in degree zero with zeroth homology group $\bar{b}[n]$. Therefore

$$\ldots \xrightarrow{d} \Gamma_n(as) \xrightarrow{d} \Gamma_{n-1}(as) \xrightarrow{d} \ldots \xrightarrow{d} \Gamma_1(as) \xrightarrow{d} \Gamma_0(as)$$

is a projective resolution of $\bar{b}$ with $d = \sum_{i=0}^n (-1)^i d_i$ and hence

$$\ldots \xrightarrow{d} \Gamma_n(as) \xrightarrow{d} \Gamma_{n-1}(as) \xrightarrow{d} \ldots \xrightarrow{d} \Gamma_1(as)$$

is a projective resolution of $t(as)$ such that the homology groups of

$$\ldots \xrightarrow{d \otimes \mathrm{id}} \Gamma_n(as) \otimes_{\Gamma(as)} F \xrightarrow{d \otimes \mathrm{id}} \Gamma_{n-1}(as) \otimes_{\Gamma(as)} F \xrightarrow{d \otimes \mathrm{id}} \ldots \xrightarrow{d \otimes \mathrm{id}} \Gamma_1(as) \otimes_{\Gamma(as)} F$$

are Hochschild homology groups. □

For instance we can consider the functor $\mathcal{L}(A;M)$ for an associative algebra A and an A-bimodule M. In degree zero $\mathrm{Tor}_*^{\Gamma(as)}(t(as),\mathcal{L}(A;M))$ gives the first Hochschild homology group of A with coefficients in M if one considers a symmetric A-module M. If M is not symmetric then the zeroth homology group differs, because one considers not just cycles modulo boundary but $M\otimes A$ modulo boundary.

Remark 6.3. Robinson and Whitehouse obtain a version of Gamma homology for associative algebras that gives shifted Hochschild homology for all degrees [17, Corollary 4.2]. So their theory differs from our version in degree zero.

6.4. **Cyclic Gamma-$\mathcal{A}s$-homology.** Starting off with $H\Gamma(as)_*(F)$ there is an obvious candidate for a cyclic variant. Let G be a functor from $\mathcal{F}(as)$ to the category of k-modules and let d be the boundary map $d\colon \mathcal{F}(as)_n \to \mathcal{F}(as)_{n-1}$. Here, d is defined as before for $\Gamma(as)$.

Definition 6.4. We define the *fake cyclic Gamma-$\mathcal{A}s$-homology of G* as

$$H\Gamma C(as)^f_*(G) = \mathrm{Tor}_*^{\mathcal{F}(as)}(r,G)$$

where r is the cokernel of $d\colon \mathcal{F}(as)_2 \to \mathcal{F}(as)_1$.

Proposition 6.5. *The fake cyclic Gamma-$\mathcal{A}s$-homology coincides with shifted Hochschild homology.*

Proof. From [12, Lemma 2.4] we see that for all $[n]$ in $\mathcal{F}(as)$ the simplicial k-module

$$\Delta^{op} \xrightarrow{\hat{C}} \mathcal{F}(as) \xrightarrow{k[\mathcal{F}(as)([n],-)]} k\text{-modules} \tag{3}$$

has homology in degrees zero and one and therefore the truncated complex

$$\ldots \xrightarrow{d} \mathcal{F}_n(as) \xrightarrow{d} \mathcal{F}_{n-1}(as) \xrightarrow{d} \ldots \xrightarrow{d} \mathcal{F}_1(as)$$

has homology concentrated in degree zero and this homology is precisely the functor r. We know that $\mathcal{F}(as)_n \otimes_{\mathcal{F}(as)} G \cong G(\underline{n})$ and the differentials in (3) give rise to the Hochschild boundary map. Therefore we obtain

$$H\Gamma C(as)^f_*(G) \cong \begin{cases} HH_{*+1}(G) & *>0, \\ r\otimes_{\mathcal{F}(as)} G & *=0. \end{cases}$$

and thus this fake cyclic variant of Gamma homology does not give anything new. □

Note that the adjunction of adding and forgetting basepoints

$$\nu\colon \mathcal{F} \rightleftarrows \Gamma : \mu$$

does not pass to the associative setting. We still have a functor $\mu\colon \Gamma(as) \to \mathcal{F}(as)$ that forgets the special role of the basepoint and a functor $\nu\colon \mathcal{F}(as) \to \Gamma(as)$ that adds an extra basepoint but these functors are no longer adjoint to each other. Possible alternatives to the above cyclic variant can be constructed as derived functors of pulled back versions of $t(as)$ as in (2). As we do not have adjointness at hand, there are several *a priori* different choices.

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