

# COMMUTATIVE RING SPECTRA

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ABSTRACT. In this survey paper on commutative ring spectra we present some basic features of commutative ring spectra and discuss model category structures. As a first interesting class of examples of such ring spectra we focus on (commutative) algebra spectra over commutative Eilenberg-Mac Lane ring spectra. We present two constructions that yield commutative ring spectra: Thom spectra associated to infinite loop maps and Segal's construction starting with bipermutative categories. We define topological Hochschild homology, some of its variants, and topological André-Quillen homology. Obstruction theory for commutative structures on ring spectra is described in two versions. The notion of étale extensions in the spectral world is tricky and we explain why. We define Picard groups and Brauer groups of commutative ring spectra and present examples.

## 1. INTRODUCTION

Since the 1990's we have several symmetric monoidal categories of spectra at our disposal whose homotopy category is the stable homotopy category. The monoidal structure is usually denoted by  $\wedge$  and is called the smash product of spectra. So since then we can talk about commutative objects in any of these categories – these are commutative ring spectra. Even before such symmetric monoidal categories were constructed, the consequences of their existence were described. In [122, §2] Friedhelm Waldhausen outlines the role of ‘rings up to homotopy’. He also coined the expression ‘brave new rings’ in a 1988 talk at Northwestern.

So what is the problem? Why don't we just write down nice commutative models of our favorite homotopy types and are down with it? Why does it make sense to have a whole chapter about this topic?

In algebra, if someone tells you to check whether a given ring is commutative, then you can sit down and check the axiom for commutativity and you should be fine. In stable homotopy theory the problem is more involved, since strict commutativity may only be satisfied by some preferred point set level model of the underlying associative ring spectrum and the operadic incarnation of commutativity is an extra structure rather than a condition.

There is one class of commutative ring spectra that is easy to construct. If you take singular cohomology with coefficients in a commutative ring  $R$ , then this is represented by the Eilenberg-Mac Lane spectrum  $HR$  and this can be represented by a commutative ring spectrum.

So it would be nice if we could have explicit models for other homotopy types that come naturally equipped with a commutative ring structure. Sometimes this is possible. If you are interested in real (or complex) vector bundles over your space, then you want to understand real (or complex) topological K-theory and Michael Joachim [56] for instance produced an explicit analytically flavoured model for periodic real topological K-theory as a commutative ring spectrum in the setting of symmetric spectra [54].

There are a few general constructions that produce commutative ring spectra for you. For instance, the construction of Thom spectra often gives rise to commutative ring spectra. We will discuss this important class of examples in Section 4. A classical construction due to Graeme Segal also produces small nice models of commutative ring spectra (see Section 5).

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Quite often, however, the spectra that we like are constructed in a synthetic way: You have some commutative ring spectrum  $R$  and you kill a regular sequence of elements in its graded commutative ring of homotopy groups,  $(x_1, x_2, \dots)$ ,  $x_i \in \pi_*(R)$ , and you consider a spectrum  $E$  with homotopy groups  $\pi_*(E) \cong \pi_*(R)/(x_1, x_2, \dots)$ . Then it is not clear that  $E$  is a commutative ring spectrum.

A notorious example is the Brown-Peterson spectrum,  $BP$ . Take the complex cobordism spectrum  $MU$ . Its homotopy groups are

$$\pi_*(MU) = \mathbb{Z}[x_1, x_2, \dots]$$

with  $x_i$  being a generator in degree  $2i$ . If you fix a large even degree, then you have a lot of possible elements in that degree, so you might wish to consider a spectrum with sparser homotopy groups. Using the theory of (commutative, 1-dimensional) formal group laws you can do that: If you consider a prime  $p$ , then there is a spectrum, called the Brown-Peterson spectrum, that corresponds to  $p$ -typical formal group laws. It can be realized as the image of an idempotent on  $MU$  and has

$$\pi_*(BP) \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots]$$

but now the algebraic generators are spread out in an exponential manner: The degree of  $v_i$  is  $2p^i - 2$ . You can actually choose the  $v_i$  as the  $x_{p^i-1}$ , so you can think of  $BP$  as a quotient of  $MU$  in the above sense. Since its birth in 1966 [27] its multiplicative properties were an important issue. In [18] it was for instance shown that  $BP$  has some partial coherence for homotopy commutativity, but in 2017 Tyler Lawson finally shows that at the prime 2 at least,  $BP$  is *not* a commutative ring spectrum [61]!

There are even worse examples: If you take the sphere spectrum  $S$  and you try to kill the non-regular element  $2 \in \pi_0(S)$  then you get the mod-2-Moore spectrum. That isn't even a ring spectrum up to homotopy. You can also kill all the generators  $v_i \in \pi_*(BP)$  including  $p = v_0$  and leaving only one  $v_n$  alive. The resulting spectrum is the connective version of Morava K-theory,  $k(n)$ . At the prime 2 this isn't even homotopy commutative. In fact, Urs Würgler shows more [125]: If  $\pi_0$  of a homotopy commutative ring spectrum has characteristic two, then it is a generalized Eilenberg-Mac Lane spectrum. Recent work of Mathew, Naumann and Noel puts severe restrictions on finite  $E_\infty$ -ring spectra [79].

So how do you decide such questions? How can you determine whether a given spectrum is a commutative ring spectrum if you don't have a construction that tells you right away that it is commutative? This is where obstruction theory comes into the story.

There is an operadic notion of an  $E_\infty$ -ring spectrum that goes back to Boardman-Vogt and May. Comparison theorems [76], [111] then tell you whether these more complicated objects are equivalent to commutative ring spectra. In the categories of symmetric spectra, orthogonal spectra and  $S$ -modules they are.

Obstruction theory might help you with a decision whether a spectrum carries a commutative monoid structure: One version [16] gives obstructions for lifting the ordinary Postnikov tower to a Postnikov tower that lives within the category of commutative ring spectra. The other kind finds some obstruction classes that tell you that you cannot extend some partial bits and pieces of a nice multiplication to a fully fledged structure of an  $E_\infty$ -ring spectrum or that some homology or homotopy operation that you observe contradicts such a structure. This can be used for a negative result (as in [61]) or for positive statements: There are result by Robinson [101] and Goerss-Hopkins [38, 39] that tell you that you have a (sometimes even unique)  $E_\infty$ -ring structure on your spectrum if all the obstruction *groups* vanish. Most notably Goerss and Hopkins used obstruction theory to prove that the Morava stabilizer groups acts on the corresponding Lubin-Tate spectrum via  $E_\infty$ -morphisms [38].

There are other things that are weird for commutative ring spectra. Quite often, we end up working with ideals in the graded commutative ring of homotopy groups, but as we saw above, this is not a suitable notion of ideal. There is a notion of an ideal in the context of

(commutative) ring spectra [53] due to Jeff Smith, but still several algebraic constructions do not have an analogue in spectra.

The algebraic behaviour on the level of homotopy groups can be quite deceiving: complexification turns a real vector bundle into a complex vector bundle. This induces a map  $\pi_*(KO) \rightarrow \pi_*(KU)$  which can be realized as a map of commutative ring spectra  $c: KO \rightarrow KU$ . On homotopy groups we get

$$(1.1) \quad \pi_*(c): \pi_*(KO) = \mathbb{Z}[\eta, y, \omega^{\pm 1}]/(2\eta, \eta^3, \eta y, y^2 - 4w) \rightarrow \mathbb{Z}[u^{\pm 1}]\pi_*(KU).$$

Here the degrees are  $|\eta| = 1$ ,  $|y| = 4$ ,  $|w| = 8$ ,  $|u| = 2$  and  $y$  is sent to  $2u^2$ . So on the algebraic level  $c$  is horrible. But John Rognes showed that the conjugation action on  $KU$  turns the map  $c: KO \rightarrow KU$  into a  $C_2$ -Galois extension of commutative ring spectra!

Even for ordinary rings, viewing a (commutative) ring  $R$  as a (commutative) ring spectrum via the Eilenberg-Mac Lane spectrum functor changes the situation completely. The ring  $R$  has a characteristic map  $\chi: \mathbb{Z} \rightarrow R$  because the integers are the initial ring. As a ring spectrum,  $H\mathbb{Z}$  is far from being initial. The map  $H\chi$  can be precomposed with the unit map of  $H\mathbb{Z}$

$$S \xrightarrow{\eta} H\mathbb{Z} \xrightarrow{H\chi} HR$$

and the sphere spectrum  $S$  is the initial ring spectrum! Now there is a lot of space between the sphere and any ring. I will discuss two consequences that this has: There is actually algebraic geometry happening between the sphere spectrum and the prime field  $\mathbb{F}_p$ : There is a Galois extension of commutative ring spectra (see 8.1)  $A \rightarrow H\mathbb{F}_p$ !

Another feature is that there exist differential graded algebras  $A_*$  and  $B_*$  that are not quasi-isomorphic, but whose associated algebra spectra over an Eilenberg-Mac Lane spectrum [117] are equivalent as ring spectra [31]. Similar phenomena happen if you consider differential graded  $E_\infty$ -algebras: There are non quasi-isomorphic ones whose associated commutative algebras over an Eilenberg-Mac Lane spectrum [98] are equivalent as commutative ring spectra [21].

**Content.** The structure of this overview is as follows: We start with some basic features of commutative ring spectra and their model category structures in Section 2. The most basic way to relate classical algebra to brave new algebra is via the Eilenberg-Mac Lane spectrum functor. We study chain algebras and algebras over Eilenberg-Mac Lane ring spectra in Section 3. As you can study the group of units of a ring we consider units of ring spectra and Thom spectra in Section 4. In Section 5 we present a construction going back to Segal. Plugging in a bipermutative category yields a commutative ring spectrum.

In Section 6 we introduce topological Hochschild homology and some of its variants and topological André-Quillen homology. In Section 7 we discuss some versions of obstruction theory that tell you whether a given multiplicative cohomology theory can be represented by a strict commutative model.

Some concepts from algebra translate directly to spectra but some others don't. We discuss the different concepts of étale maps for commutative algebra spectra in Section 8. Picard and Brauer groups for commutative ring spectra are important invariants and feature in Section 9.

**Disclaimers.** For more than 30 years, the phrase *commutative ring spectrum* meant a commutative monoid in the homotopy category of spectra. Since the 90's this has changed. At the beginning of this new era people were careful not to use this name, in order to avoid confusion with the homotopy version. In this paper we reserve the phrase *commutative ring spectrum* for a commutative monoid in some symmetric monoidal category of spectra.

The second disclaimer is that for this paper a space is always compactly generated weak Hausdorff. I denote the corresponding category just by **Top**.

Last but not least: Of course, this overview is not complete. I had to omit important aspects of the field due to space constraints. Most prominently probably is the omission of not discussing topological cyclic homology and its wonderful applications to algebraic K-theory.

I try to give adequate references, but often it was just not feasible to describe the whole development of a topic and much worse, I probably have forgotten to cite important contributions. If you read this and it affects you, then I can only apologize.

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## 2. FEATURES OF COMMUTATIVE RING SPECTRA

**2.1. Some basics.** Before we actually start with model structures, we state some basic facts about commutative ring spectra.

Let  $R$  be a commutative ring spectrum. Then the category of  $R$ -module spectra is closed symmetric monoidal: For two such  $R$ -module spectra  $M, N$  the smash product over  $R$ ,  $M \wedge_R N$ , is again an  $R$ -module and the usual axioms of a symmetric monoidal category are satisfied. There is an  $R$ -module spectrum  $F_R(M, N)$ , the function spectrum of  $R$ -module maps from  $M$  to  $N$ .

We denote the category of  $R$ -module spectra by  $\mathcal{M}_R$ . The category of commutative  $R$ -algebras is the category of commutative monoids in  $\mathcal{M}_R$  and we denote this category by  $\mathcal{C}_R$ .

By definition, every object  $A$  of  $\mathcal{C}_R$  receives a unit map from  $R$  and hence  $R$  is initial in  $\mathcal{C}_R$ . In particular, the sphere spectrum is the initial commutative ring spectrum. Every discrete ring is a  $\mathbb{Z}$ -algebra, similarly, every (commutative) ring spectrum is a (commutative)  $S$ -algebra. If  $R$  is a commutative ring spectrum, then the category of commutative  $R$ -algebra is isomorphic to the category of commutative ring spectra under  $R$ , *i.e.*, such commutative ring spectra  $A$  with a distinguished map  $\eta: R \rightarrow A$  in that category.

We allow the trivial  $R$ -algebra corresponding to the one-point spectrum  $*$  and this spectrum is a terminal object in  $\mathcal{C}_R$ .

In any symmetric monoidal category  $(\mathcal{C}, \otimes, 1, \tau)$  the coproduct of two commutative monoids  $A$  and  $B$  in  $\mathcal{C}$  is  $A \otimes B$ . So, for two commutative  $R$ -algebras  $A$  and  $B$ , their coproduct is  $A \wedge_R B$ .

**2.2. Model structures on commutative monoids.** I will assume that you are familiar with the concept of model categories and that you have seen some examples and read Chapter 4 in this book. Good general reference are Hovey's [52] or Hirschhorn's [48] book. You could also just skip this section and have in mind that there are some serious model category issues lurking in the dark.

For this section I will mainly focus on two models for spectra: Symmetric spectra [54] and  $S$ -modules [34]. They are different concerning their model structures. In the model structure in [54] on symmetric spectra the sphere spectrum is cofibrant whereas in the one for  $S$ -modules it is not, but all objects are fibrant.

The model structures on commutative monoids in either of the categories [34, 54] are special cases of a *right induced model structure*: We have a functor  $P_R$  from  $R$ -module spectra to commutative  $R$ -algebra spectra assigning the free commutative  $R$ -algebra spectrum on  $M$  to any  $R$ -module spectrum  $M$ , explicitly

$$P_R(M) = \bigvee_{n \geq 0} M^{\wedge_{R^n}} / \Sigma_n.$$

The symbol  $P_R$  should remind you of a polynomial algebra. This functor has a right adjoint, the forgetful functor  $U$ . In a right-induced model structure one determines the fibrations and weak equivalences by the right adjoint functor. In our cases, a map of commutative  $R$ -algebra spectra is a fibration or a weak equivalence, if it is one in the underlying category of  $R$ -module spectra. Note that establishing right induced model structures on commutative monoids in some model category does not always work. The standard example is the category of  $\mathbb{F}_p$ -chain complexes (say  $p$  is an odd prime). Then the chain complex  $\mathbb{D}^2$  is acyclic, having  $\mathbb{F}_p$  in degrees 1 and 2 with the identity map as differential, but the free graded commutative monoid generated

by it is  $\Lambda_{\mathbb{F}_p}(x_1) \otimes \mathbb{F}_p[x_2]$  with  $|x_i| = i$  and the induced differential is determined by  $d(x_2) = x_1$  and the Leibniz rule. But then  $d(x_2^p)$  is a cycle that is not a boundary, so the resulting object is not acyclic.

If  $R$  is a commutative  $S$ -algebra in the setting of EKMM [34], then the categories of associative  $R$ -algebras and of commutative  $R$ -algebras possess a right induced model structure [34, Corollary VII.4.10]. The existence of the model structure for commutative monoids is a special case of the existence of right-induced model structures for  $\mathbb{T}$ -algebras ([34, Theorem VII.4.9]) where  $\mathbb{T}$  is a continuous monad on the category of  $R$ -module spectra that preserves reflexive coequalizers and satisfies the cofibration hypothesis [34, VII.4]. The category of commutative  $S$ -algebras is identified [34, Proposition II.4.5] with the category of algebras for the monad  $P_S$  as above on the category of  $S$ -modules.

In diagram categories such as symmetric spectra and orthogonal spectra the situation is different: In the standard model structures on these categories the sphere spectrum is cofibrant. If one would take a right-induced model structure on the category of commutative monoids, *i.e.*, the model structure such that a map of commutative ring spectra  $f: A \rightarrow B$  is a fibration or weak equivalence if it is one in the underlying category, then the sphere would still be cofibrant. If we take a fibrant replacement of the sphere  $S \rightarrow S^{\text{fib}}$ , then in particular  $S^{\text{fib}}$  would be fibrant in the model category of symmetric spectra, hence it would be an Omega-spectrum and its zeroth level would be a strictly commutative model for  $QS^0$ . However, Moore shows [90, Theorem 3.29] that this implied that  $QS^0$  had the homotopy type of a product of Eilenberg-MacLane spaces – but this is false.

The usual way to avoid this problem is to consider a positive model structure on  $Sp^\Sigma$  (see [76, Definition 6.1] for the general approach). Here the positive level fibrations (weak equivalences) are maps  $f \in Sp^\Sigma(X, Y)$  such that  $f(n)$  is a fibration (weak equivalence) for all levels  $n \geq 1$ . The positive cofibrations are then cofibrations in  $Sp^\Sigma$  that are isomorphisms in level zero. The positive stable model category is then obtained by a Bousfield localization that forces the stable equivalences to be the weak equivalences and the right-induced model structure on the commutative monoids in  $Sp^\Sigma$  then has the desired properties.

There is another nice model for connective spectra, given by  $\Gamma$ -spaces [114, 70]. This category is built out of functors from finite pointed sets to spaces, so it is a very hands-on category with explicit constructions. It is also a symmetric monoidal category with a suitable model structure. We refer to [70, 110] for background on this. Its (commutative) monoids are called (commutative)  $\Gamma$ -rings. Beware that commutative  $\Gamma$ -rings, however, do *not* model all connective commutative ring spectra. Tyler Lawson proves in [62] that commutative  $\Gamma$ -rings satisfy a vanishing condition for Dyer-Lashof operations of positive degree on classes in their zeroth mod- $p$ -homology (for all primes  $p$ ) and that for instance the free  $E_\infty$ -ring spectrum generated by  $\mathbb{S}^0$  cannot be modelled as a commutative  $\Gamma$ -ring.

**2.3. Behaviour of the underlying modules.** In the setting of EKMM it is shown that the underlying  $R$ -modules of cofibrant commutative  $R$ -algebras have a well-behaved smash product in the derived category of  $R$ -modules:

**Theorem 2.1.** [34, Theorem VII.6.7] *If  $A$  and  $B$  are two cofibrant commutative  $R$ -algebras, and if  $\varphi_A: \Gamma A \xrightarrow{\sim} A$  and  $\varphi_B: \Gamma B \xrightarrow{\sim} B$  are chosen cell  $R$ -module spectra approximations then*

$$\varphi_A \wedge_R \varphi_B: \Gamma A \wedge_R \Gamma B \rightarrow A \wedge_R B$$

*is a weak equivalence.*

Brooke Shipley developed a model structure for commutative symmetric ring spectra in [116] in which the underlying symmetric spectrum of a cofibrant commutative ring spectrum is also cofibrant as a symmetric spectrum [116, Corollary 4.3].

She starts with introducing a different model structure on symmetric spectra. Let  $M$  denote the class of monomorphisms of symmetric sequences in pointed simplicial sets and let  $S \otimes M$  denote the set  $\{S \otimes f, f \in M\}$  where  $\otimes$  denotes the tensor product of symmetric sequences. An

$S$ -cofibration is a morphism in  $(S \otimes M)$ -cof, *i.e.*, a morphism in  $Sp^\Sigma$  that has the left lifting property with respect to maps that have the right lifting property with respect to  $S \otimes M$ . She shows that the classes of  $S$ -cofibrations and stable equivalences determine a model structure with the  $S$ -fibrations being the class of morphisms with the right lifting property with respect to  $S$ -cofibrations that are also stable equivalences [116, Theorem 2.4]. This model structure was already mentioned in [54, 5.3.6]. Shipley proves that this model structure is cofibrantly generated, is monoidal and satisfies the monoid axiom [116, 2.4, 2.5].

Note that symmetric spectra are  $S$ -modules in symmetric sequences. This allows for a version of an  $R$ -model structure for every associative symmetric ring spectrum  $R$  with  $R$ -cofibrations,  $R$ -fibrations and stable equivalences [116, Theorem 2.6]. In the positive variant of this model structure the positive  $R$ -cofibrations are  $R$ -cofibrations that are isomorphisms in level zero. Together with the stable equivalences this determines the *positive  $R$ -model structure*.

The corresponding right induced model structure on commutative  $R$ -algebra spectra for a commutative symmetric ring spectrum  $R$  is then the *convenient* model structure: The weak equivalences are stable equivalences, the fibrations are positive  $R$ -fibrations and the cofibrations are determined by the structure.

She then shows the remarkable property of this model structure on commutative  $R$ -algebra spectra:

**Theorem 2.2.** [116, Corollary 4.3] *If  $A$  is cofibrant as a commutative  $R$ -algebra then  $A$  is  $R$ -cofibrant in the  $R$ -model structure. If  $A$  is fibrant as a commutative  $R$ -algebra, then  $A$  is fibrant in the positive  $R$ -model structure on  $R$ -module spectra.*

The positive  $R$ -model structure ensures that  $R$  is *not* cofibrant, hence cofibrant commutative  $R$ -algebras will not be positively  $R$ -cofibrant!

**2.4. Comparison, rigidification and  $E_n$ -structures.** Stefan Schwede proves [111, Theorem 5.1] that the homotopy category of commutative  $S$ -algebras from [34] is equivalent to the homotopy category of commutative symmetric ring spectra by establishing a Quillen equivalence between the corresponding model categories. In [76, Theorem 0.7] the analogous comparison result is proven for commutative orthogonal ring spectra and commutative symmetric ring spectra.

Even before any symmetric monoidal category of spectra was constructed, the notion of operadically defined  $E_\infty$ -ring spectra [81] was available. An  $E_\infty$ -structure on a spectrum is a multiplication that is homotopy commutative in a coherent way. See Chapter 5 of this book for background on operads and their role in the study of spectra with additional structure.

There is an explicit comparison of the good old  $E_\infty$ -ring spectra and commutative ring spectra, see [34, Proposition II.4.5] or [76, Remark 0.14], in particular, every  $E_\infty$ -ring spectrum  $\bar{R}$  can be rigidified to a commutative ring spectrum  $R$  in such a way that the homotopy type is preserved.

There are several popular  $E_\infty$ -operads that will show up later: for instance the linear isometries operad (see (4.3)) and the Barratt-Eccles operad. The  $n$ -ary part of the latter is easy to describe: You take  $\mathcal{O}(n) = E\Sigma_n$ , a contractible space with free  $\Sigma_n$ -action. For compatibility reasons it is advisable to take the realization of the standard simplicial model of  $E\Sigma_n$  whose set of  $q$ -simplices is  $(\Sigma_n)^{q+1}$ .

An operad with a nice geometric description is the little  $m$ -cubes operad, that in arity  $n$  consists of the space of  $n$ -tuples of linearly embedded  $m$ -cubes in the standard  $m$ -cube with disjoint interiors and with axes parallel to that of the ambient cube. [23, Example 5]. We call this (and every equivalent) operad in spaces  $E_m$ . For  $m = 1$  this operad parametrizes  $A_\infty$ -structures and the colimit is an  $E_\infty$ -operad. Hence the intermediate  $E_m$ 's for  $1 < m < \infty$  interpolate between these structures, they give  $A_\infty$ -structures with homotopy-commutative multiplications that are coherent up to some order.

**2.5. Power operations.** The extra structure of an  $E_\infty$ -ring spectrum gives homology operations. The general setting allows for  $H_\infty$ -ring spectra [28]; for simplicity we assume that  $E$  and

$R$  are two  $E_\infty$ -ring spectra whose structure is given by the Barratt-Eccles operad, *i.e.*, there are structure maps

$$(2.1) \quad \xi_R^n: (E\Sigma_n)_+ \wedge_{\Sigma_n} R^{\wedge n} \rightarrow R$$

for  $R$  and also for  $E$ . McClure describes the general setting of power operations in [28, IX §1]. Fix a prime  $p$  and abbreviate  $(E\Sigma_p)_+ \wedge_{\Sigma_p} R^{\wedge p}$  by  $D_p(R)$ ;  $D_p R$  is often known as the  *$p$ th extended power construction on  $R$* . A *power operation* assigns to every class  $[x] \in E_n R$  and every class  $e \in E_m(D_p S^n)$  a class  $Q^e[x] \in E_m R$ , hence we can view  $Q^e$  as a map

$$Q^e: E_n R \rightarrow E_m R.$$

The construction is as follows. Take a representative  $x: S^n \rightarrow E \wedge R$  of  $[x]$  and  $e \in E_m(D_p S^n)$  and apply the following composition to  $e$ :

$$(2.2) \quad \begin{array}{ccc} E_m(D_p S^n) & \xrightarrow{E_m(D_p x)} & E_m(D_p(E \wedge R)) \\ & \searrow \text{dotted} & \downarrow \delta \\ & & E_m(D_p E \wedge D_p R) \\ & & \downarrow E_m(\xi_E^p \wedge \text{id}) \\ & & E_m(E \wedge D_p R) \\ & & \downarrow \mu_* \\ & & E_m(D_p R) \\ & \searrow \text{dotted} & \downarrow E_m(\xi_R^p) \\ & & E_m(R). \end{array}$$

Here,

$$\delta: (E\Sigma_p)_+ \wedge_{\Sigma_p} (E \wedge R)^{\wedge p} \rightarrow (E\Sigma_p)_+ \wedge_{\Sigma_p} E^{\wedge p} \wedge (E\Sigma_p)_+ \wedge_{\Sigma_p} R^{\wedge p}$$

is the canonical map induced by the diagonal on the space  $E\Sigma_p$  and  $\mu$  denotes the multiplication in  $E$ , so it induces

$$\mu_*: \pi_*(E \wedge E \wedge D_p R) \rightarrow \pi_*(E \wedge D_p R).$$

There are several important special cases of this construction:

- (a) For  $E$  the sphere spectrum one obtains operations on the homotopy groups of an  $E_\infty$ -ring spectrum, see [28, IV §7].
- (b) For  $E = H\mathbb{F}_p$  the power operations for certain classes  $e_i \in H_i(\Sigma_p; \mathbb{F}_p)$  are often called (Araki-Kudo-)Dyer-Lashof operations. These are natural homomorphisms

$$(2.3) \quad Q^i: (H\mathbb{F}_p)_n(R) \rightarrow (H\mathbb{F}_p)_{n+2i(p-1)}(R)$$

for odd primes and  $Q^i: (H\mathbb{F}_2)_n(R) \rightarrow (H\mathbb{F}_2)_{n+i}(R)$  at the prime 2 that satisfy a list of axioms [28, Theorem III.1.1] and compatibility relations with the homology Bockstein and the dual Steenrod operations.

- (c) There are also important  $K(n)$ -local versions of such operations and we will encounter them later.

### 3. CHAIN ALGEBRAS AND ALGEBRAS OVER EILENBERG-MAC LANE SPECTRA

The derived category of a ring is an important object in many subjects. The initial ring is the ring of integers. Every ring  $R$  has an associated Eilenberg-Mac Lane spectrum,  $HR$ .

**3.1. *HR*-module and algebra spectra.** We collect some results that compare the category of chain complexes of  $R$ -modules with the category of module spectra over  $HR$ . We start with additive statements and move to comparison results for flavors of differential graded  $R$ -algebras. For an overview of algebraic applications of these equivalences see for instance [41].

In the eighties, so before any strict symmetric monoidal category of spectra was constructed, Alan Robinson developed the notion of the derived category,  $\mathcal{D}(E)$ , of right  $E$ -module spectra for every  $A_\infty$ -ring spectrum  $E$ . He showed the following result.

**Theorem 3.1.** [99, Theorem 3.1] *For every associative ring  $R$  there is an equivalence of categories between the derived category of  $R$ ,  $\mathcal{D}(R)$ , and the derived category of the associated Eilenberg-Mac Lane spectrum,  $\mathcal{D}(HR)$ .*

Later, in the context of  $S$ -modules this corresponds to [34, IV, Theorem 2.4]. Work of Schwede and Shipley strengthened the result to a Quillen equivalence of the corresponding model categories:

**Theorem 3.2.** [113, Theorem 5.1.6] *The model category of unbounded chain complexes of  $R$ -modules is Quillen equivalent to the model category of *HR*-module spectra.*

Stefan Schwede uses the setting of  $\Gamma$ -spaces [110] to embed simplicial rings and modules into the stable world: He constructs a lax symmetric monoidal Eilenberg-Mac Lane functor  $H$  from simplicial abelian groups to  $\Gamma$ -spaces together with a linearization functor  $L$  in the opposite direction and proves the following comparison result:

**Theorem 3.3.** [110, Theorems 4.4, 4.5] *If  $R$  is a simplicial ring, then the adjoint functors  $H$  and  $L$  constitute a Quillen equivalence between the categories of simplicial  $R$ -modules and *HR*-module spectra. If  $R$  is in addition commutative, then  $H$  and  $L$  induce a Quillen equivalence between the categories of simplicial  $R$ -algebras and *HR*-algebra spectra.*

Here, the functor  $L$  is actually left inverse to  $H$  and induces an isomorphism of  $\Gamma$ -spaces

$$\mathrm{Hom}(HA, HB) \cong H(\mathrm{Hom}_{\mathbf{sAb}}(A, B))$$

[110, Lemma 2.1], thus  $H$  embeds algebra into brave new algebra.

Brooke Shipley extends this equivalence to corresponding categories of monoids in the differential graded setting:

**Theorem 3.4.** [117, Theorem 1.1] *For any commutative ring  $R$ , the model categories of unbounded differential graded  $R$ -algebras and *HR*-algebra spectra are Quillen equivalent.*

Dugger and Shipley show in [31] that there are examples of *HR*-algebras that are weakly equivalent as  $S$ -algebras, but that are not quasi-isomorphic. A concrete example is the differential graded ring  $A_*$  which is generated by an element in degree 1,  $e_1$ , and has  $d(e_1) = 2$  and satisfies  $e_1^4 = 0$ . The corresponding  $H\mathbb{Z}$ -algebra spectrum is equivalent as a ring spectrum to the one on the exterior algebra  $B_* = \Lambda_{\mathbb{F}_2}(x_2)$  (with  $|x_2| = 2$ ) but  $A_*$  and  $B_*$  are *not* quasi-isomorphic. You find more examples and proofs in [31, §§4,5].

We cannot expect that commutative *HR*-algebra spectra correspond to commutative differential graded  $R$ -algebras unless  $R$  is of characteristic zero, because of cohomology operations, but we get the following result:

**Theorem 3.5.** [98, Corollary 8.3] *Let  $R$  be a commutative ring. Then there is a chain of Quillen equivalences between the model category of commutative *HR*-algebra spectra and  $E_\infty$ -monoids in the category of unbounded  $R$ -chain complexes.*

Haldun Özgür Bayındır shows [21] that one can find  $E_\infty$ -differential graded algebras that are not quasi-isomorphic, but whose corresponding commutative *HR*-algebra spectra are equivalent as commutative ring spectra.



**3.2. Cochain algebras.** A prominent class of examples of commutative  $HR$ -algebra spectra consists of function spectra  $F(X_+, HR)$ . Here,  $X$  is an arbitrary space and  $R$  is a commutative ring. The diagonal  $\Delta: X \rightarrow X \times X$  and the multiplication on  $HR$ ,  $\mu_{HR}$ , induce a multiplication

$$\begin{array}{ccc} F(X_+, HR) \wedge F(X_+, HR) & \longrightarrow & F(X_+ \wedge X_+, HR \wedge HR) \cong F((X \times X)_+, HR \wedge HR) \\ & & \downarrow \Delta^*, \mu_{HR} \\ & & F(X_+, HR) \end{array}$$

that turns  $F(X_+, HR)$  into a  $HR$ -algebra spectrum. As the diagonal is cocommutative and as  $\mu_{HR}$  is commutative, the resulting multiplication is commutative.

These function spectra are models for the singular cochains of a space  $X$  with coefficients in  $R$ :

$$\pi_*(F(X_+, HR)) \cong H^{-*}(X; R).$$

Beware that the homotopy groups of  $F(X_+, HR)$  are concentrated in non-positive degrees – *i.e.*,  $F(X_+, HR)$  is coconnective.

Studying the singular cochains of a space  $S^*(X; R)$  as a differential graded  $R$ -module is not enough in order to recover the homotopy type of  $X$ . If we work over the rational numbers, then Quillen showed that rational homotopy theory is algebraic in the sense that one can use rational differential graded Lie algebras or coalgebras as models for rational homotopy theory [94]. Sullivan [118] constructed a functor, assigning a rational differential graded commutative algebra to a space, that is closely related to the singular cochain functor with rational coefficients. He used this to classify rational homotopy types.

For a general commutative ring  $R$ , the singular cochains are an  $E_\infty$ -algebra. Mike Mandell proves [72, Main Theorem] that the singular cochain functor with coefficients in an algebraic closure of  $\mathbb{F}_p$ ,  $\bar{\mathbb{F}}_p$ , induces an equivalence between the homotopy category of connected  $p$ -complete nilpotent spaces of finite  $p$ -type and a full subcategory of the homotopy category of  $E_\infty$ - $\bar{\mathbb{F}}_p$ -algebras. He also characterizes those  $E_\infty$ - $\bar{\mathbb{F}}_p$ -algebras that arise as cochain algebras of 1-connected  $p$ -complete spaces of finity  $p$ -type explicitly [72, Characterization Theorem]. There is also an integral version of this result, stating that finite type nilpotent spaces are weakly equivalent if and only if their  $E_\infty$ -algebras of integral cochains are quasi-isomorphic [74, Main Theorem].

#### 4. UNITS OF RING SPECTRA AND THOM SPECTRA

One construction that can give rise to highly structured multiplications on a spectrum is the Thom spectrum construction: For instance, complex bordism,  $MU$ , obtains a commutative ring structure this way. Early on Mahowald emphasized [71] that multiplicative properties of the structure maps for Thom spectra translate to multiplicative structures on the resulting Thom spectra. Their properties and the corresponding orientation theory is systematically studied in [81]. There is the following general result by Lewis:

**Theorem 4.1.** [64, Theorem IX.7.1 and Remark IX.7.2] *Assume that  $f$  is a map of spaces from  $X$  to the classifying space for stable spherical fibrations,  $BG$ , that is a  $\mathcal{C}$ -map for some operad  $\mathcal{C}$  over the linear isometries operad. Then the Thom spectrum  $M(f)$  associated to  $f$  carries a  $\mathcal{C}$ -structure. In particular, infinite loops maps from  $X$  to  $BG$  give rise to  $E_\infty$ -ring spectra.*

Note that  $BG$  is the classifying space of the units of the sphere spectrum,  $GL_1(S)$ . A naive definition of  $GL_1(S)$  is given by the pullback of the diagram

$$(4.1) \quad \begin{array}{ccc} GL_1 S & \xrightarrow{\quad} & \Omega^\infty S \\ \downarrow \text{dotted} & & \downarrow \\ \pi_0(S)^\times = \{\pm 1\} & \longrightarrow & \pi_0(S) = \mathbb{Z}, \end{array}$$

9

so by the components of  $QS^0$  corresponding to  $\pm 1 \in \mathbb{Z}$ .

In the following we give a short overview of Thom spectra that arise in a more general context, where the target of the map is the space of units,  $GL_1(R)$ , for a commutative ring spectrum  $R$ . The first idea is to define the space  $GL_1(R)$  as the space that represents the functor that sends a space  $X$  to the units in  $R^0(X)$ . Copying the definition from (4.1) above with  $S$  replaced by  $R$  gives a valid first definition of  $GL_1(R)$  and it was shown [81] that for commutative  $R$  this model is an  $E_\infty$ -space.

In the approaches [1] and [20], the idea is to replace the naive model of  $GL_1(R)$  with its  $E_\infty$ -structure with a strictly commutative model. As spaces with an  $E_\infty$ -structure are not equivalent to strictly commutative spaces (that's the problem again that then  $QS^0$  would be a product of Eilenberg-Mac Lane spaces [90]), one has to find a different category with the property that there is a Quillen equivalence between commutative monoids in that category and  $E_\infty$ -monoids in spaces and such that there are models of  $\Omega^\infty(R)$  and  $GL_1(R)$  in this category.

In [1] the authors work with  $*$ -modules and in [20] the authors use Schlichtkrull's model of  $GL_1(R)$  in commutative  $I$ -spaces where  $I$  is the skeleton of the category of finite sets and injections.

The idea is to construct a spectrum version of the assembly map for discrete rings: If  $R$  is a discrete ring and if  $R^\times$  is its group of units, then there is a canonical map

$$(4.2) \quad \mathbb{Z}[R^\times] \rightarrow R$$

from the group ring  $\mathbb{Z}[R^\times]$  to  $R$  that takes an element  $\sum_{i=1}^n a_i r_i$  of  $\mathbb{Z}[R^\times]$  (with  $a_i \in \mathbb{Z}$  and  $r_i \in R^\times$ ) to the same sum, but now we use the ring structure of  $R$  to convert the formal sum into an actual sum  $\sum_{i=1}^n a_i r_i \in R$ . Note that  $R^\times$  is an abelian group if  $R$  is a commutative ring.

We will sketch both constructions of Thom spectra and briefly discuss the application in [1] to the question when a Thom spectrum allows for an  $E_\infty$ -map to some other  $E_\infty$ -ring spectrum, for instance, whether one can realize an  $E_\infty$ -version of the string orientation  $MO\langle 8 \rangle \rightarrow \mathrm{tmf}$  [3] or an  $E_\infty$ -version of a complex orientation [49].

The focus in [20] is on multiplicative properties of the Thom spectrum functor and on applications to topological Hochschild homology. We present the results about multiplicative structures and discuss their results on THH of Thom spectra in Section 6. We'll also describe how the concept of  $I$ -spaces can be generalized to a setting in which the units can be adapted to non-connective ring spectra.

**4.1. Thom spectra via  $\mathbb{L}$ -spaces and orientations.** Fix a countably infinite-dimensional real vector space  $U$  and consider

$$\mathbb{L} = \mathcal{L}(1) = \mathcal{L}(U, U),$$

the space of linear isometries from  $U$  to itself. The notation  $\mathcal{L}(1)$  is due to the fact that  $\mathcal{L}(1)$  is the 1-ary part of the famous linear isometries operad [23, §1] with arity  $n$  term

$$(4.3) \quad \mathcal{L}(n) = \mathcal{L}(U^n, U).$$

See [23] or [1] for details. Note that  $\mathbb{L}$  is a monoid with respect to composition.

**Definition 4.2.** The *category of  $\mathbb{L}$ -spaces*,  $\mathrm{Top}[\mathbb{L}]$ , is the category of spaces with a left action of the monoid  $\mathbb{L}$ .

Using the 2-ary part of the linear isometries operad, one can manufacture a product on  $\mathrm{Top}[\mathbb{L}]$ : For objects  $X, Y$  of  $\mathrm{Top}[\mathbb{L}]$  their product  $X \times_{\mathbb{L}} Y$  is the coequalizer

$$\mathcal{L}(2) \times (\mathcal{L}(1) \times \mathcal{L}(1)) \times X \times Y \rightrightarrows \mathcal{L}(2) \times X \times Y \longrightarrow X \times_{\mathbb{L}} Y.$$

Here, one map uses the  $\mathcal{L}(1)$ -action on the spaces  $X$  and  $Y$  and the other map uses the operad product  $\mathcal{L}(2) \times \mathcal{L}(1) \times \mathcal{L}(1) \rightarrow \mathcal{L}(2)$ .

As  $\mathcal{L}(2) = \mathcal{L}(U^2, U)$  has a left  $\mathcal{L}(1)$ -action,  $X \times_{\mathbb{L}} Y$  is an  $\mathcal{L}(1)$ -space. The product is associative and has a symmetry, but it is only weakly unital. See [22, §4] for a careful discussion.

By [22, Proposition 4.7] there is an isomorphism of categories between commutative monoids with respect to  $\times_{\mathbb{L}}$  and  $E_{\infty}$ -spaces whose  $E_{\infty}$ -structure is parametrized by the linear isometries operad.

For strict unitality, one restricts to the full subcategory  $\mathcal{M}_*$  of objects of  $\text{Top}[\mathbb{L}]$  for which the unit map is a homeomorphism. Such objects are called *\*-modules*. The commutative monoids in  $\mathcal{M}_*$  again model  $E_{\infty}$ -spaces [22, Proposition 4.11].

For an associative ring spectrum  $R$ , there is a strictly associative model in  $\mathcal{M}_*$  of the space of units  $GL_1(R)$  and the functor  $GL_1$  is right adjoint to the inclusion of grouplike objects. One can form a bar construction,  $B_{\times_{\mathbb{L}}}$ , of a cofibrant replacement of  $GL_1(R)$ ,  $GL_1(R)^c$ , with respect to the monoidal product  $\times_{\mathbb{L}}$ , where  $B_{\times_{\mathbb{L}}}(GL_1(R)^c)$  is the geometric realization of the simplicial  $\mathcal{M}_*$  object

$$[n] \mapsto * \times_{\mathbb{L}} \underbrace{GL_1^c(R) \times_{\mathbb{L}} \dots \times_{\mathbb{L}} GL_1^c(R)}_n \times_{\mathbb{L}} *.$$

Similarly,  $E_{\times_{\mathbb{L}}}(GL_1(R)^c)$  is constructed out of the simplicial object

$$[n] \mapsto * \times_{\mathbb{L}} \underbrace{GL_1^c(R) \times_{\mathbb{L}} \dots \times_{\mathbb{L}} GL_1^c(R)}_{n+1}.$$

Adapted to the situation there are suspension spectrum and underlying infinite loop space functors [65, Lemma 7.5]

$$(4.4) \quad \mathcal{M}_* \begin{array}{c} \xleftarrow{(\Sigma_{\mathbb{L}}^{\infty})_+} \\ \xrightarrow{\Omega_S^{\infty}} \end{array} \mathcal{M}_S$$

that are a Quillen adjoint pair of functors. Here, the suspension functor is strong symmetric monoidal and the underlying loop space functor is lax symmetric monoidal.

The spectrum version of the assembly map from (4.2) is

$$(\Sigma_{\mathbb{L}}^{\infty})_+(GL_1^c(R)) \rightarrow (\Sigma_{\mathbb{L}}^{\infty})_+(GL_1(R)) \rightarrow R$$

where the first map comes from the cofibrant replacement of the units and the second one is the counit of an adjunction [1, (3.1)].

**Definition 4.3.** [1, Definition 3.12] The *Thom spectrum* for a map  $f: X \rightarrow B_{\times_{\mathbb{L}}}(GL_1^c(R))$  in  $\mathcal{M}_*$  is the  $R$ -module spectrum (in the world of [34])

$$(4.5) \quad M(f) = (\Sigma_{\mathbb{L}}^{\infty})_+ P^c \wedge_{(\Sigma_{\mathbb{L}}^{\infty})_+ GL_1^c(R)} R.$$

Here,  $P^c$  is a cofibrant replacement as a right  $GL_1^c(R)$ -module of the pullback

$$\begin{array}{ccc} P & \longrightarrow & E_{\times_{\mathbb{L}}}(GL_1^c(R)) \\ \downarrow & & \downarrow \\ X & \longrightarrow & B_{\times_{\mathbb{L}}}(GL_1^c(R)) \end{array}$$

**Remark 4.4.** Because of the cofibrancy of  $P^c$ , the smash product in (4.5) is actually a derived smash product. See [1, §3] for the necessary background on the model structures involved.

In the commutative case, [1, §4, §5] is set in the classical setting of  $E_{\infty}$  ring spectra and  $E_{\infty}$ -spaces as in [81]. If  $R$  is a commutative ring spectrum or an  $E_{\infty}$  ring spectrum then the naive space of units,  $GL_1(R)$ , is a group-like  $E_{\infty}$ -space and hence is an infinite loop space that has an associated connective spectrum,  $gl_1(R)$ , with  $\Omega^{\infty} gl_1(R) = GL_1(R)$ .

The crucial ingredient in this case is the pair of functors  $(\Sigma_+^{\infty} \Omega^{\infty}, gl_1)$  that is an adjunction between the homotopy category of connective spectra and the homotopy category of  $E_{\infty}$ -ring spectra in the sense of Lewis-May-Steinberger.

In particular, one gets a version of the assembly map from (4.2)

$$\Sigma_+^{\infty} \Omega^{\infty}(gl_1(R)) \rightarrow R$$

for every  $E_\infty$ -ring spectrum. By [34] one can replace  $E_\infty$ -ring spectra with commutative  $S$ -algebras, *i.e.*, with commutative ring spectra. This simplifies the discussion of pushouts and allows us to replace  $\Sigma_+^\infty \Omega^\infty$  by  $(\Sigma_{\mathbb{L}}^\infty)_+ \Omega_S^\infty$  from (4.4) to get

$$(\Sigma_{\mathbb{L}}^\infty)_+ \Omega_S^\infty (gl_1(R)) \rightarrow R$$

Note, that a map of infinite loop spaces  $f: B \rightarrow BGL_1(R)$  encodes the same data as a map of spectra  $f: b \rightarrow bgl_1(R)$  where the lower case letters denote the associated connective spectra. As before we consider the pullback  $p$

$$\begin{array}{ccc} p & \longrightarrow & egl_1(R) \\ \downarrow & & \downarrow \\ b & \xrightarrow{f} & bgl_1(R) \end{array}$$

and form the corresponding derived smash product:

**Definition 4.5.** Let  $f: b \rightarrow bgl_1(R)$  be a map of connective spectra. Then the *Thom spectrum associated to  $f$* ,  $M(f)$ , is the homotopy pushout in the category of commutative  $S$ -algebras

$$M(f) = (R \wedge (\Sigma_{\mathbb{L}}^\infty)_+ \Omega_S^\infty p) \wedge_{R \wedge (\Sigma_{\mathbb{L}}^\infty)_+ \Omega_S^\infty gl_1(R)}^L R$$

As the (homotopy) pushout is the (derived) smash product, this resembles the construction from (4.5)

In the commutative ring spectrum setting the question about orientations is the following problem: Assume that there is a map of commutative ring spectra  $\alpha: R \rightarrow A$ , then  $A$  is a commutative  $R$ -algebra spectrum. For a map of spectra  $f: b \rightarrow bgl_1(R)$  as above we can ask whether there is a morphism of commutative  $R$ -algebra spectra from  $M(f)$  to  $A$ . As  $M(f)$  is defined as a (homotopy) pushout, we see that we get a condition that says that we get maps from the ingredients of the derived smash product. As we start with a map  $\alpha$  from  $R$  to  $A$ , we get an induced map

$$gl_1(\alpha): gl_1(R) \rightarrow gl_1(A).$$

So what is missing is a map

$$(\Sigma_{\mathbb{L}}^\infty)_+ \Omega_S^\infty p \rightarrow A$$

that is compatible with the map  $(\Sigma_{\mathbb{L}}^\infty)_+ \Omega_S^\infty gl_1(R) \rightarrow A$ . With the help of the adjunction this means that we need a map

$$p \rightarrow gl_1(A)$$

such that precomposing it with the map  $gl_1(R) \rightarrow p$  gives  $gl_1(\alpha)$ . This argument can be turned into a proof for the following result:

**Theorem 4.6.** [1, Theorem 4.6] *The derived mapping space of commutative  $R$ -algebras from  $M(f)$  to  $A$ ,  $Map_{\mathcal{C}_R}(M(f), A)$ , is weakly equivalent to the fiber in the map between derived mapping spaces*

$$Map_{\mathcal{M}_S}(p, gl_1(A)) \rightarrow Map_{\mathcal{M}_S}(gl_1(R), gl_1(A))$$

at the basepoint  $gl_1(\alpha)$  of  $Map_{\mathcal{M}_S}(gl_1(R), gl_1(A))$ .

An important example is the question of the string orientation of the spectrum of topological modular forms,  $\mathrm{tmf}$ . For background on  $\mathrm{tmf}$  and its variants see [120]. In particular, [120, Chapter 10] contains André Henriques' notes of Mike Hopkins' lecture on the string orientation. Let  $BO\langle 8 \rangle$  be the 7-connected cover of  $BO$  and let  $bo\langle 8 \rangle$  be the associated spectrum with the canonical map  $f: bo\langle 8 \rangle \rightarrow bgl_1(S)$ . So we are in the situation where  $R = S$  and we take  $A = \mathrm{tmf}$ . Ando, Hopkins and Rezk [3] establish the existence of an  $E_\infty$ -map

$$M\mathrm{String} = MO\langle 8 \rangle \rightarrow \mathrm{tmf}$$

by showing a fiber property as above.

An approach to orientations of the form  $MU \rightarrow E$  is described in [49]: You start with an  $E_\infty$  ring spectrum  $E$  and an ordinary complex orientation of  $E$  [97, §6.1] and want to know whether

you can refine this to an  $E_\infty$ -map  $MU \rightarrow E$ . Hopkins and Lawson establish a filtration of  $MU$  by  $E_\infty$ -Thom spectra

$$S \rightarrow MX_1 \rightarrow MX_2 \rightarrow \dots \rightarrow MU$$

and for a given  $E_\infty$ -map  $MX_n \rightarrow E$  they identify the space of extensions to an  $E_\infty$ -map  $MX_{n+1} \rightarrow E$  [49, Theorem 1].

**Remark 4.7.** In [1] the authors present a different approach to Thom spectra and questions about orientations that uses  $\infty$ -categorical techniques. In certain cases it is unrealistic to hope for  $E_\infty$ -maps out of Thom spectra, for instance if one doesn't know that the target spectrum carries an  $E_\infty$  structure. The space of  $E_n$ -maps out of Thom spectra is described in [29, Theorem 4.2], [7, Corollary 3.18].

**4.2. Thom spectra via  $I$ -spaces.** Let  $I$  be the skeleton of the category of finite sets and injective maps. As objects we choose the sets  $\mathbf{n} = \{1, \dots, n\}$  for  $n \geq 0$  with the convention that  $\mathbf{0}$  denotes the empty set. A morphism  $f \in I(\mathbf{n}, \mathbf{m})$  is an injective function from  $\mathbf{n}$  to  $\mathbf{m}$ . Hence  $\mathbf{0}$  is an initial object of  $I$  and the permutation group  $\Sigma_n$  is the group of automorphisms of  $\mathbf{n}$  in  $I$ . The category  $I$  is symmetric monoidal with respect to the disjoint union:  $\mathbf{n} \sqcup \mathbf{m} = \mathbf{n} + \mathbf{m}$  with unit  $\mathbf{0}$  and non-trivial symmetry  $\mathbf{n} + \mathbf{m} \rightarrow \mathbf{m} + \mathbf{n}$  given by the shuffle permutation that moves the first  $n$  elements to the positions  $m + 1, \dots, m + n$ .

The functor category of  $I$ -spaces,  $\mathbf{Top}^I$ , *i.e.*, the category of functors  $X: I \rightarrow \mathbf{Top}$  together with natural transformations as morphisms, inherits a symmetric monoidal structure from  $I$  and  $\mathbf{Top}$  via the Day convolution product. Explicitly, one gets:

**Definition 4.8.** Let  $X, Y$  be two  $I$ -spaces. Their product  $X \boxtimes Y$  is the  $I$ -space given by

$$(X \boxtimes Y)(\mathbf{n}) = \operatorname{colim}_{\mathbf{p} \sqcup \mathbf{q} \rightarrow \mathbf{n}} X(\mathbf{p}) \times Y(\mathbf{q}).$$

The unit  $1_I$  is the discrete  $I$ -space  $\mathbf{n} \mapsto 1_I(\mathbf{0}, \mathbf{n})$ .

As  $\mathbf{0}$  is initial, the unit  $1_I$  is the terminal object in  $\mathbf{Top}^I$ . Commutative monoids in  $\mathbf{Top}^I$  are called *commutative  $I$ -space monoids* in [20] and their category is denoted by  $C(\mathbf{Top}^I)$ . A general fact about Day convolution products is that commutative monoids correspond to lax symmetric monoidal functors.

For an  $I$ -space  $X$  let's denote by  $X_{hI}$  the Bousfield-Kan homotopy colimit of  $X$ .

**Definition 4.9.** [20, Definition 2.2] A map of  $I$ -spaces  $f: X \rightarrow Y$  is an  *$I$ -equivalence*, if the induced map on homotopy colimits  $f_{hI}: X_{hI} \rightarrow Y_{hI}$  is a weak homotopy equivalence in  $\mathbf{Top}$ .

With the corresponding  $I$ -model structure the category of  $I$ -spaces is actually Quillen equivalent to the category of spaces [107, Theorem 3.3], but there is a *positive flat model structure* on  $I$ -spaces (see [20, §2]) that lifts to a right-induced model structure on  $C(\mathbf{Top}^I)$  that makes it Quillen equivalent to  $E_\infty$ -spaces.

Let  $\mathbf{Sp}^\Sigma$  denote the category of symmetric spectra. There is a canonical Quillen adjoint functor pair

$$(4.6) \quad \mathbf{Top}^I \begin{array}{c} \xleftarrow{\mathbb{S}^I} \\ \xrightarrow{\Omega^I} \end{array} \mathbf{Sp}^\Sigma$$

modelling the suspension spectrum functor and the underlying infinite loop space functor with

$$\mathbb{S}^I X(n) = \mathbb{S}^n \wedge X(\mathbf{n}), \quad \Omega^I(E)(\mathbf{n}) = \Omega^n E_n$$

Where  $\mathbb{S}^n$  is the  $n$ -fold smash product of the 1-sphere with  $\Sigma_n$ -action given by permutation of the smash factors.

Stable equivalences in symmetric spectra do in general not agree with stable homotopy equivalences, but there is a notion of *semistable* symmetric spectra, that has the feature that a map  $f: E \rightarrow F$  between two semistable symmetric spectra is a stable equivalence if and only if it is a stable homotopy equivalence. See [54, §5.6] for details and other characterizations.

**Definition 4.10.** For a commutative semistable symmetric ring spectrum  $R$  the commutative  $I$ -space monoid of units,  $GL_1^I(R)$ , has as  $GL_1^I(R)(\mathbf{n})$  those components of the commutative  $I$ -space monoid  $\Omega^I(R)(\mathbf{n}) = \Omega^n R_n$  that represent units in  $\pi_0(R)$ .

The adjunction from (4.6) gives a version of the assembly map from (4.2) as

$$\mathbb{S}^I(GL_1^I(R)) \rightarrow \mathbb{S}^I\Omega^I(R) \rightarrow R.$$

For technical reasons one has to work with a cofibrant replacement of  $GL_1^I(R)$ ,  $G \rightarrow GL_1^I(R)$  in the positive flat model structure on  $C(\mathbf{Top}^I)$ . The construction of a Thom spectrum associated to a map  $f: X \rightarrow BG$  is now similar to the approach in [1]: One defines  $BG$  and  $EG$  via two sided-bar constructions and takes a suitable pushout:

**Definition 4.11.** [20, Definitions 2.10, 2.12, 3.6]

- Let  $BG = B_{\boxtimes}(1_I, G, 1_I)$  and let  $EG$  be defined via a functorial factorization

$$B_{\boxtimes}(1_I, G, G) \xrightarrow{\sim} EG \twoheadrightarrow BG.$$

- For any  $I$ -space  $X$  over  $BG$  define  $U(X)$  as the  $I$ -space with  $G$ -action given by the pullback

$$\begin{array}{ccc} U(X) & \xrightarrow{\quad} & X \\ \downarrow & & \downarrow \\ EG & \longrightarrow & BG. \end{array}$$

Here,  $X$  and  $BG$  are considered as  $I$ -spaces with trivial  $G$ -action.

- Let  $R$  be a semistable commutative symmetric ring spectrum that is  $S$ -cofibrant.

The *Thom spectrum associated with a map of  $I$ -spaces*  $f: X \rightarrow BG$  is

$$(4.7) \quad M^I(f) = B_{\boxtimes}(\mathbb{S}^I(UX), \mathbb{S}^I G, R).$$

You should think of this two-sided bar construction as

$$\mathbb{S}^I(UX) \boxtimes_{\mathbb{S}^I G}^L R$$

and then you have to admit that this looks very similar to (4.5). This Thom spectrum functor is homotopically meaningful (see [20, Proposition 3.8]). Concerning multiplicative structures one gets the following result.

**Proposition 4.12.** [20, Proposition 3.10, Corollary 3.11] *The functor  $M^I(-)$  is lax symmetric monoidal and if  $\mathcal{D}$  is an operad in spaces, then it sends  $\mathcal{D}$ -algebras in  $\mathbf{Top}^I$  over  $BG$  to  $\mathcal{D}$ -algebras in  $R$ -modules in symmetric spectra over  $M^I GL_1(R) = B_{\boxtimes}(\mathbb{S}^I(EG), \mathbb{S}^I(G), R)$ .*

If you don't like diagram categories for some reason, then there is also an  *$I$ -spacification functor* [20, §4.1] that transforms a map of topological spaces

$$(4.8) \quad f: X \rightarrow BG_{hI}$$

to a map of  $I$ -spaces over  $BG$ , so you can associate a Thom spectrum to such a map as well. By abuse of notation, we will still denote this Thom spectrum by  $M^I(f)$ . This construction respects actions of operads augmented over the Barratt-Eccles operad and hence it also provides an  $E_\infty$  Thom spectrum functor.

An important question is: Given a ring spectrum  $A$ , can it be realized as a Thom spectrum with respect to a loop map, *i.e.*, in the setting of [20] is  $A$  equivalent to  $M^I(f)$  with  $f$  a loop map to  $BG_{hI}$ ? A striking result is that one can identify certain quotients as such Thom spectra!

**Theorem 4.13.** [20, Theorem 5.6] *Let  $R$  be a commutative ring spectrum whose homotopy groups are concentrated in even degrees and let  $u_i \in \pi_{2i}(R)$  be arbitrary elements with  $1 \leq i \leq n-1$ . Then the iterative cofiber of the multiplication maps by the  $u_i$ 's,  $R/(u_1, \dots, u_{n-1})$  can be realized as the Thom spectrum of a loop map from  $SU(n)$  to  $BG_{hI}$ . In particular,  $R/(u_1, \dots, u_{n-1})$  can be realized as an associative ring spectrum.*

An example of such a quotient is  $R = ku \rightarrow ku/u = H\mathbb{Z}$ . Note that there is no assumption on the regularity of the elements  $u_i$  in the above statement. For periodic ring spectra the assumptions on the degree of the elements can be relaxed and the two-periodic version of Morava K-theory can be constructed as a Thom spectrum relative to  $R = E_n$ , the  $n$ -th Morava  $E$ -theory or Lubin-Tate spectrum [20, Corollary 5.7]. A related but different construction of quotients of Lubin-Tate spectra modelling versions of Morava K-theory is carried out in [50, §3].

**4.3. Graded units.** There is one problem with the constructions of spaces and spectra of units as above. As they are constructed from the underlying infinite loop space of a spectrum and just take into account the units in  $\pi_0$ , they ignore graded units coming from periodicity elements in the homotopy groups of a spectrum. So for instance, the Bott class  $u \in \pi_2(KU)$  is not represented in  $GL_1(KU)$  or  $GL_1^I(KU)$ .

There is a construction of *graded units*. We'll sketch the construction and mention two of its applications: graded Thom spectra and logarithmic ring spectra.

**Definition 4.14.** [107, Definition 4.2] The category  $J$  has as objects pairs of objects of  $I$ . A morphism in  $J((\mathbf{n}_1, \mathbf{n}_2), (\mathbf{m}_1, \mathbf{m}_2))$  is a triple  $(\alpha, \beta, \sigma)$  where  $\alpha \in I(\mathbf{n}_1, \mathbf{m}_1)$ ,  $\beta \in I(\mathbf{n}_2, \mathbf{m}_2)$  and  $\sigma$  is a bijection

$$\sigma: \mathbf{m}_1 \setminus \alpha(\mathbf{n}_1) \rightarrow \mathbf{m}_2 \setminus \beta(\mathbf{n}_2).$$

For another morphism  $(\gamma, \delta, \xi) \in J((\mathbf{m}_1, \mathbf{m}_2), (\mathbf{l}_1, \mathbf{l}_2))$  the composition is the morphism  $(\gamma \circ \alpha, \delta \circ \beta, \tau(\xi, \sigma))$  where  $\tau(\xi, \sigma)$  is the permutation

$$\tau(\xi, \sigma)(s) = \begin{cases} \xi(s), & \text{if } s \in \mathbf{l}_1 \setminus \gamma(\mathbf{m}_1), \\ \delta(\sigma(t)), & \text{if } s = \gamma(t) \in \gamma(\mathbf{m}_1 \setminus \alpha(\mathbf{n}_1)). \end{cases}$$

Note that  $\mathbf{l}_1 \setminus \gamma(\alpha(\mathbf{n}_1))$  is the disjoint union of  $\mathbf{l}_1 \setminus \gamma(\mathbf{m}_1)$  and  $\gamma(\mathbf{m}_1 \setminus \alpha(\mathbf{n}_1))$ .

With these definitions  $J$  is actually a category and it inherits a symmetric monoidal structure from  $I$  via componentwise disjoint union [107, Proposition 4.3]. In particular, the category of  $J$ -spaces,  $\mathbf{Top}^J$ , is symmetric monoidal with the Day convolution product. Note, however, that the unit for the monoidal structure  $\boxtimes_J$  is  $J((\mathbf{0}, \mathbf{0}), (-, -))$  and this is not a constant functor but  $J((\mathbf{0}, \mathbf{0}), (\mathbf{n}, \mathbf{n}))$  can be identified with the symmetric group  $\Sigma_n!$

**Proposition 4.15.** [107, 4.4, 4.5] *For every  $J$ -space  $X$  the homotopy colimit,  $X_{hJ}$ , is a space over  $QS^0$ .*

*Proof.* It is not hard to see that  $J$  is isomorphic to Quillen's category  $\Sigma^{-1}\Sigma$  [107, Proposition 4.4] and its classifying space is  $QS^0$  by the Barratt-Priddy-Quillen result. Therefore  $BJ$  is  $QS^0$ . Every  $J$ -space has a map to the terminal  $J$ -space that is the constant  $J$ -diagram on a point and this induces a map

$$X_{hJ} \rightarrow *_hJ = BJ \simeq QS^0.$$

□

For any  $I$ -space  $X$  we also get that  $X_{hI}$  is a space over  $BI$ , but as  $I$  has an initial object this just gives a map to  $BI \simeq *$ , the terminal object.

Let  $C(\mathbf{Top}^J)$  denote the category of commutative  $J$ -space monoids, *i.e.*, commutative monoids in  $\mathbf{Top}^J$ . The following result is crucial:

**Theorem 4.16.** [107, Theorem 4.11] *There is a model structure on  $C(\mathbf{Top}^J)$  such that there is a Quillen equivalence between  $C(\mathbf{Top}^J)$  and the category of  $E_\infty$ -spaces over  $BJ$ .*

Here, the  $E_\infty$ -structure is parametrized by the Barratt-Eccles operad.

For a (commutative)  $J$ -space monoid, one can associate units:

**Definition 4.17.** [107, §4] Let  $A$  be a  $J$ -space monoid. Then let  $A^\times$  be the  $J$ -space monoid with  $A^\times(\mathbf{n}_1, \mathbf{n}_2)$  being the union of those components of  $A(\mathbf{n}_1, \mathbf{n}_2)$  that represent units in  $\pi_0(A_{hJ})$ .

So now one has to construct a functor from spectra to  $J$ -spaces that sees all the homotopy groups, not just the ones in non-negative degrees:

**Definition 4.18.** [107, (4.5)]

- Let  $\Omega^J$  be the functor from symmetric spectra to  $J$ -spaces that takes a symmetric spectrum  $E$  and sends it to the  $J$ -space with

$$\Omega^J(\mathbf{n}_1, \mathbf{n}_2) = \Omega^{n_2} E_{n_1}.$$

- If  $R$  is a symmetric ring spectrum, then its  $J$ -space of units is

$$GL_1^J(R) = (\Omega^J(R))^\times.$$

Sagave and Schlichtkrull show that this is homotopically meaningful and that for a commutative symmetric ring spectrum  $R$ , the units  $GL_1^J(R)$  are actually in  $C(\text{Top}^J)$  [107, §4]. Most importantly, the inclusion  $GL_1^J(R) \hookrightarrow \Omega^J(R)$  realizes the inclusion of graded units  $\pi_*(R)^\times$  into  $\pi_*(R)$  for positively fibrant  $R$ .

Hence, for instance  $GL_1^J(KU)$  (and any other model of the 'usual' units) only detects the units  $\pm 1$  in  $\pi_0(KU)$  whereas  $GL_1^J(KU)$  also detects the Bott class.

**Remark 4.19.**

- John Rognes developed the concept of logarithmic ring spectra and in [106] and [105] this concept is fully explored with the help of graded units. The idea is that you want a spectrum that sits between a commutative ring spectrum like  $ku$  and its localization  $KU$ , so you remember the Bott class as the extra datum of a logarithmic structure. This concept has its origin in algebraic geometry and is useful in stable homotopy theory for instance for obtaining localization sequences in topological Hochschild homology [105].
- In [108] Sagave and Schlichtkrull use graded units adapted to the setting of orthogonal spectra,  $GL_1^W$ , to construct *graded Thom spectra* associated to virtual vector bundles, *i.e.*, associated to a map  $f: X \rightarrow \mathbb{Z} \times BO$  in such a way that uses the  $E_\infty$ -structure on  $\mathbb{Z} \times BO$ . They use this for orientation theory and relate  $GL_1^W$ -orientations to logarithmic structures. They provide an  $E_\infty$ -Thom isomorphism that allows to compute the homology of spectra appearing in connection with logarithmic ring spectra [108, §§ 7,8].

## 5. CONSTRUCTING COMMUTATIVE RING SPECTRA FROM BIPERMUTATIVE CATEGORIES

In section 4 we saw that Thom spectra give rise to commutative ring spectra. Algebraic K-theory is another machine that takes a commutative ring (spectrum)  $R$  and produces a commutative ring spectrum  $K(R)$ . In this section we focus on a classical construction that takes a small bipermutative category  $\mathcal{C}$  and turns it into a commutative ring spectrum. This construction goes back to Segal [114]; its multiplicative properties were investigated by May [81, 82, 83, 84], Shimada-Shimakawa [115], Woolfson [124] and Elmendorf-Mandell [35].

We sketch a simplified version of the construction, present some important examples and refer to [35] for a discussion of the multiplicative properties.

**Definition 5.1.** A *permutative category*  $(\mathcal{C}, \oplus, 0, \tau)$  is a category  $\mathcal{C}$  together with an object  $0$  of  $\mathcal{C}$ , a functor  $\oplus: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  and a natural isomorphism  $\tau_{C_1, C_2}: C_1 \oplus C_2 \rightarrow C_2 \oplus C_1$  for all objects  $C_1, C_2$  of  $\mathcal{C}$  such that

- $\oplus$  is strictly associative, *i.e.*, for all objects  $C_1, C_2, C_3$  of  $\mathcal{C}$

$$C_1 \oplus (C_2 \oplus C_3) = (C_1 \oplus C_2) \oplus C_3.$$

- $0$  is a strict unit, *i.e.*, for all objects  $C$  of  $\mathcal{C}$ :  $C \oplus 0 = C = 0 \oplus C$ .
- $\tau^2$  is the identity, *i.e.*, for all objects  $C_1, C_2$  of  $\mathcal{C}$  the composite

$$C_1 \oplus C_2 \xrightarrow{\tau_{C_1, C_2}} C_2 \oplus C_1 \xrightarrow{\tau_{C_2, C_1}} C_1 \oplus C_2$$

is the identity on  $C_1 \oplus C_2$ .



- The diagrams

$$\begin{array}{ccc}
C \oplus 0 & \xrightarrow{\tau_{C,0}} & 0 \oplus C \\
\parallel & & \parallel \\
& C & 
\end{array}
\quad
\begin{array}{ccc}
C_1 \oplus C_2 \oplus C_3 & \xrightarrow{\tau_{C_1 \oplus C_2, C_3}} & C_3 \oplus C_1 \oplus C_2 \\
\searrow \text{id}_{C_1} \oplus \tau_{C_2, C_3} & & \nearrow \tau_{C_1, C_3} \oplus \text{id}_{C_2} \\
& C_1 \oplus C_3 \oplus C_2 & 
\end{array}$$

commute for all objects  $C, C_1, C_2, C_3$  of  $\mathcal{C}$ .

We work with small permutative categories, *i.e.*, we require that the objects of  $\mathcal{C}$  for a set (and not a proper class). We recall Segal's construction from [114, §2]:

**Definition 5.2.** Let  $\mathcal{C}$  be a small permutative category and let  $X$  be a finite set with basepoint  $+$   $\in X$ . Let  $\mathcal{C}(X)$  be the category whose objects are families  $(C_S, \varrho_{S,T})$  where

- $S \subset X, + \notin S$ ,
- $S$  and  $T$  are pairs of such subsets that are disjoint,
- the  $C_S$  are objects of  $\mathcal{C}$  and  $\varrho_{S,T}$  is an isomorphism in  $\mathcal{C}$

$$\varrho_{S,T}: C_S \oplus C_T \rightarrow C_{S \cup T}.$$

- For  $S = \emptyset$ :  $C_\emptyset = 0$  and  $\varrho_{\emptyset, T} = \text{id}_{C_T}$  for all  $T$  and
- for pairwise disjoint  $S, T, U$  that don't contain  $+$  the following diagrams commute:

$$\begin{array}{ccc}
C_S \oplus C_T & \xrightarrow{\varrho_{S,T}} & C_{S \cup T} \\
\tau \downarrow & & \parallel \\
C_T \oplus C_S & \xrightarrow{\varrho_{T,S}} & C_{T \cup S},
\end{array}
\quad
\begin{array}{ccc}
C_S \oplus C_T \oplus C_U & \xrightarrow{\varrho_{S,T} \oplus \text{id}_{C_U}} & C_{S \cup T} \oplus C_U \\
\text{id}_{C_S} \oplus \varrho_{T,U} \downarrow & & \downarrow \varrho_{S \cup T, U} \\
C_S \oplus C_{T \cup U} & \xrightarrow{\varrho_{S, T \cup U}} & C_{S \cup T \cup U}.
\end{array}$$

Morphisms  $\alpha: (C_S, \varrho_{S,T}) \rightarrow (C'_S, \varrho'_{S,T})$  consist of a family of morphisms  $\alpha_S \in \mathcal{C}(C_S, C'_S)$  for all  $S \subset X$  with  $+$   $\notin S$  such that  $\alpha_\emptyset = \text{id}_0$  and for all  $S, T \in X$  with  $+$   $\notin S, T$  and  $S \cap T = \emptyset$  the diagram

$$\begin{array}{ccc}
C_S \oplus C_T & \xrightarrow{\varrho_{S,T}} & C_{S \cup T} \\
\alpha_S \oplus \alpha_T \downarrow & & \downarrow \alpha_{S \cup T} \\
C'_S \oplus C'_T & \xrightarrow{\varrho'_{S,T}} & C'_{S \cup T}
\end{array}$$

commutes.

Thus up to isomorphism, every object  $C_S$  for  $S = \{x_1, \dots, x_n\}$  can be decomposed as

$$C_S \cong C_{\{x_1\}} \oplus \dots \oplus C_{\{x_n\}}$$

by an iterated application of the isomorphisms  $\varrho$ , but these isomorphisms are part of the data. Segal shows [114, Corollary 2.2] that this construction gives rise to a so-called  $\Gamma$ -space (see [114, Definition 1.2] for a definition) that sends a finite pointed set  $X$  to the classifying space of  $\mathcal{C}(X)$ . Every  $\Gamma$ -space gives rise to a spectrum, and we denote the spectrum associated to  $\mathcal{C}$  by  $HC$ .

**Remark 5.3.** Segal's construction actually works for symmetric monoidal categories and it produces a spectrum whose associated infinite loop space is the group completion of the classifying space of the category  $\mathcal{C}$ ,  $B\mathcal{C}$ , and the latter is the geometric realization of the nerve of  $\mathcal{C}$ .

**Definition 5.4.** A *bipermutative category*  $\mathcal{R}$  is a category with two permutative category structures:  $(\mathcal{R}, \oplus, 0_{\mathcal{R}}, \tau_{\oplus})$  and  $(\mathcal{R}, \otimes, 1_{\mathcal{R}}, \tau_{\otimes})$  that are compatible in the following sense:

- (a)

$$0_{\mathcal{R}} \otimes C = 0_{\mathcal{R}} = C \otimes 0_{\mathcal{R}}$$

for all objects  $C$  of  $\mathcal{R}$ .

- (b) For all objects  $A, B, C$  we have an equality between  $(A \oplus B) \otimes C$  and  $A \otimes C \oplus B \otimes C$  and the diagram

$$\begin{array}{ccc} (A \oplus B) \otimes C & \xlongequal{\quad} & A \otimes C \oplus B \otimes C \\ \tau_{\oplus} \otimes \text{id} \downarrow & & \downarrow \tau_{\oplus} \\ (B \oplus A) \otimes C & \xlongequal{\quad} & B \otimes C \oplus A \otimes C \end{array}$$

commutes.

- (c) We define the distributivity isomorphism  $d_{\ell}: A \otimes (B \oplus C) \rightarrow A \otimes B \oplus A \otimes C$  for all  $A, B, C$  in  $\mathcal{R}$  via the following diagram

$$\begin{array}{ccc} A \otimes (B \oplus C) & \xrightarrow{\tau_{\otimes}} & (B \oplus C) \otimes A \\ \downarrow d_{\ell} & & \parallel \\ A \otimes B \oplus A \otimes C & \xleftarrow{\tau_{\otimes} \oplus \tau_{\otimes}} & B \otimes A \oplus C \otimes A \end{array}$$

then the diagram

$$\begin{array}{ccc} (A \oplus B) \otimes (C \oplus D) & \xrightarrow{d_{\ell}} & (A \oplus B) \otimes C \oplus (A \oplus B) \otimes D \\ \parallel & & \parallel \\ A \otimes (C \oplus D) \oplus B \otimes (C \oplus D) & & A \otimes C \oplus B \otimes C \oplus A \otimes D \oplus B \otimes D \\ \downarrow d_{\ell} \oplus d_{\ell} & \nearrow \text{id} \oplus \tau_{\oplus} \oplus \text{id} & \\ A \otimes C \oplus A \otimes D \oplus B \otimes C \oplus B \otimes D & & \end{array}$$

commutes.

This definition is taken from [81, Definition VI.3.3, p. 154]. The definition in [35] is less strict, but bipermutative categories in the above sense are also bipermutative in the sense of [35, Definition 3.6]. We refer to Elmendorf and Mandell for a proof that for a bipermutative category  $\mathcal{R}$ , one actually obtains a commutative ring spectrum  $H\mathcal{R}$ :

**Theorem 5.5.** [35, Corollary 3.9] *If  $\mathcal{R}$  is a bipermutative category, then  $H\mathcal{R}$  is equivalent to a strictly commutative symmetric ring spectrum.*

Segal's construction enables us to find small and explicit models for certain connective commutative ring spectra. Famous examples of bipermutative categories and their associated commutative ring spectra are the following:

- (a) If  $R$  is a commutative discrete ring, then the category  $\mathcal{R}_R$  which has the elements of  $R$  as objects and only identity morphisms is a bipermutative category with the addition in the ring being  $\oplus$  and the multiplication being  $\otimes$ . The associated spectrum,  $H\mathcal{R}_R$  is the Eilenberg-Mac Lane spectrum of the ring  $R$ ,  $HR$ .
- (b) Let  $\mathcal{E}$  denote the bipermutative category of finite sets whose objects are the finite sets  $\mathbf{n} = \{1, \dots, n\}$  for  $n \in \mathbb{N}_0$ . By convention  $\mathbf{0}$  is the empty set. The morphisms in  $\mathcal{E}$  are

$$\mathcal{E}(\mathbf{n}, \mathbf{m}) = \begin{cases} \emptyset, & n \neq m, \\ \Sigma_n, & n = m. \end{cases}$$

For the full structure see [81, VI, Example 5.1]. Here,  $H\mathcal{E}$  is the sphere spectrum,  $S$ .

- (c) The bipermutative category of complex vector spaces,  $\mathcal{V}_{\mathbb{C}}$ , with objects the natural numbers with zero and morphisms

$$\mathcal{V}_{\mathbb{C}}(n, m) = \begin{cases} \emptyset, & n \neq m, \\ U(n), & n = m \end{cases}$$

is bipermutative. On objects we set  $n \oplus m = n + m$  and  $n \otimes m = nm$  and on morphism we use the block sum and the tensor product of matrices. The associated spectrum is  $H\mathcal{V}_{\mathbb{C}} = ku$ , the connective version of topological complex K-theory. Its real analog,  $\mathcal{V}_{\mathbb{R}}$ , gives a model for connective topological real K-theory,  $ko$ . You can also work with the general linear group instead of the unitary or orthogonal group.

- (d) If  $R$  is a discrete commutative ring, then we define the category  $F_R$  as the one with objects  $\mathbb{N}_0$  again. As morphisms we have

$$F_R(n, m) = \begin{cases} \emptyset, & n \neq m, \\ GL_n(R), & n = m. \end{cases}$$

This category is often called *the small category of free  $R$ -modules*. Its spectrum is the *free algebraic K-theory of  $R$* ,  $K^f(R)$ . Its homotopy groups agree with the algebraic K-groups of  $R$  from degree 1 on.

## 6. FROM TOPOLOGICAL HOCHSCHILD TO TOPOLOGICAL ANDRÉ-QUILLEN HOMOLOGY

For rings and algebras Hochschild homology contains a lot of information. For commutative rings and algebras André-Quillen homology is the adequate tool. There are spectrum level versions of these homology theories: topological Hochschild homology, THH, and topological André-Quillen homology, TAQ.

We can determine classes in the algebraic K-theory of a ring spectrum using the trace to topological Hochschild homology or to topological cyclic homology:

$$(6.1) \quad tr: K(R) \rightarrow THH(R).$$

For instance the trace from  $K(\mathbb{Z})$  to  $THH(\mathbb{Z})$  detects important classes. Bökstedt, Madsen and Rognes [26, 103] show for instance that

$$tr: K_{2p-1}(\mathbb{Z}) \rightarrow THH_{2p-1}(\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$$

is surjective for all primes  $p$ .

We give a construction of topological Hochschild homology and more generally, for commutative ring spectra  $R$  we define  $X \otimes R$  for  $X$  a finite pointed simplicial set. We give some examples of calculations of such  $X$ -homology groups of  $R$  and tell you about topological Hochschild cohomology as a derived center of an algebra spectrum. We define topological André-Quillen homology and we will see applications to Postnikov towers for commutative ring spectra later in 7.

**6.1. THH and friends.** Let  $X$  be a finite pointed simplicial set and let  $R$  be a cofibrant commutative ring spectrum.

**Definition 6.1.** We denote by  $X \otimes R$  the simplicial spectrum with

$$(X \otimes R)_n = \bigwedge_{x \in X_n} R.$$

By slight abuse of notation we will use the same symbol for the geometric realization of  $X \otimes R$ .

**Remark 6.2.**

- As the smash product is the coproduct in  $\mathcal{C}_S$ , the simplicial structure maps of  $X \otimes R$  are induced from the ones on  $X$ .
- As  $X$  is pointed,  $X \otimes R$  comes with maps

$$R \rightarrow X \otimes R \rightarrow R$$

whose composition is the identity.

- The commutative multiplication on  $R$  induces a commutative multiplication on  $X \otimes R$ , hence  $X \otimes R$  is an augmented commutative (simplicial)  $R$ -algebra spectrum.
- One could also work with the fact that the spectra of [34] are tensored over topological spaces or similarly, that symmetric spectra [54] in topological spaces are enriched over simplicial sets and over topological spaces. This gives an equivalent situation. It is for instance shown in [34, Corollary VII.3.4] that  $|X \otimes A| \simeq |X| \otimes A$  for simplicial spaces  $X$  and commutative  $R$ -algebra spectra  $A$ .
- The above definition can be extended to tensoring with an arbitrary pointed simplicial sets by expressing such a simplicial set as the colimit of its finite pointed simplicial subcomplexes.

There are many important special cases of this construction.

**Definition 6.3.**

- For the simplicial 1-sphere  $X = \mathbb{S}^1$  the commutative  $R$ -algebra spectrum  $\mathbb{S}^1 \otimes R$  is the *topological Hochschild homology of  $R$*  and is denoted by  $\mathrm{THH}(R)$ .
- More generally, for an  $n$ -sphere, we denote by  $\mathrm{THH}^{[n]}(R)$  the spectrum  $\mathbb{S}^n \otimes R$  and this is called *topological Hochschild homology of order  $n$* .
- If  $\mathbb{T}^n$  denotes the torus  $(\mathbb{S}^1)^n$ , then  $\mathbb{T}^n \otimes R$  is the  *$n$ -torus homology of  $R$* .

For the small model of the simplicial 1-sphere with just one non-degenerate 0 and 1 simplex we have  $(\mathbb{S}^1)_n = \{0, 1, \dots, n\}$  and the simplicial spectrum  $\mathbb{S}^1 \otimes R$  is precisely the cyclic bar construction on  $R$ :

$$R \begin{array}{c} \leftarrow \\ \rightleftarrows \\ \rightarrow \end{array} R \wedge R \begin{array}{c} \leftarrow \\ \rightleftarrows \\ \rightarrow \end{array} R \wedge R \wedge R \begin{array}{c} \leftarrow \\ \rightleftarrows \\ \rightarrow \end{array} \dots$$

where the degeneracy map  $s_i: R^{n+1} \rightarrow R^{n+2}$  inserts the unit map  $\eta: S \rightarrow R$  after the  $i$ th factor of  $R$  and the face maps  $d_i: R^{n+1} \rightarrow R^n$  for  $0 \leq i < n$  are given by the multiplication in  $R$  of the  $i$ th and  $(i + 1)$ st smash factor. The last face map  $d_n$  cyclically permutes the smash factors to bring the last one to the front and then it multiplies the former factors with number  $n$  and 0.

As for Hochschild homology you should think about this as a genuine cyclic object:

$$\begin{array}{c} R \wedge R \wedge \dots \\ \wedge \\ R \wedge \dots \end{array}$$

The original definition of  $\mathrm{THH}$  is due to Marcel Bökstedt [24]. McClure, Schwänzl and Vogt [86] show that for an  $E_\infty$ -ring spectrum  $R$ ,  $\mathrm{THH}(R)$ , is equivalent to tensoring  $R$  with the topological 1-sphere. Kuhn systematically studies constructions like  $X \otimes R$  in a reduced setting [60] for pointed spaces  $X$ . So the above definition is an unreduced variant of this that uses simplicial sets instead of topological spaces.

**Lemma 6.4.** *Let  $X$  and  $Y$  be finite simplicial pointed sets. Then*

$$(X \times Y) \otimes R \simeq X \otimes (Y \otimes R).$$

*Proof.* Observe that

$$((X \times Y) \otimes R)_n = \bigwedge_{(x,y) \in X_n \times Y_n} R \cong \bigwedge_{x \in X_n} \left( \bigwedge_{y \in Y_n} R \right)$$

and this is the diagonal of the bisimplicial spectrum

$$([m], [\ell]) \mapsto (X \otimes ((Y \otimes R)_\ell))_m$$

in degree  $n$ . □

One of the important features of  $\mathrm{THH}(R)$  is that it receives a trace map from algebraic K-theory (see (6.1)) that we can now write as

$$\mathrm{tr}: K(R) \rightarrow \mathbb{S}^1 \otimes R.$$

Taking higher dimensional tori gives targets for iterated trace maps. Algebraic K-theory of a commutative ring spectrum is again a commutative ring spectrum and the trace map is a map of commutative ring spectra, hence one can iterate the process of forming K-theory and traces. If we denote by  $K^n(R)$  the  $n$ -fold iteration, then – as we have the product formula from Lemma 6.4 – we get an iterated trace to  $\mathbb{T}^n \otimes R$ . Explicitly, for  $n = 2$  this is

$$K(K(R)) \rightarrow \mathbb{S}^1 \otimes (\mathbb{S}^1 \otimes R) \simeq (\mathbb{S}^1 \times \mathbb{S}^1) \otimes R = \mathbb{T}^2 \otimes R.$$

There are variants of Definition 6.1: As we work with pointed simplicial sets, we can glue an  $R$ -module to the base point and use the  $R$ -module structure for the face maps. A second variant is to work relative to some commutative ring spectrum  $R$ : In 6.1 the smash products were over the sphere spectrum, but if we work with a commutative  $R$ -algebra spectrum  $A$ , then we can take smash products over  $R$  instead of  $S$ . Recall that  $\wedge_R$  is the coproduct in the category of commutative  $R$ -algebra spectra,  $\mathcal{C}_R$ .

**Definition 6.5.** Let  $R$  be a cofibrant commutative ring spectrum,  $A$  a cofibrant commutative  $R$ -algebra spectrum,  $M$  an  $A$ -module spectrum over  $R$  and let  $X$  be a finite pointed simplicial set. We denote by  $\mathcal{L}_X^R(A; M)$  the simplicial spectrum with

$$\mathcal{L}_X^R(A; M)_n = M \wedge_R \bigwedge_{x \in X_n \setminus *} A.$$

We call  $\mathcal{L}_X^R(A; M)$  the *Loday construction of  $A$  over  $R$  with coefficients in  $M$* .

As  $M$  is just an  $A$ -module spectrum, the resulting simplicial spectrum and also its realization carries an  $A$ -module structure over  $R$ , but no multiplicative structure in general. However, if we place a commutative  $A$ -algebra  $C$  at the basepoint, then the resulting spectrum is an augmented commutative  $C$ -algebra spectrum.

We will see later in Section 8 that for instance

$$\mathrm{THH}^R(A) := \mathcal{L}_{\mathbb{S}^1}^R(A)$$

measures properties of  $A$  as a commutative  $R$ -algebra spectrum. The case of  $X = \mathbb{S}^0$  gives

$$\mathcal{L}_{\mathbb{S}^0}^R(A) = A \wedge_R A$$

so there is a Künneth spectral sequence [34, IV.4.1] for calculating its homotopy groups.

An important example of a Loday construction is Pirashvili's construction of *higher order Hochschild homology*. He works with discrete commutative  $k$ -algebras  $A$  and  $A$ -modules  $M$  and defines  $\mathrm{HH}_X^k(A; M)$  [92, §5.1]. For  $X = \mathbb{S}^n$  this is his notion of higher order Hochschild homology (in his notation  $H^{[n]}(A; M)$ ). In our setting this corresponds to  $\mathcal{L}_X^{\mathrm{HH}^k}(HA; HM)$  if  $A$  is flat over  $k$ .

## 6.2. Examples.

- (a) A classical example of a  $\mathrm{THH}$ -calculation is the one of  $H\mathbb{Z}$  and  $H\mathbb{F}_p$  by Marcel Bökstedt ([25], see [67, Chapter 13] and the references for published accounts of these results):

**Proposition 6.6.**

$$\mathrm{THH}_*(H\mathbb{F}_p) \cong \mathbb{F}_p[\mu], \quad |\mu| = 2.$$

$$\mathrm{THH}_i(H\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & i = 0, \\ \mathbb{Z}/j\mathbb{Z}, & i = 2j - 1, \\ 0, & \text{otherwise.} \end{cases}$$

A crucial ingredient for these and for many other calculations of THH is Bökstedt's spectral sequence: If  $R$  is a commutative ring spectrum and if  $E_*$  is a homotopy commutative ring spectrum such that  $E_*(R)$  is flat over  $E_*$  then there is a multiplicative spectral sequence

$$E_{p,q}^2 = \mathrm{HH}_{p,q}^{E_*}(E_*(R)) \Rightarrow E_{p+q}\mathrm{THH}(R).$$

Here  $\mathrm{HH}_{p,q}$  denotes Hochschild homology in homological degree  $p$  and internal degree  $q$  ([25], [34, Theorem IV.1.9]).

- (b) If we apply THH to Eilenberg-MacLane spectra of number rings, Lindenstrauss and Madsen show that THH detects arithmetic properties:

**Proposition 6.7.** [66, Theorem 1.1] *Let  $K$  be a number field and let  $\mathcal{O}_K$  be its ring of integers. Then*

$$\mathrm{THH}_n(H\mathcal{O}_K) = \begin{cases} \mathcal{O}_K, & n = 0, \\ \mathcal{D}_{\mathcal{O}_K}^{-1}/\ell\mathcal{O}_K, & n = 2\ell - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Here,  $\mathcal{D}_{\mathcal{O}_K}^{-1}$  is the inverse different. This is the set of all  $x \in K$  such that the trace  $\mathrm{tr}(xy)$  is an integer for all  $y \in \mathcal{O}_K$ . The inverse different detects ramified primes and their ramification type.

We calculate higher order THH of number rings with reduced coefficients in [32, Theorem 4.3].

- (c) For a suspension spectrum on a based (Moore) loop space,  $\Sigma_+^\infty \Omega_M X$ , the cyclic bar construction reduces to the suspension spectrum of the cyclic bar construction on  $\Omega_M X$  and Goodwillie [40, Proof of Theorem V.1.1] identifies the latter with the free loop space on  $X$ ,  $LX$ . Hence one obtains

$$\mathrm{THH}(\Sigma_+^\infty \Omega_M X) \simeq \Sigma_+^\infty LX.$$

- (d) Let  $R$  be a ring spectrum and let  $\Pi$  be a pointed monoid. Hesselholt and Madsen show that  $\mathrm{THH}(R[\Pi])$  splits as

$$\mathrm{THH}(R[\Pi]) \simeq \mathrm{THH}(R) \wedge |N^{cy}\Pi|$$

where  $|N^{cy}\Pi|$  denotes the cyclic nerve of  $\Pi$  [45, Theorem 7.1].

- (e) As a sample calculation for second order THH we get [33, Theorem 2.1]:

$$(6.2) \quad \mathrm{THH}_*^{[2]}(H\mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}[x_1, x_2, \dots]/(p^n x_n, x_n^p - px_{n+1}, n \geq 1)$$

with  $|x_1| = 2p$ .

- (f) At an odd prime  $KU_{(p)}$  splits as

$$KU_{(p)} \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} L.$$

Here,  $L$  is the Adams summand of  $KU_{(p)}$  with  $\pi_*(L) \cong \mathbb{Z}_{(p)}[v_1^{\pm 1}]$  and  $|v_1| = 2p - 2$ . For consistency we set  $L = KU_{(2)}$  at the prime 2. We denote by  $ku$ ,  $\ell$  and  $ko$  the connective covers of  $KU$ ,  $L$  and  $KO$ .

McClure and Staffeldt determine the mod  $p$ -homotopy of  $\mathrm{THH}(\ell)$  at odd primes [88] and they show that  $\mathrm{THH}(L)_p \simeq L_p \vee (\Sigma L_p)_{\mathbb{Q}}$  [88, Corollary 7.2, Theorem 8.1].

Ausoni [9] determines the mod  $p$  and mod  $v_1$  homotopy of  $\mathrm{THH}(ku)$  as an input for his work on  $K(ku)$ .

Angeltveit, Hill and Lawson show [5, Theorem 2.6] that for all primes, as an  $\ell_*$ -module

$$\mathrm{THH}_*(\ell) \cong \ell_* \oplus \Sigma^{2p-1} F \oplus T$$

where  $F$  is a torsionfree summand and  $T$  is an infinite direct sum of torsion modules concentrated in even degrees. They describe  $F$  explicitly using a rational calculation.

Determining the torsion is way more involved [5, Theorem 2.8]. The calculation of  $\mathrm{T HH}_*(\ell)$  uses the method of *duelling Bockstein spectral sequences* for the Bockstein spectral sequences associated to

$$\begin{array}{ccc} \ell & \longrightarrow & \ell/p \\ \downarrow & & \downarrow \\ \ell/v_1 = H\mathbb{Z}_{(p)} & \longrightarrow & H\mathbb{F}_p = \ell/(p, v_1). \end{array}$$

They describe the 2-local homotopy groups of  $\mathrm{T HH}(ko)$  [5, §7] by first determining  $\mathrm{T HH}_*(ko; ku)$  and then using the Bockstein spectral sequence associated to the cofiber sequence  $\Sigma ko \rightarrow ko \rightarrow ku$ .

Again, things are way easier for the periodic versions [9, Proposition 7.13], [5, Corollary 7.9]:

$$\mathrm{T HH}(KO) \simeq KO \vee \Sigma KO_{\mathbb{Q}}, \quad \mathrm{T HH}(KU) \simeq KU \vee \Sigma KU_{\mathbb{Q}}.$$

- (g) John Greenlees uses a generalization of the concept of Gorenstein maps of commutative rings to the spectral world in order to determine Gorenstein descent properties for cofiber sequences of connective commutative ring spectra [42, Theorem 7.4].

**6.3. Topological Hochschild homology of Thom spectra.** We start with a general statement about  $X \otimes M^I(f)$  if  $M^I(f)$  is a Thom spectrum associated to an  $E_{\infty}$ -map to  $BG_{hI}$  with  $BG_{hI}$  as in (4.8) with  $R = S$ , hence  $G$  is a cofibrant replacement of  $GL_1^I(S)$ .

**Theorem 6.8.** [109, Theorem 1.1] *For any pointed simplicial set  $X$  and any map of grouplike  $E_{\infty}$ -spaces  $f: A \rightarrow BG_{hI}$  there is an equivalence of  $E_{\infty}$ -ring spectra*

$$X \otimes M^I(f) \simeq M^I(f) \wedge \Omega^{\infty}(a \wedge |X|)_+$$

where  $a$  is the spectrum associated to  $A$  with  $\Omega^{\infty}a = A$ .

This result generalizes [22], where the case of  $X = \mathbb{S}^1$  is covered. In general, for  $X = \mathbb{S}^n$  the above result determines the higher order topological Hochschild homology of  $M^I(f)$  [109, (1.6)] as

$$\mathrm{T HH}^{[n]}(M^I(f)) \simeq M^I(f) \wedge B^n A_+.$$

As an example consider the canonical map  $f: BU \rightarrow BG_{hI}$ . Then one obtains

$$X \otimes MU \simeq MU \wedge \Omega^{\infty}(bu \wedge |X|)$$

and

$$\mathrm{T HH}^{[n]}(MU) \simeq MU \wedge \Omega^{\infty}(bu \wedge \mathbb{S}^n) \simeq MU \wedge B^n BU_+.$$

There is also a statement about  $\mathrm{T HH}$  of Thom spectra associated to single loop maps in [22, Theorem 1]. We state the relative version of this, so  $G$  is a cofibrant replacement of  $GL_1^I(R)$ .

**Theorem 6.9.** [20, Theorem 6.6] *Assume that  $R$  is a commutative symmetric ring spectrum that is semistable and  $S$ -cofibrant. Let  $M^I(f)$  be a Thom spectrum associated to a map  $f: M \rightarrow BG_{hI}$  of topological monoids, where  $M$  is grouplike and well-pointed. Then*

$$\mathrm{T HH}^R(M^I(f)) \simeq M^I(L^{\eta}(B(f))).$$

Here,  $M^I(L^{\eta}(B(f)))$  is the Thom spectrum associated to the map

$$\begin{array}{ccc} L(B(M)) & \xrightarrow{L(B(f))} & LBBG_{hI} \simeq BG_{hI} \times BBG_{hI} \\ \downarrow L^{\eta}(B(f)) & & \downarrow \mathrm{id} \times \eta \\ BG_{hI} & \xleftarrow{\mu} & BG_{hI} \times BG_{hI} \end{array}$$

Note that  $BBG_{hI}$  is an  $H$ -group, so we can split the free loop space  $LBBG_{hI}$  into the base space and the based loops

$$LBBG_{hI} \simeq BBG_{hI} \times \Omega BBG_{hI}$$

and the second factor is equivalent to  $BG_{hI}$ . As usual,  $\eta$  denotes the Hopf map  $\eta: \mathbb{S}^3 \rightarrow \mathbb{S}^2$  and it induces a map  $\eta: BBG_{hI} \rightarrow BG_{hI}$  as above via

$$BBG_{hI} \simeq \Omega^2 B^4 G_{hI} \rightarrow \Omega^3 B^4 G_{hI} \simeq BG_{hI}$$

by reducing the loop coordinates by precomposition.

For quotient spectra, this results gives a new way of calculating  $\mathrm{THH}^R(R/I)$ . For related results see [4] and in the case where  $R/I$  happens to be commutative see [33, §7].

A second example comes from viewing  $H\mathbb{Z}_{(p)}$  as a Thom spectrum associated as a 2-fold loop map  $\Omega^2(\mathbb{S}^3\langle 3 \rangle) \rightarrow BG_{hI}$  which allows for a determination of  $\mathrm{THH}(H\mathbb{Z}_{(p)})$  as  $H\mathbb{Z}_{(p)} \wedge \Omega(\mathbb{S}^3\langle 3 \rangle)_+$  [22, Theorem 3.8] and an additive equivalence

$$\mathrm{THH}^{[2]}(H\mathbb{Z}_{(p)}) \simeq H\mathbb{Z}_{(p)} \wedge \mathbb{S}^3\langle 3 \rangle_+.$$

This gives a geometric interpretation of (6.2), but without an identification of the multiplicative structure. See also [59, §4], where Klang presents related results, using the framework of factorization homology.

**6.4. Topological Hochschild cohomology as a derived center.** In the discrete case, *i.e.*, for a commutative ring  $k$  and a  $k$ -algebra  $A$  one can describe the center of  $A$ ,

$$Z(A) = \{b \in A, ab = ba \text{ for all } a \in A\}$$

as the set of  $A$ -bimodule maps from  $A$  to  $A$ . If  $f$  is such a map,  $f: A \rightarrow A$  with  $f(cad) = cf(a)d$  for all  $a, c, d \in A$ , then  $f$  is determined by  $f(1) =: b$  and this  $b$  satisfies

$$ab = af(1) = f(a \cdot 1) = f(a) = f(1 \cdot a) = f(1)a = ba$$

so the set of such morphisms gives rise to an element in the center and vice versa, for any  $b \in Z(A)$  we get such an  $f$  by setting  $f(1) = b$ .

Hochschild cohomology of  $A$  over  $k$  can be described as

$$\mathrm{HH}_k^*(A) = \mathrm{Ext}_{A \otimes_k A^o}^*(A, A)$$

if  $A$  is  $k$ -projective. Hence  $\mathrm{HH}_k^0(A) = Z(A)$  and the Hochschild cohomology of  $A$  is the *derived center of  $A$* . Hochschild cohomology has a graded commutative algebra structure via a cup product, but the solved Deligne conjecture [87] says that the Hochschild cochain complex is in general not a differential graded commutative algebra, but that it has an  $E_2$ -algebra structure.

For ring spectra there is no homotopically meaningful definition of a center: requiring equality translates to an equalizer diagram and this wouldn't be homotopy invariant. For a commutative ring spectrum  $R$  and an  $R$ -algebra spectrum  $A$  this equalizer corresponds precisely to taking not just  $R$ -module endomorphisms but  $A$ -bimodule endomorphisms. So a homotopy invariant version is as follows.

**Definition 6.10.** For a commutative ring spectrum  $R$  and an  $R$ -algebra spectrum  $A$ , the *topological Hochschild cohomology groups of  $A$  over  $R$*  are

$$\mathrm{THH}_R^*(A) = \pi_* \mathrm{Ext}_{A \wedge_R A^o}(A, A)$$

and the *derived center of  $A$  over  $R$*  is

$$\mathrm{THH}_R(A) = \mathrm{Ext}_{A \wedge_R A^o}(A, A).$$

Here,  $\mathrm{Ext}_{A \wedge_R A^o}(A, A)$  denotes the derived endomorphism spectrum of  $A$  as an  $A$ -bimodule [34, IV §1].

McClure and Smith's proof of the Deligne conjecture also provides a spectrum version for topological Hochschild cohomology, giving the derived center an  $E_2$ -structure:

**Theorem 6.11.** [87] *If  $A$  is an associative  $R$ -algebra spectrum, then  $\mathrm{THH}_R(A)$  is an  $E_2$ -ring spectrum.*



An important example of a calculation of such a derived center is Angeltveit's calculation of  $\mathrm{THH}_{E_n}(K_n)$ . Here  $E_n$  denotes Morava  $E$ -theory with

$$\pi_*(E_n) \cong W(\mathbb{F}_q)[[u_1, \dots, u_{n-1}]]\langle u^{\pm 1} \rangle$$

where the  $u_i$  are deformation parameters for the height  $n$  Honda formal group law with  $|u_i| = 0$  and  $u$  is a periodicity element with  $|u| = 2$ . The sequence of elements  $(p, u_1, \dots, u_{n-1})$  is a regular sequence and  $K_n$  is the 2-periodic version of Morava  $K$ -theory

$$K_n = E_n/(p, u_1, \dots, u_{n-1}), \quad (K_n)_* = \mathbb{F}_q\langle u^{\pm 1} \rangle.$$

Angeltveit shows that the derived center of  $K_n$  over  $E_n$  depends on the chosen  $A_\infty$ -algebra structure of  $K_n$  over  $E_n$ :

**Theorem 6.12.** [4, Theorems 5.21, 5.22]

- (a) For any prime  $p$  and any  $n \geq 1$  there is an  $A_\infty$ -structure on  $K_n$  such that  $\mathrm{THH}_{E_n}(K_n) \simeq E_n$ .
- (b) For  $n = 1$  and any  $d$  with  $1 \leq d < p - 1$  and any  $a$  with  $1 \leq a \leq p - 1$  there is an  $A_\infty$ -structure on  $K_1$  with

$$\mathrm{THH}_{E_1}^*(K_1) \cong \pi_*(E_1)[[q]]/(p + a(uq)^d).$$

Here, the first structure in (a) occurs as the one coming from the *least commutative*  $A_\infty$  structure on  $K_n$  (see [4, Theorem 5.8] for a precise statement). The case  $n = 1, p = 2$  of (a) is due to Baker and Lazarev [11, Proof of Theorem 3.1] who show that at the prime 2

$$\mathrm{THH}_{KU_2}(K(1)) \simeq KU_2.$$

**6.5. Topological André-Quillen homology.** We will first sketch the definition of ordinary André-Quillen homology. See [95] for the original account and [55] for a very readable modern introduction.

**Definition 6.13.** Let  $k$  be a commutative ring with unit and let  $A$  be a commutative  $k$ -algebra. The  $A$ -module of Kähler differentials of  $A$  over  $k$  is the  $A$ -module generated by elements  $d(a)$  for  $a \in A$  subject to the relations that  $d$  is  $k$ -linear and satisfies the Leibniz rule:

$$d(ab) = d(a)b + ad(b).$$

This  $A$ -module is denoted by  $\Omega_{A|k}^1$ .

The conditions imply  $d(1) = d(1 \cdot 1) = 2d(1)$  and hence  $d(1) = 0$ . For a polynomial algebra  $A = k[x_1, \dots, x_n]$  the  $A$ -module  $\Omega_{k[x_1, \dots, x_n]|k}^1$  is freely generated by  $dx_1, \dots, dx_n$ . By induction one can show that  $d(x_i^m) = mx_i^{m-1}d(x_i)$  for all  $m \geq 2$ .

Consider for instance the  $\mathbb{F}_p$ -algebra  $\mathbb{F}_p[x]/(x^p - x)$ . Then the module of Kähler differentials is generated by  $d(x)$ . However, as we are in characteristic  $p$  we get

$$d(x) = d(x^p) = px^{p-1}d(x) = 0$$

and hence  $\Omega_{\mathbb{F}_p[x]/(x^p-x)|\mathbb{F}_p}^1 = 0$ .

**Remark 6.14.** For a commutative  $k$ -algebra  $A$  there is an isomorphism between  $\Omega_{A|k}^1$  and the first Hochschild homology group of  $A$  over  $k$ : Every  $a \otimes b$  in Hochschild chain degree one is a cycle and if you send  $a \otimes b$  to  $ad(b)$  then this gives a well-defined map modulo Hochschild boundaries and it induces an isomorphism  $\mathrm{HH}_1^k(A) \cong \Omega_{A|k}^1$  [67, Proposition 1.1.10].

**Definition 6.15.** Let  $M$  be an  $A$ -module. A  $k$ -linear derivation from  $A$  to  $M$  is a  $k$ -linear map  $\delta: A \rightarrow M$  which satisfies the Leibniz rule.

The set of all such derivations,  $\mathrm{Der}_k(A, M)$ , is an  $A$ -submodule of the  $A$ -module of all  $k$ -linear maps. The symbol  $d$  in the definition of  $\Omega_{A|k}^1$  satisfies the conditions of a derivation, hence the map

$$d: A \rightarrow \Omega_{A|k}^1, \quad a \mapsto da$$

is a derivation, in fact, it is the *universal derivation*:

**Proposition 6.16.** [55] For all  $A$ -modules  $M$  the canonical map

$$\mathrm{Hom}_A(\Omega_{A|k}^1, M) \rightarrow \mathrm{Der}_k(A, M), \quad f \mapsto f \circ d$$

is an  $A$ -linear isomorphism.

There is another crucial reformulation of the above isomorphism:  $\mathrm{Der}_k(A, M)$  can also be identified with the morphisms of commutative  $k$ -algebras from  $A$  to the square-zero extension  $A \oplus M$ . The latter is the commutative  $A$ -algebra with underlying module  $A \oplus M$  with multiplication

$$(a_1, m_1)(a_2, m_2) = (a_1 a_2, a_1 m_2 + a_2 m_1), \quad a_1, a_2 \in A, m_1, m_2 \in M.$$

A derivation  $\delta: A \rightarrow M$  corresponds to the map into the second component of  $A \oplus M$ .

The idea of André-Quillen homology is to take the derived functor of  $A \mapsto M \otimes_A \Omega_{A|k}^1$ . But in which sense? As  $A$  is a commutative algebra, we need a resolution of  $A$  as such an algebra. The category of differential graded commutative  $k$ -algebras in general doesn't have a (right-induced) model structure, so instead one works with *simplicial resolutions*. The category of simplicial commutative  $k$ -algebras *does* have a nice model structure. Let  $P_\bullet \rightarrow A$  be a cofibrant resolution. Each  $P_n$  can be chosen to be a polynomial algebra [55, §4].

**Definition 6.17.** The *André-Quillen homology of  $A$  over  $k$  with coefficients in  $M$*  is

$$\mathrm{AQ}_*(A|k : M) = \pi_*(M \otimes_{P_\bullet} \Omega_{P_\bullet|k}^1).$$

A definition of  $\Omega_{A|k}^1$  in terms of generators and relations is not suitable for a generalization to commutative ring spectra. Instead we use the following description:

**Lemma 6.18.** Denote by  $I$  the kernel of the multiplication map

$$\mu: A \otimes_k A \rightarrow A.$$

Then  $\Omega_{A|k}^1$  is isomorphic to  $I/I^2$ .

*Proof.* Note that  $I$  is generated by elements of the form  $a \otimes 1 - 1 \otimes a$ . Such an element is identified with  $d(a)$ . Taking the quotient by  $I^2$  corresponds to the Leibniz rule for  $d$ .  $\square$

The ideal  $I$  can also be viewed as a non-unital  $k$ -algebra and  $I/I^2$  is the *module of indecomposables of  $I$* .

This definition translates to brave new commutative rings. Basterra's work is formulated in the setting of [34]:

**Definition 6.19.** Let  $A$  be a commutative  $R$ -algebra spectrum.

- We define  $I(A \wedge_R A)$  as the pullback

$$\begin{array}{ccc} I(A \wedge_R A) & \xrightarrow{\quad} & A \wedge_R A \\ \downarrow & & \downarrow \mu \\ * & \xrightarrow{\quad} & A \end{array}$$

- If  $N$  is a non-unital commutative  $R$ -algebra spectrum, then its  *$R$ -module of indecomposables*,  $Q(N)$ , is defined as the pushout

$$\begin{array}{ccc} N \wedge_R N & \xrightarrow{\quad} & * \\ \mu_N \downarrow & & \downarrow \\ N & \xrightarrow{\quad} & Q(N) \end{array}$$

- For an  $A$ -module spectrum  $M$  we define the *topological André-Quillen homology of  $A$  over  $R$  with coefficients in  $M$*  as

$$(6.3) \quad \mathrm{TAQ}(A|R : M) = \mathbf{LQ}(\mathbf{R}I(A \wedge_R A))$$

and denote its homotopy groups as  $\mathrm{TAQ}_*(A|R : M)$ . We abbreviate  $\mathbf{LQ}(\mathbf{R}I(A \wedge_R A))$  by  $\Omega_{A|R}$ .

Thus for  $\Omega_{A|R}$  we take homotopy invariant versions of the kernel of the multiplication map followed by taking indecomposables by applying the right derived functor of  $I$  and the left derived functor of  $Q$ .

**Definition 6.20.** Dually, *topological André-Quillen cohomology of  $A$  over  $R$  with coefficients in  $M$*  is  $F_A(\Omega_{A|R}, M)$  and we set  $\mathrm{TAQ}^n(A|R; M) = \pi_{-n}F_A(\Omega_{A|R}, M)$ .

Basterra proves [16, Proposition 3.2] that maps from  $\Omega_{A|R}$  to  $M$  in the homotopy category of  $A$ -modules correspond to maps in the homotopy category of commutative  $R$ -algebra maps over  $A$  from  $A$  to  $A \vee M$  where  $A \vee M$  carries the square-zero multiplication.

For example, if  $f: B \rightarrow BGL_1(S)$  is an infinite loop map and  $M(f)$  is the associated Thom spectrum, then Basterra and Mandell show [17, Theorem 5 and Corollary] that

$$\mathrm{TAQ}(M(f)) \simeq M(f) \wedge b$$

where  $\Omega^\infty b \simeq B$ . In the case of an  $E_\infty$ -space  $B$  the spherical group ring  $\Sigma_+^\infty B$  has

$$\mathrm{TAQ}(\Sigma_+^\infty B) \simeq \Sigma_+ B \wedge b.$$

## 7. HOW DO WE RECOGNIZE RING SPECTRA AS BEING (NON) COMMUTATIVE?

If you have a concrete model of a homotopy type, say in symmetric spectra, then you can be lucky and this model possesses a commutative structure and you should be able to check this by hand. Of course you could also try to disprove commutativity by showing that your spectrum doesn't have power operations as in (2.2) and this has been done in many cases, but sometimes you might need a different approach.

**7.1. Obstructions via filtrations and resolutions.** An obstruction theory for  $A_\infty$ -structures on homotopy ring spectra was developed as early as 1989 [100] by Alan Robinson. Obstruction theories for  $E_\infty$ -structures came much later: Goerss-Hopkins and Robinson [38], [101] independently developed one with obstruction groups that later turned out to be isomorphic [19]. The idea is to use a filtration or resolution of an operad such that the corresponding filtration quotients or the corresponding spectral sequence gives rise to obstruction groups that contain obstructions for lifting a partial structure to a full  $E_\infty$ -ring structure [101, Theorem 5.6], [38, Corollary 5.9]. The Goerss-Hopkins approach also allows to calculate the homotopy groups of the derived  $E_\infty$  mapping space between two such  $E_\infty$ -ring spectra [38, Theorem 4.5].

The obstruction groups have as input the algebra of cooperations  $E_*E$  of a spectrum  $E$  and they compute André-Quillen cohomology groups of the graded commutative  $E_*$ -algebra  $E_*E$  in the setting of differential graded (or simplicial)  $E_\infty$ -algebras. See [73], [39, §2.4] for background on these cohomology groups and see [19, §2] for the comparison results. In Robinson's setting these groups are called Gamma-cohomology. The obstruction groups vanish if for instance  $E_*E$  is étale as an  $E_*$ -algebra.

If you prefer to work with explicit chain complexes, then there are several equivalent ones computing Gamma cohomology groups in Robinson's setting [101, §2.5], [102, §6], [93, §2] and therefore, by the comparison result from [19, Theorem 2.6], computing the obstruction groups in the Goerss-Hopkins setting as well.

There is another version of obstruction theory for promoting a homotopy  $T$ -algebra structure to an actual one by Johnson and Noel [58] where  $T$  is a monad. This includes obstructions for operadic structures on spectra but also includes for instance group actions. Noel shows that in certain situations the obstruction theory [58] can be compared to the one of [38].

We list some important applications:

- (a) The development of the Hopkins-Miller and Goerss-Hopkins obstruction theory was motivated by the Morava- $E$ -theory spectra, also known as Lubin-Tate spectra,  $E_n$  and variants of those. These are Landweber exact cohomology theories that govern the deformation theory of height  $n$  formal group laws. In [97] an obstruction theory was established that allowed to show that the  $E_n$  are  $A_\infty$  spectra and that the Morava stabilizer group  $\mathbb{G}_n$  acts on  $E_n$  via maps of  $A_\infty$  spectra. In [38] the corresponding obstruction

theory for  $E_\infty$ -structures was developed and [38, Corollaries 7.6, 7.7] shows that the  $\mathbb{G}_n$ -action is via  $E_\infty$ -maps.

- (b) It was known that  $KU$  and  $KO$  are  $E_\infty$ -spectra and it was also known that the  $p$ -completed Adams summand  $L_p$  is  $E_\infty$ . In [12] Andy Baker and I use Robinson's version of the  $E_\infty$ -obstruction theory to show that these  $E_\infty$ -structures are unique and that the  $p$ -local Adams summand also has a unique  $E_\infty$ -structure. Uniqueness also holds for the connective covers [14]. It is important to have uniqueness results for  $E_\infty$ -structures because calculations can depend on a choice of such a structure.
- (c) For an  $E_\infty$ -ring spectrum  $R$  there is a  $\theta$ -algebra structure on its  $p$ -adic K-theory,  $\pi_*L_{K(1)}(KU_p \wedge R)$  [39, Theorem 2.2.4] and in good cases

$$\pi_*L_{K(1)}(KU_p \wedge R) \cong \lim_k (KU_p)_*(R \wedge M(p^k))$$

where  $M(p^k)$  is the mod- $p^k$  Moore spectrum. The study of such structures was initiated by McClure in [28, Chapter IX]. There is a variant of the Goerss-Hopkins obstruction theory for realizing for instance a  $\theta$ -algebra (see [39, §2.4.4] and [63, Theorem 5.14]) as a  $K(1)$ -local  $E_\infty$ -ring spectrum.

There is one for realizing an  $E_\infty$ - $Hk$ -algebra spectrum with a fixed Dyer-Lashof structure on its homotopy [91, Proposition 2.2] (for  $k$  a field of characteristic  $p$ ). Other variants can be found in the literature.

The  $\theta$ -algebra version was successfully applied by Lawson and Naumann [63] to show that  $BP\langle 2 \rangle$  at 2 has an  $E_\infty$ -structure. By a different method Hill and Lawson [46, Theorem 4.2] find a commutative model for  $BP\langle 2 \rangle$  at the prime 3.

- (d) Mathew, Naumann and Noel use operations in Morava- $E$ -theory to prove May's nilpotence conjecture:

**Theorem 7.1.** [79, Theorem A] *If  $R$  is an  $H_\infty$ -ring spectrum and if  $x \in \pi_*(R)$  is in the kernel of the Hurewicz homomorphism  $\pi_*(R) \rightarrow H_*(R; \mathbb{Z})$ , then  $x$  is nilpotent.*

They use this – among many other applications – for the following result about  $E_\infty$ -ring spectra:

**Theorem 7.2.** [79, Proposition 4.2] *If  $R$  is an  $E_\infty$ -ring spectrum and if there is an  $m \in \mathbb{Z}$ ,  $m \neq 0$  with  $m \cdot 1 = 0 \in \pi_0(R)$ , then for all primes  $p$  and for all  $n \geq 1$ :*

$$K(n)_*(R) \cong 0.$$

Lawson observed that using  $K(n)$ -techniques (see [96] for background) this implies that for finite  $E_\infty$ -ring spectra  $R$  either the rational homology is non-trivial or  $R$  is weakly contractible, because if  $H_*(R; \mathbb{Q}) \cong 0$ , then by the above result all the Morava K-theories also vanish on  $R$ , but then the finiteness of  $R$  implies weak contractibility (see [79, Remark 4.3] for the full argument).

The Dyer-Lashof variant is for instance important when one wants to decide whether a given  $H_\infty$ -map can be upgraded to an  $E_\infty$ -map: Roughly speaking, an  $H_\infty$ -spectrum is like an  $E_\infty$ -spectrum in the homotopy category. You can find applications of this approach for instance in Noel's work [91] and in [58].

Other spectra, for instance,  $BP$ , come with homology operations just because they sit in the right place: analyzing the maps  $MU \rightarrow BP \rightarrow H\mathbb{F}_p$  gives [28, p. 63] that  $(H\mathbb{F}_p)_*(BP)$  embeds into the dual of the Steenrod algebra such that  $(H\mathbb{F}_p)_*(BP)$  is closed under the action of the Dyer-Lashof algebra – even without establishing a structured multiplication on  $BP$ . This led Lawson [61] to look for the right obstructions for an  $E_\infty$ -structure of  $BP$  at 2 via secondary operations (see Theorem 7.5).

**7.2. Obstructions via Postnikov towers.** A different approach to obstruction theory is to consider Postnikov towers in the world of commutative ring spectra [16] or in the setting of  $E_n$ -algebras [18].

To this end Basterra uses TAQ-cohomology to lift ordinary  $k$ -invariants of a connective commutative ring spectrum to  $k$ -invariants in a multiplicative Postnikov tower:

Assume that  $R$  is a connective commutative ring spectrum. Then there is a map of commutative ring spectra

$$p_0: R \rightarrow H(\pi_0(R))$$

and without loss of generality we can assume that  $p_0$  is a cofibration of commutative ring spectra that realizes the identity on  $\pi_0$ , *i.e.*,  $\pi_0(p_0) = \text{id}_{\pi_0(R)}$ .

If we abbreviate  $\pi_0(R)$  to  $B$  and if  $M$  is a  $B$ -module, then an element in  $\text{TAQ}^n(A|R; HM)$  corresponds to a morphism  $\varphi: A \rightarrow A \vee \Sigma^n HM$  in the homotopy category of  $R$ -algebra spectra over  $A$  and we can form the pullback of

$$\begin{array}{ccc} & & A \\ & & \downarrow i_A \\ A & \xrightarrow{\varphi} & A \vee \Sigma^n HM \end{array}$$

If we postcompose  $\varphi$  with the projection map to  $\Sigma^n HM$

$$(7.1) \quad A \xrightarrow{\varphi} A \vee \Sigma^n HM \longrightarrow \Sigma^n HM$$

such a TAQ-class forgets to an Ext-class in  $\text{Ext}_R^n(A; HM)$ , *i.e.*, if  $R$  is the sphere spectrum, to an ordinary cohomology class. Basterra shows that this projection maps  $k$ -invariants in the world of commutative ring spectra to ordinary  $k$ -invariants of the underlying spectrum.

**Theorem 7.3.** [16, Theorem 8.1] *For any connective commutative ring spectrum  $A$  there is a sequence of commutative ring spectra  $A_i$ ,  $\pi_0(A)$ -modules  $M_i$  and elements*

$$\tilde{k}_i \in \text{TAQ}^{i+2}(A_i|S; HM_{i+1})$$

such that

- $A_0 = H\pi_0(A)$  and  $A_{i+1}$  is the pullback of  $A_i$  with respect to  $\tilde{k}_i$ ,
- $\pi_j A_i = 0$  for all  $j > i$ ,
- there are maps of commutative ring spectra  $\lambda_i: A \rightarrow A_i$  which induce an isomorphism in homotopy groups up to degree  $i$  such that the diagram

$$\begin{array}{ccc} & & A_{i+1} \\ & \nearrow \lambda_{i+1} & \downarrow \\ A & \xrightarrow{\lambda_i} & A_i \end{array}$$

commutes in the homotopy category of commutative ring spectra.

You start with  $A_0 = H\pi_0(A)$  and then you have to find a suitable map  $A_0 \rightarrow A_0 \vee \Sigma^2 H(\pi_1(A))$  as a starting point of the multiplicative Postnikov tower.

Basterra's result can be used as an obstruction theory as follows. If  $A$  is a connective spectrum then it has an ordinary Postnikov tower with  $k$ -invariants living in ordinary cohomology groups

$$k_i \in H^{i+2}(A_i; \pi_{i+1}(A)).$$

You can then investigate whether it is possible to find multiplicative  $k$ -invariants

$$\tilde{k}_i \in \text{TAQ}^{i+2}(A_i|S; H\pi_{i+1}(A))$$

that forget to the  $k_i$ 's under the map from (7.1).

Basterra and Mandell show the following partial result using Postnikov towers for  $E_n$ -algebra spectra.

**Theorem 7.4.** [18, Theorem 1.1] *The Brown Peterson spectrum,  $BP$ , has an  $E_4$ -structure at every prime.*

This ensures by the main result of [75] that the derived category of  $BP$ -module spectra has a symmetric monoidal smash product. Tyler Lawson, however, showed that there are certain secondary operations in the  $\mathbb{F}_2$ -homology of every such spectrum with an  $E_{12}$ -structure and he could show that these are not present in the  $\mathbb{F}_2$ -homology of  $BP$  at 2. Let  $BP\langle n \rangle$  denote the spectrum  $BP/(v_{n+1}, v_{n+2}, \dots)$ .

**Theorem 7.5.** [61, Theorem 1.1.2] *The Brown-Peterson spectrum at the prime 2 does not possess an  $E_n$ -structure for any  $n$  with  $12 \leq n \leq \infty$ . The truncated Brown-Peterson spectrum  $BP\langle n \rangle$  for  $n \geq 4$  cannot have an  $E_n$ -structure for any  $n$  with  $12 \leq n \leq \infty$ .*

**7.3. Realization of  $E_\infty$ -spectra via derived algebraic geometry.** There is a completely different important and highly successful approach to realization problems, namely using *derived algebraic geometry*. You can learn about derived algebraic geometry in the next Chapter of the book.

## 8. WHAT ARE ÉTALE MAPS?

We first recall the algebraic notion of an étale  $k$ -algebra from [67, E.1]: Let  $k$  be a commutative ring and let  $A$  be a finitely generated commutative  $k$ -algebra. Then  $A$  is *étale*, if  $A$  is flat over  $k$  and if the module of Kähler differentials  $\Omega_{A|k}^1$  is trivial. If  $\Omega_{A|k}^1 = 0$ , then  $k \rightarrow A$  is called *unramified*. A  $k$ -algebra  $B$  (not necessarily commutative) is called *separable*, if the multiplication map

$$B \otimes_k B^o \rightarrow B$$

has a section as a  $B$ -bimodule map. In algebra, a commutative separable algebra has Hochschild homology concentrated in homological degree zero, in particular the module of Kähler differentials is trivial.

### 8.1. Rognes' Galois extensions of commutative ring spectra.

**Definition 8.1.** [104, Definition 4.1.3] Let  $A \rightarrow B$  be a map of commutative ring spectra and let  $G$  be a finite group acting on  $B$  via commutative  $A$ -algebra maps. Assume that  $S \rightarrow A \rightarrow B$  is a sequence of cofibrations in the model structure on commutative ring spectra of [34, Corollary VII.4.10]. Then  $A \rightarrow B$  is a  $G$ -Galois extension if

- (a) the canonical map  $\iota: A \rightarrow B^{hG}$  is a weak equivalence and
- (b)

$$(8.1) \quad h: B \wedge_A B \rightarrow \prod_G B$$

is a weak equivalence.

The first condition is the familiar fixed points condition from classical Galois theory of fields. The map  $\iota$  comes from taking the adjoint of the map

$$A \wedge EG_+ \xrightarrow{\text{id} \wedge p} A \wedge S^0 \cong A \longrightarrow B$$

where  $p: EG_+ \rightarrow S^0$  collapses  $EG$  to the non-base point of  $S^0$ .

The map  $h$  is adjoint to the composite

$$B \wedge_A B \wedge G_+ \rightarrow B \wedge_A B \rightarrow B$$

that comes from the  $G$ -action on the right factor of  $B \wedge_A B$  followed by the multiplication in  $B$ . (Informally, if smashes were tensors, then  $h(b_1 \otimes b_2) = (b_1 \cdot g(b_2))_{g \in G}$ .) Note that  $\prod_G B$  is isomorphic to  $F(G_+, B)$ , so we could rewrite the condition in (8.1) as the requirement that

$$h: B \wedge_A B \rightarrow F(G_+, B)$$

is a weak equivalence.

The condition that the map  $h$  from (8.1) is a weak equivalence is crucial. It is also necessary for Galois extensions of discrete commutative rings in order to ensure that the extension is

unramified. For instance  $\mathbb{Z} \subset \mathbb{Z}[i]$  satisfies that  $\mathbb{Z}[i]^{C_2} = \mathbb{Z}$ , but  $h: \mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{Z}[i] \rightarrow \mathbb{Z}[i] \times \mathbb{Z}[i]$  is not surjective:  $h$  detects the ramification at the prime 2. Therefore  $\mathbb{Z} \rightarrow \mathbb{Z}[i]$  is *not* a  $C_2$ -Galois extension but  $\mathbb{Z}[\frac{1}{2}] \rightarrow \mathbb{Z}[\frac{1}{2}, i]$  is  $C_2$ -Galois.

Galois extensions of commutative ring spectra can have rather bad properties as modules. So the following definition is actually an additional assumption (this does not happen in the discrete setting).

**Definition 8.2.** [104, Definition 4.3.1] A Galois extension  $A \rightarrow B$  is *faithful* if it is faithful as an  $A$ -module, *i.e.*, for every  $A$ -module  $M$  with  $M \wedge_A B \simeq *$  we have  $M \simeq *$ .

Important examples of Galois extensions of commutative ring spectra are the following. By  $C_n$  we denote the cyclic group of order  $n$ .

- (a) The concept of Galois extensions of commutative ring spectra corresponds to the one for commutative rings via the Eilenberg-Mac Lane spectrum functor [104, Proposition 4.2]: Let  $R \rightarrow T$  be a homomorphism of discrete commutative rings and let  $G$  be a finite group acting on  $T$  via  $R$ -algebra homomorphisms. Then  $R \rightarrow T$  is a  $G$ -Galois extension of commutative rings if and only if  $HR \rightarrow HT$  is a  $G$ -Galois extension of commutative ring spectra.
- (b) The complexification of real vector bundles gives rise to a map of commutative ring spectra  $KO \rightarrow KU$  from real to complex topological K-theory. There is a  $C_2$ -action on  $KU$  corresponding to complex conjugation of complex vector bundles. Rognes shows [104, Proposition 5.3.1] that this turns  $KO \rightarrow KU$  into a  $C_2$ -Galois extension.
- (c) At an odd prime  $p$  there is a  $p$ -adic Adams operation on  $KU_p$  that gives rise to a  $C_{p-1}$ -action on  $KU_p$  such that  $L_p \rightarrow KU_p$  is a  $C_{p-1}$ -Galois extension (see [104, 5.5.4]).
- (d) There is a notion of pro-Galois extensions of commutative ring spectra and  $L_{K(n)}S \rightarrow E_n$  is a  $K(n)$ -local pro-Galois extension with the extended Morava stabilizer group as the Galois group [104, Theorem 5.4.4].
- (e) Let  $p$  be an arbitrary prime. The projection map  $\pi: EC_p \rightarrow BC_p$  induces a map on function spectra

$$F(\pi_+, H\mathbb{F}_p): F((BC_p)_+, H\mathbb{F}_p) \rightarrow F((EC_p)_+, H\mathbb{F}_p) \sim H\mathbb{F}_p$$

which identifies  $H\mathbb{F}_p$  as a  $C_p$ -Galois extension over  $F((BC_p)_+, H\mathbb{F}_p)$  [104, Proposition 5.6.3]. Hence in the world of commutative ring spectra group cohomology sits between  $S$  and  $H\mathbb{F}_p$  as the base of a Galois extension! Beware, this Galois extension is not faithful. This observation is due to Ben Wieland: the Tate construction  $H\mathbb{F}_p^{tC_p}$  isn't trivial and it is actually killed by the Galois extension (in the spectral sequence you augment a Laurent generator to zero).

- (f) Studying elliptic curves with level structures gives  $C_2$ -Galois extensions  $\mathrm{TMF}_0(3) \rightarrow \mathrm{TMF}_1(3)$  and  $\mathrm{Tmf}_0(3) \rightarrow \mathrm{Tmf}_1(3)$  [78, Theorems 7.6, 7.12]. For  $\mathrm{TMF}_1(3)$  (and  $\mathrm{Tmf}_1(3)$ ) you consider elliptic curves with one chosen point of exact order 3 and for  $\mathrm{TMF}_0(3)$  (and  $\mathrm{Tmf}_0(3)$ ) you only remember a subgroup of order 3. As  $C_2 \cong \mathbb{Z}/3\mathbb{Z}^\times$  this gives a  $C_2$ -action. This can be made rigorous (see [47, 46, 78]).

**8.2. Notions of étale morphisms.** Weibel-Geller [123] show that for an étale extension of commutative rings  $\varphi: A \rightarrow B$  Hochschild homology satisfies *étale descent*: The map  $\mathrm{HH}(\varphi)_*$  induces an isomorphism

$$(8.2) \quad B \otimes_A \mathrm{HH}_*(A) \cong \mathrm{HH}_*(B)$$

and for finite  $G$ -Galois extensions  $\varphi: A \rightarrow B$  one obtains *Galois descent*:

$$(8.3) \quad \mathrm{HH}_*(A) \cong \mathrm{HH}_*(B)^G$$

It is easy to see that for a  $G$ -Galois extension of discrete commutative rings  $\varphi: A \rightarrow B$  with finite  $G$ , the induced extension of graded commutative rings  $\mathrm{HH}_*(\varphi): \mathrm{HH}_*(A) \rightarrow \mathrm{HH}_*(B)$  is

again  $G$ -Galois. In addition to having the right fixed-point property it satisfies

$$\begin{aligned} \mathrm{HH}_*(B) \otimes_{\mathrm{HH}_*(A)} \mathrm{HH}_*(B) &\cong B \otimes_A \mathrm{HH}_*(A) \otimes_{\mathrm{HH}_*(A)} B \otimes_A \mathrm{HH}_*(A) \\ &\cong B \otimes_A B \otimes_A \mathrm{HH}_*(A) \\ &\cong \prod_G B \otimes_A \mathrm{HH}_*(A) \\ &\cong \prod_G \mathrm{HH}_*(B). \end{aligned}$$

If  $\varphi: A \rightarrow B$  is étale, then the module of Kähler differentials  $\Omega_{B|A}^1$  is trivial and it can be easily seen that the map  $B \rightarrow \mathrm{HH}_*^A(B)$  is an isomorphism and that André-Quillen homology of  $B$  over  $A$  is trivial, because étale algebras are smooth.

For commutative ring spectra the situation is different. There are several non-equivalent notions of étale maps:

**Definition 8.3.** Let  $\varphi: A \rightarrow B$  be a morphism of commutative ring spectra.

- (a) [69, Definition 7.5.1.4] We call  $\varphi$  *Lurie-étale*, if  $\pi_0(\varphi): \pi_0(A) \rightarrow \pi_0(B)$  is an étale map of commutative rings and if the canonical map

$$\pi_*(A) \otimes_{\pi_0(A)} \pi_0(B) \rightarrow \pi_*(B)$$

is an isomorphism.

- (b) [85, Definition 3.2], [104, Definition 9.2.1] The morphism  $\varphi$  is (*formally*) *THH-étale*, if  $B \rightarrow \mathrm{THH}^A(B)$  is a weak equivalence.
- (c) [85, Definition 3.2], [104, Definition 9.4.1] We define  $\varphi$  to be (*formally*) *TAQ-étale*, if  $\mathrm{TAQ}(B|A)$  is weakly equivalent to  $*$ .

**Remark 8.4.**

- Rognes [104] reserves the labels THH-étale and TAQ-étale only for such maps that – in addition to the conditions above – identify  $B$  as a dualizable  $A$ -module.
- The condition of being Lurie-étale is strong and is a very algebraic one. It is for instance not satisfied by the  $C_2$ -Galois extension  $KO \rightarrow KU$  because on the level of homotopy groups this extension is rather appalling, compare (1.1).
- McCarthy and Minasian show that THH-étale implies TAQ-étale and they show that for  $n > 1$  the map  $H\mathbb{F}_p \rightarrow F(K(\mathbb{Z}/p\mathbb{Z}, n)_+, H\mathbb{F}_p)$  is a TAQ-étale morphism that is not THH-étale. They attribute this example to Mike Mandell [85, Example 3.5]. Minasian [89, Corollary 2.8] proves that both notions are equivalent for morphisms between connective commutative ring spectra.
- For connective spectra, the notion of Lurie-étaleness has several good features [69, §7.5] and Mathew shows in [77, Corollary 3.1] that one can use [68, Lemma 8.9] to show that under some finiteness condition TAQ-étaleness implies Lurie-étaleness in the connective case.

**Definition 8.5.** [104, Definition 9.1.1] Let  $C$  be a cofibrant associative  $A$ -algebra spectrum. Then  $C$  is *separable* if the multiplication map  $\mu: C \wedge_A C^o \rightarrow C$  has a section in the homotopy category of  $C$ -bimodule spectra.

**Proposition 8.6.** [104, 9.2.6] *If  $C$  is a commutative separable  $A$ -algebra spectrum, then  $C$  is THH-étale.*

*Proof.* Recall from Remark 6.2 that  $\mathrm{THH}^A(C)$  is an augmented commutative  $C$ -algebra spectrum, so the composite of the unit map  $C \rightarrow \mathrm{THH}^A(C)$  with the augmentation

$$C \rightarrow \mathrm{THH}^A(C) \rightarrow C$$

is the identity. In addition, we get a splitting in the homotopy category of  $C$ -bimodule spectra,

$$C \xrightarrow{s} C \wedge_A C \xrightarrow{\mu} C,$$



*i.e.*, the above composite is the identity on  $C$ . Taking the derived smash product  $C \wedge_{C \wedge_A C}^L (-)$  of the above sequence gives the sequence

$$\mathrm{THH}^A(C) \rightarrow C \rightarrow \mathrm{THH}^A(C)$$

in which the last map is equivalent to the unit map of  $\mathrm{THH}^A(C)$  and whose composite is the identity. So the unit map  $C \rightarrow \mathrm{THH}^A(C)$  has a right and a left inverse in the homotopy category of  $C$ -module spectra.  $\square$

**Definition 8.7.** Let  $A \rightarrow B$  be a map of commutative ring spectra and let  $G$  be a finite group acting on  $B$  via maps of commutative  $A$ -algebra spectra. Assume that  $S \rightarrow A \rightarrow B$  is a sequence of cofibrations in the model structure on commutative ring spectra of [34, Corollary VII.4.10]. Then  $A \rightarrow B$  is *unramified* if

$$h: B \wedge_A B \rightarrow \prod_G B$$

is a weak equivalence.

**Proposition 8.8.** (*compare* [104, Lemma 9.1.2]) *If  $A \rightarrow B$  is unramified, then  $B$  is separable over  $A$ .*

*Proof.* The canonical inclusion map  $i: B \rightarrow F(G_+, B)$  can be modelled by the pointed map from  $G_+$  to  $S^0$ , that sends the neutral element  $e \in G$  to the non-basepoint of  $S^0$  and sends all other elements to the basepoint. We define a section to the multiplication map of  $B$  to be

$$B \xrightarrow{i} F(G_+, B) \xleftarrow{h, \sim} B \wedge_A B.$$

Note that  $h$  is not a  $B$ -bimodule map, but we are only interested in its  $e$ -component of  $F(G_+, B)$ .  $\square$

Thus we can conclude that unramified maps of commutative ring spectra are  $\mathrm{THH}$ -étale and that the failure of the map  $B \rightarrow \mathrm{THH}^A(B)$  to being a weak equivalence detects ramification. This idea was exploited in [33] in order to identify for instance  $ko \rightarrow ku$  as a wildly ramified extension whereas the inclusion of the Adams summand  $\ell \rightarrow ku_{(p)}$  is tamely ramified [33, Theorems 4.1, 5.2]. Sagave also identified this map as being log-étale [106, Theorem 1.6].

**8.3. Versions of étale descent.** Transferring the Geller-Weibel result to the setting of commutative ring spectra, it seems natural to define two versions of descent:

**Definition 8.9.** In the following  $\varphi: A \rightarrow B$  is a cofibration and  $A$  is cofibrant.

- The morphism  $\varphi: A \rightarrow B$  satisfies *étale descent* if the canonical morphism

$$(8.4) \quad B \wedge_A \mathrm{THH}(A) \rightarrow \mathrm{THH}(B)$$

is a weak equivalence.

- If  $\varphi: A \rightarrow B$  is a map of commutative ring spectra and if  $G$  is a finite group acting on  $B$  via commutative  $A$ -algebra maps, then we say that  $\varphi$  satisfies *Galois descent* if the map

$$(8.5) \quad \mathrm{THH}(A) \rightarrow \mathrm{THH}(B)^{hG}$$

is a weak equivalence.

Akhil Mathew clarifies the relationship between the different notions of étale morphisms and the notions of descent. He proves that Lurie-étale morphisms satisfy étale descent [77, Theorem 1.3] and he shows that for a faithful  $G$ -Galois extension with finite Galois group  $G$ , both descent properties are equivalent [77, Proposition 4.3] and they are in turn equivalent to the property that  $\mathrm{THH}(A) \rightarrow \mathrm{THH}(B)$  is again a  $G$ -Galois extension.

Moreover, he shows that the morphism

$$\varphi: F(\mathbb{S}_+^1, H\mathbb{F}_p) \rightarrow F(\mathbb{S}_+^1, H\mathbb{F}_p)$$

that is induced by the degree- $p$  map on  $\mathbb{S}^1$  is a faithful  $C_p$ -Galois extension, but that it does *not* satisfy étale descent [77, Theorem 2.1] and hence it doesn't satisfy Galois descent.

The Hopf fibration  $\mathbb{S}^1 \rightarrow \mathbb{S}^3 \rightarrow \mathbb{S}^2$  is a principal  $\mathbb{S}^1$ -bundle. The corresponding morphism of commutative  $H\mathbb{Q}$ -algebra spectra of cochains

$$F(\eta, H\mathbb{Q}): F(\mathbb{S}_+^2, H\mathbb{Q}) \rightarrow F(\mathbb{S}_+^3, H\mathbb{Q})$$

is therefore an  $\mathbb{S}^1$ -Galois extension [104, Proposition 5.6.3].

In joint work with Christian Ausoni we show that the morphism  $F(\eta, H\mathbb{Q})$  does not satisfy Galois descent, *i.e.*,

$$\mathrm{THH}(F(\mathbb{S}_+^2, H\mathbb{Q})) \not\approx \mathrm{THH}(F(\mathbb{S}_+^3, H\mathbb{Q}))^{h\mathbb{S}^1} :$$

The homotopy groups of  $\mathrm{THH}(F(\mathbb{S}_+^2, H\mathbb{Q}))$  contain an element in degree  $-1$  that is not present in  $\pi_*(\mathrm{THH}(F(\mathbb{S}_+^3, H\mathbb{Q}))^{h\mathbb{S}^1})$ .

Mathew identifies the problem with étale descent of finite faithful Galois extensions for  $\mathrm{THH}$  as being caused by the non-trivial fundamental group of  $\mathbb{S}^1$ . He shows the following result.

**Theorem 8.10.** [77, Proposition 5.2] *Let  $X$  be a simply connected pointed space and let  $A \rightarrow B$  be a faithful  $G$ -Galois extension of commutative ring spectra with finite  $G$ . Then the map*

$$B \wedge_A (X \otimes A) \rightarrow X \otimes B$$

*is a weak equivalence.*

In particular, higher order topological Hochschild homology,  $\mathrm{THH}^{[n]}$  for  $n \geq 2$ , *does* satisfy étale descent for faithful finite Galois extensions. However, étale descent remains for instance an issue for torus homology.

Note that sometimes  $\mathrm{THH}$  *does* satisfy descent, even for ramified maps of commutative ring spectra. For instance, Christian Ausoni shows in [9, Theorem 10.2] that  $\mathrm{THH}(\ell_p)$  is  $p$ -adically equivalent to  $\mathrm{THH}(ku_p)^{hC_{p-1}}$  and even that  $K(\ell_p)$  is  $p$ -adically equivalent to  $K(ku_p)^{hC_{p-1}}$ .

**Remark 8.11.** In [30] Clausen, Mathew, Naumann and Noel prove far-reaching Galois descent results for topological Hochschild homology and algebraic K-theory; in particular they confirm a Galois descent conjecture for algebraic K-theory by Ausoni and Rognes in many important cases. They identify  $\mathrm{THH}$  as a *weakly additive invariant* (see [30, Definition 3.11]) and prove descent in the form of [30, Theorems 5.1, 5.6].

## 9. PICARD AND BRAUER GROUPS

**9.1. Picard groups in the setting of a symmetric monoidal category.** Let  $(\mathcal{C}, \otimes, 1, \tau)$  be a symmetric monoidal category. An important class of objects in  $\mathcal{C}$  are those objects  $C$  that have an inverse with respect to  $\otimes$ , *i.e.*, there is an object  $C'$  of  $\mathcal{C}$  such that

$$C \otimes C' \cong 1.$$

One wants to gather such objects in a category and build a space and spectrum out of them:

**Definition 9.1.** The *Picard groupoid* of  $\mathcal{C}$ ,  $\mathrm{Picard}(\mathcal{C})$ , is the category whose objects are the invertible objects of  $\mathcal{C}$  and whose morphisms are isomorphisms between invertible objects.

In general, this category might not be small. Note that if  $C_1$  and  $C_2$  are objects of  $\mathrm{Picard}(\mathcal{C})$ , then so is  $C_1 \otimes C_2$ ; in fact,  $\mathrm{Picard}(\mathcal{C})$  is itself a symmetric monoidal category.

**Definition 9.2.** Let  $\mathcal{C}$  be as above and assume that  $\mathrm{Picard}(\mathcal{C})$  is small.

Then  $\mathrm{PIC}(\mathcal{C})$  is the classifying space of the symmetric monoidal category  $\mathrm{Picard}(\mathcal{C})$  and let  $\mathrm{pic}(\mathcal{C})$  denote the connective spectrum associated to the infinite loop space associated to  $\mathrm{PIC}(\mathcal{C})$ .

The *Picard group* of  $\mathcal{C}$ ,  $\mathrm{Pic}(\mathcal{C})$ , is  $\pi_0 \mathrm{PIC}(\mathcal{C})$ .

If the Picard groupoid of  $\mathcal{C}$  is small, then the Picard group can also be described as the set of isomorphism classes of invertible objects of  $\mathcal{C}$  with the product

$$[C_1] \otimes [C_2] := [C_1 \otimes C_2].$$

The neutral element is the isomorphism class of the unit,  $[1]$ .

**Definition 9.3.** Let  $R$  be a (discrete) commutative ring, then we denote by  $\text{Pic}(R)$  the Picard group of the symmetric monoidal category of the category of  $R$ -modules and by  $\text{PIC}(R)$  (and  $\text{pic}(R)$ ) the Picard space (and Picard spectrum) of this category.

For instance the Picard group of a ring of integers in a number ring is its ideal class group.

**9.2. Picard group for commutative ring spectra.** For commutative ring spectra  $R$ , the above definition of  $\text{PIC}(R)$  and  $\text{pic}(R)$  would either be much too rigid (if one would choose  $\mathcal{C}$  to be the category of  $R$ -module spectra and isomorphisms) or not strict enough (if one would take  $\mathcal{C}$  to be the homotopy category of  $R$ -module spectra). See [80, §2] for an adequate background for a suitable definition and see [36, §4] for a dictionary how to pass from a commutative ring spectrum  $R$  and its category of modules to the  $\infty$ -categorical setting. Instead of working with symmetric monoidal categories, one uses presentable symmetric monoidal  $\infty$ -categories  $\mathcal{C}$ . Then the Picard  $\infty$ -groupoid of  $\mathcal{C}$  is the maximal subgroupoid of the underlying  $\infty$ -category of  $\mathcal{C}$  spanned by the invertible objects. This groupoid is equivalent to a grouplike  $E_\infty$ -space  $\text{PIC}(\mathcal{C})$  and hence there is a connective ring spectrum,  $\text{pic}(\mathcal{C})$ , associated to  $\mathcal{C}$  [36, §5].

Let  $R$  be a commutative ring spectrum. The operadic nerve of the category of cofibrant-fibrant  $R$ -modules is a stable presentable symmetric monoidal  $\infty$ -category [69, Proposition 4.1.3.10] and we will abbreviate this as the  $\infty$ -category of  $R$ -modules,  $R\text{mod}$ .

**Definition 9.4.** The *Picard group of a commutative ring spectrum*  $R$ ,  $\text{Pic}(R)$ , is the group  $\pi_0(\text{PIC}(R\text{mod}))$ .

Again, these Picard groups can also be described as the set of isomorphism classes of invertible  $R$ -modules in the homotopy category of  $R$ -module spectra.

The Picard space  $\text{PIC}(R)$  is a delooping of the units of  $R$  ([80, §2.2], [119, §5]): There is an equivalence

$$\text{PIC}(R) \simeq \text{Pic}(R) \times BGL_1(R).$$

**Remark 9.5.** There is a map  $\text{Pic}(\pi_*R) \rightarrow \text{Pic}(R)$  that realizes an element in the algebraic Picard group of invertible graded  $\pi_*R$ -modules as a module over  $R$  and in many cases this map is actually an isomorphism [13, Theorem 43]. In this case we call  $\text{Pic}(R)$  *algebraic*. A notable exception comes from Galois extensions of ring spectra: As in algebra, if  $A \rightarrow B$  is a  $G$ -Galois extension of commutative ring spectra with abelian Galois group  $G$ , then  $[B] \in \text{Pic}(A[G])$  [104, Proposition 6.5.2]. But for instance  $[KU_*]$  is certainly *not* an element in the algebraic Picard group  $\text{Pic}(KO_*[C_2])$ , see (1.1).

The equivalence classes of suspensions of  $R$  are always in  $\text{Pic}(R)$ , but if  $R$  is periodic, these suspensions don't generate a free abelian group. Let us mention some crucial examples of Picard groups of commutative ring spectra:

- The Picard group of the initial commutative ring spectrum  $S$  is  $\text{Pic}(S) \cong \mathbb{Z}$  where  $n \in \mathbb{Z}$  corresponds to the class of  $S^n$  [51].
- For connective commutative ring spectra the Picard group of  $R$  is algebraic [13, Theorem 21], [80, Theorem 2.4.4].
- For periodic real and complex K-theory the Picard groups just notices the suspensions of the ground ring: The Picard group of  $KU$  is algebraic:  $\text{Pic}(KU) \cong \mathbb{Z}/2\mathbb{Z}$ , and  $\text{Pic}(KO) \cong \mathbb{Z}/8\mathbb{Z}$  (Hopkins, [80, Example 7.1.1] and [36, §7]).
- The same applies to the periodic version of the spectrum of topological modular forms:  $\text{Pic}(\text{TMF}) \cong \mathbb{Z}/576\mathbb{Z}$  [80, Theorem A]. But for  $\text{Tmf}$ , the spectrum of topological forms that mediates between  $\text{TMF}$  and its connective version  $\text{tmf}$  one gets [80, Theorem B]

$$\text{Pic}(\text{Tmf}) \cong \mathbb{Z} \oplus \mathbb{Z}/24\mathbb{Z}$$

where the copy of the integers comes from the suspensions of  $\text{Tmf}$  and the generator of the  $\mathbb{Z}/24\mathbb{Z}$ -summand is described in [80, Construction 8.4.2]

- For any odd prime and any finite subgroup  $G$  of the full Morava stabilizer group  $\mathbb{G}_{p-1}$  Heard, Mathew and Stojanoska [44, Theorem 1.5] prove – using Galois descent techniques for  $\text{pic}$  – that the Picard group of  $E_{p-1}^{hG}$  is a cyclic group generated by the suspension of  $E_{p-1}^{hG}$ .

A Picard group that contains more elements than just the ones coming from suspensions of the commutative ring spectrum says that there are more self-equivalences of the homotopy category of  $R$ -modules than the standard suspensions. One might view this as twisted suspensions. Gepner and Lawson explore the concept of having a Picard-grading on the category of  $R$ -module spectra and they develop a Pic-resolution model category structure in the sense of Bousfield [36, §3.2].

**9.3. Descent method and local versions.** A crucial method for calculating Picard groups is Galois descent. If  $A \rightarrow B$  is a  $G$ -Galois extension (for  $G$  finite), then for the Picard spectra and spaces the following equivalences hold [36, 80]:

$$(9.1) \quad \text{pic}(A) \simeq \tau_{\geq 0} \text{pic}(B)^{hG} \text{ and } \text{PIC}(A) \simeq \text{PIC}(B)^{hG}.$$

Here,  $\tau_{\geq 0}$  denotes the connective cover of a spectrum. In general, the extension  $B$  is easier to understand than  $A$ , for instance in the case of the  $C_2$ -Galois extension  $KO \rightarrow KU$ , one obtains information about  $\text{pic}(A)$  using the homotopy fixed point spectral sequence

$$H^{-s}(G; \pi_t \text{pic}(B)) \Rightarrow \pi_{t-s}(\text{pic}(B)^{hG}).$$

In [47, §6] for instance, Hill and Meier use Galois descent to determine the Picard groups of  $\text{TMF}_0(3)$  and  $\text{Tmf}_0(3)$ :

**Theorem 9.6.** [47, Theorems 6.9, 6.12]

$$\text{Pic}(\text{TMF}_0(3)) \cong \mathbb{Z}/48\mathbb{Z} \text{ and } \text{Pic}(\text{Tmf}_0(3)) \cong \mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}.$$

Hopkins-Mahowald-Sadofsky started the investigation of the Picard groups of the  $K(n)$ -local homotopy categories for varying  $n$  [51]. They denote these Picard groups by  $\text{Pic}_n$ . Note that the relevant symmetric monoidal product for fixed  $n$  is

$$X \otimes Y = L_{K(n)}(X \wedge Y)$$

for  $K(n)$ -local  $X$  and  $Y$ . They determined  $\text{Pic}_1$  for all primes  $p$ :

**Theorem 9.7.** [51, Theorem 3.3, Proposition 2.7]

- At the prime 2:  $\text{Pic}_1 \cong \mathbb{Z}_2^\times \times \mathbb{Z}/4\mathbb{Z}$ .
- For all odd primes  $p$ ,  $\text{Pic}_1 \cong \mathbb{Z}_p \times \mathbb{Z}/q\mathbb{Z}$  with  $q = 2p - 2$ .

In the  $K(n)$ -local setting the notion of algebraic elements in  $\text{Pic}_n$  is slightly more involved. Hopkins, Mahowald and Sadofsky show [51] (see also [37, Theorem 2.4]) that a  $K(n)$ -local spectrum  $X$  is  $K(n)$ -locally invertible if and only if  $\pi_*(L_{K(n)}(E_n \wedge X))$  is a free  $(E_n)_*$ -module of rank one and if and only if  $\pi_*(L_{K(n)}(E_n \wedge X))$  is invertible as a continuous module over the completed group ring  $(E_n)_*[[\mathbb{G}_n]]$ . Here,  $\mathbb{G}_n$  is the full Morava-stabilizer group. Hence applying  $\pi_*(L_{K(n)}(E_n \wedge -))$  gives a map from  $\text{Pic}_n$  to the Picard group of continuous  $(E_n)_*[[\mathbb{G}_n]]$ -modules and this group is called  $\text{Pic}_n^{\text{alg}}$ . The kernel of the map,  $\kappa_n$ , collects the exotic elements in  $\text{Pic}_n$ :

$$0 \rightarrow \kappa_n \rightarrow \text{Pic}_n \rightarrow \text{Pic}_n^{\text{alg}}.$$

For odd primes, all elements in  $\text{Pic}_1$  can be detected algebraically but for  $p = 2$  one has a non-trivial element in  $\kappa_1$ . See [37] for  $\text{Pic}_2$  at  $p = 3$  and a general overview. There is ongoing work on  $\text{Pic}_2$  at  $p = 2$  by Agnès Beaudry, Irina Bobkova, Paul Goerss and Hans-Werner Henn.

**9.4. Brauer groups of commutative rings.** Probably most of you will know the definition of the Brauer group of a field. But as for many features that we want to transfer to the spectral world we need to consider algebraic concepts developed for commutative rings (not fields).

Azumaya started to think about general Brauer groups [10] in the setting of local rings. A general definition of the Brauer group of a commutative ring  $R$  was given by Auslander and Goldman [8] as Morita equivalence classes of Azumaya algebras. The Brauer group was then globalized to schemes by Grothendieck [43]. He also shows that the Brauer group of the initial ring  $\mathbb{Z}$  is trivial; this is a byproduct of his identification of Brauer groups of number rings in [43, III, Proposition (2.4)].

**9.5. Brave new Brauer groups.** Baker and Lazarev define in [11] what an Azumaya algebra spectrum is. We use one version of this definition in [15] to develop Brauer groups for commutative ring spectra. Related concepts can be found in [57] and [121].

Fix a cofibrant commutative ring spectrum  $R$ .

**Definition 9.8.** A cofibrant associative  $R$ -algebra  $A$  is called an *Azumaya  $R$ -algebra spectrum* if  $A$  is dualizable and faithful as an  $R$ -module spectrum and if the canonical map

$$A \wedge_R A^\circ \rightarrow F_R(A, A)$$

is a weak equivalence.

We list some crucial properties of Azumaya algebra spectra. For the first property recall the discussion of derived centers from Definition 6.10.

**Proposition 9.9.**

- (a) [11, Proposition 2.3] *If  $A$  is an Azumaya  $R$ -algebra spectrum, then  $A$  is homotopically central over  $R$ , i.e.,  $R \rightarrow \mathrm{THH}_R(A)$  is a weak equivalence.*
- (b) [15, Proposition 1.3] *Every Azumaya  $R$ -algebra spectrum  $A$  is separable over  $R$ .*
- (c) [15, Proposition 1.5] *If  $A$  is Azumaya over  $R$  and if  $C$  is a cofibrant commutative  $R$ -algebra then  $A \wedge_R C$  is Azumaya over  $C$ . Conversely, if  $C$  is as above and dualizable and faithful as an  $R$ -module, then  $A \wedge_R C$  being Azumaya over  $C$  implies that  $A$  is Azumaya over  $R$ .*

*If  $A$  and  $B$  are Azumaya over  $R$ , then  $A \wedge_R B$  is also Azumaya over  $R$ .*

- (d) [15, 2.2] *If  $M$  is a faithful, dualizable, cofibrant  $R$ -module, then (a cofibrant replacement of)  $F_R(M, M)$  is an  $R$ -Azumaya algebra spectrum.*

Thus the endomorphism Azumaya algebras are the ones that are always there and you want to ignore them.

**Definition 9.10.** Let  $A$  and  $B$  be two Azumaya  $R$ -algebra spectra. We call them *Brauer equivalent* if there are dualizable, faithful  $R$ -modules  $N$  and  $M$  such that there is an  $R$ -algebra equivalence

$$A \wedge_R F_R(M, M) \simeq B \wedge_R F_R(N, N).$$

We denote by  $Br(R)$  the set of Brauer equivalence classes of  $R$ -Azumaya algebra spectra.

Note that  $Br(R)$  is an abelian group with multiplication induced by the smash product over  $R$ . Johnson shows [57, Lemma 5.7] that one can reduce the above relation to what he calls *Eilenberg-Watts equivalence*. This implies that one can still think about the Brauer group of a commutative ring spectrum as the Morita equivalence classes of Azumaya algebra spectra.

We showed a Galois descent result [15, Proposition 3.2], saying that under some natural condition you can descent an Azumaya algebra  $C$  over  $B$  to an Azumaya algebra  $C^{hG}$  over  $A$  if  $A \rightarrow B$  is a faithful  $G$ -Galois extension with finite Galois group  $G$ .

**9.6. Examples of Brauer groups.** As we know that  $Br(\mathbb{Z}) = 0$ , we conjecture [15] that the Brauer group of the initial ring spectrum is also trivial. This conjecture was proven in [6, Corollary 7.17]. They actually showed a much stronger result:

**Theorem 9.11.** [6, Theorem 7.16] *If  $R$  is a connective commutative ring spectrum such that  $\pi_0(R)$  is either  $\mathbb{Z}$  or the Witt vectors  $W(\mathbb{F}_q)$ , then the Brauer group of  $R$  is trivial.*

Different approaches can be used to construct a *Brauer space* for a commutative ring spectrum  $R$ ,  $Br_R$ , [6, Definition 7.1], [36, §5], [119] and to show that this space is a delooping of the Picard space,  $PIC$

$$\Omega Br_R \simeq PIC(R)$$

with  $\pi_0(Br_R) \cong Br(R)$ .

An important question in the classical context of Brauer groups of schemes is to which extend these groups can be controlled by the second étale cohomology group. See the introduction of [121] for a nice overview. Toën shows that for quasi-compact and quasi-separated schemes  $X$  one can identify the *derived Brauer group of  $X$*  with  $H_{\text{ét}}^1(X; \mathbb{G}_m) \times H_{\text{ét}}^2(X; \mathbb{G}_m)$ . The work of Antieau and Gepner [6, §7.4] relates Brauer groups of connective commutative ring spectra to étale cohomology groups by establishing a spectral sequence starting from étale cohomology groups for étale sheaves over a connective commutative ring spectrum converging to the homotopy groups of the Brauer space [6, Theorem 7.12].

It is not hard to see that the integral version of the quaternions gives a non-trivial element in  $Br(S[\frac{1}{2}])$  [15, Proposition 6.3]; Antieau and Gepner show [6, Corollary 7.18]

$$Br(S[\frac{1}{p}]) \cong \mathbb{Z}/2\mathbb{Z} \quad \text{for all primes } p$$

and they prove the existence of a short exact sequence

$$0 \rightarrow Br(S_{(p)}) \rightarrow \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{q \neq p} \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

by applying [6, Corollary 7.13] where they calculate the homotopy groups of the Brauer space of any connective commutative ring spectrum  $R$  in terms of étale cohomology groups and the homotopy groups of  $R$ .

They use the classical exact sequence for the Brauer group of the rationals [43, §2] coming from the Albert-Brauer-Hasse-Noether theorem:

$$0 \rightarrow Br(\mathbb{Q}) \rightarrow \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{p \text{ prime}} Br(\mathbb{Q}_p) \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

with  $Br(\mathbb{Q}_p) = \mathbb{Q}/\mathbb{Z}$ . This determines  $Br(\mathbb{Z}[\frac{1}{p}])$  and  $Br(\mathbb{Z}_{(p)})$  and this in turn gives the above result for the sphere spectra with  $p$  inverted or localized at  $p$ .

In [15, Theorem 10.1] we show that the  $K(n)$ -local Brauer group of the  $K(n)$ -local sphere is non-trivial at least for odd primes and  $n > 1$ .

Gepner and Lawson prove a version of Galois descent for a suitable  $\infty$ -category of Azumaya algebras:

**Theorem 9.12.** [36, Theorem 6.15] *There is an equivalence of symmetric monoidal  $\infty$ -categories*

$$Az_A \rightarrow (Az_B)^{hG}$$

for every  $G$ -Galois extension  $A \rightarrow B$  with finite  $G$ .

They also construct a map of  $\infty$ -groupoids  $Az_R \rightarrow Br_R$  for any commutative ring spectrum  $R$  and show that this map is essentially surjective, such that equality in  $\pi_0(Br_R)$  corresponds precisely to Morita equivalence. They investigate the algebraic Brauer groups (*i.e.*, the Morita classes of Azumaya algebras over the coefficients) [36, §7.1] of 2-periodic commutative ring spectra with vanishing odd homotopy groups, such as  $KU$  or  $E_n$ , by relating them to the classical Brauer-Wall group of  $\pi_0$  of the ring spectrum and they identify a non-trivial Morita class of a quaternion  $KO$ -algebra that becomes Morita-trivial over  $KU$ .

There is recent work by Hopkins and Lurie [50] who identify the  $K(n)$ -local Brauer group of a Lubin-Tate spectrum  $E$  at all primes. For odd primes they obtain:

**Theorem 9.13.** [50, Theorem 1.0.11] *The  $K(n)$ -local Brauer group of  $E$  is the product of the Brauer-Wall group of the residue field  $\pi_0(E)/m$  and a group  $Br'(E)$  which in turn can be expressed as an inverse limit of abelian groups  $Br'_\ell$  such that the kernel of  $Br'_\ell \rightarrow Br'_{\ell-1}$  is non-canonically isomorphic to  $m^{\ell+2}/m^{\ell+3}$ .*

One ingredient is their construction of *atomic  $E$ -algebra spectra* [50, Definition 1.0.2] via a Thom spectrum construction relative to  $E$  for polarizations of lattices [50, Definition 3.2.1] using the machinery from [1, 2]. Here, the starting point is a lattice  $\Lambda$  of finite rank together with a *polarization map*  $Q: K(\Lambda, 1) \rightarrow \text{PIC}(E) \simeq \text{Pic}(E) \times BGL_1(E)$ .

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