A classification of Goodwillie towers

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Structured Ring Spectra - TNG
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  $\partial_n F$: spectrum with Σ_n-action (n^{th} Taylor coefficient of F at $*$).
- **Question**: What additional structure do the Taylor coefficients $\partial_* F = \{\partial_n F\}_{n \geq 1}$ possess that would allow the Taylor tower to be reconstructed?

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- **Consequence**: For any F, the symmetric sequence $\partial_* F$ is a coalgebra over the cotriple

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  - $P_n F \simeq \mathrm{Tot}(\Phi(\partial_*\Phi)^\bullet \partial_{\le n} F)$.
- **In fact**: There is an equivalence of homotopy categories

$$\boxed{n\text{-excisive functors}} \xrightarrow[\simeq]{\partial_*} \boxed{n\text{-truncated } K\text{-coalgebras}}$$

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- **This means**: There are natural maps

$$\partial_k F \to \mathrm{Map}(\mathrm{Lie}_{n_1} \wedge \ldots \wedge \mathrm{Lie}_{n_k}, \partial_n F)_{h\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}}$$

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- **Simple example**: A 2-excisive functor $F : \text{Top}_*^{\text{fin}} \to \text{Spectra}$ is determined uniquely by the single structure map

$$\partial_1 F \to (\Sigma \partial_2 F)_{h\Sigma_2}.$$