Towards an understanding of ramified extensions of structured ring spectra

Birgit Richter
Joint work with Bjørn Dundas, Ayelet Lindenstrauss

Women in Homotopy Theory and Algebraic Geometry

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We are interested in commutative monoids (commutative ring spectra) and their algebraic properties.

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- Take your favorite commutative ring R and consider singular cohomology with coefficients in R, $H^*(-;R)$. The representing spectrum is the Eilenberg-MacLane spectrum of R, HR.

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- Topological complex K-theory, $KU^0(X)$, measures how many different complex vector bundles of finite rank live over your space X. You consider isomorphism classes of complex vector bundles of finite rank over X, $Vect_{\mathbb{C}}(X)$. This is an abelian monoid wrt the Whitney sum of vector bundles. Then group completion gives $KU^0(X)$:

$$KU^0(X) = Gr(Vect_{\mathbb{C}}(X)).$$

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- The homotopy groups of KO are more complicated.

$$\pi_*(KO) = \mathbb{Z}[\eta, y, w^{\pm 1}]/2\eta, \eta^3, \eta y, y^2 - 4w, \quad |\eta| = 1, |w| = 8.$$

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The map that assigns to a real vector bundle its complexified vector bundle induces a ring map $c\colon KO \to KU$. Its effect on homotopy groups is $\eta \mapsto 0$, $y \mapsto 2u^2$, $w \mapsto u^4$. In particular, $\pi_*(KU)$ is a graded commutative $\pi_*(KO)$-algebra.

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This definition is a direct generalization of the definition of Galois extensions of commutative rings (due to Auslander-Goldman).

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Let $\mathbb{Q} \subset K$ be a finite G-Galois extension of fields and let $\mathcal{O}_K$ denote the ring of integers in K. Then $\mathbb{Z} \to \mathcal{O}_K$ is never unramified, hence $H\mathbb{Z} \to H\mathcal{O}_K$ is never a G-Galois extension.

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$\mathbb{Z}[\frac{1}{2}] \to \mathbb{Z}[i, \frac{1}{2}]$, however, is C_2-Galois.

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is a C_{p-1}-Galois extension. Here, the C_{p-1}-action is generated by an Adams operation.

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BUT: Trace methods work for connective spectra, these are spectra with trivial negative homotopy groups.

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We have to live with ramification!

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How do we measure ramification?

Relative THH

If we have a G-action on a commutative A-algebra B and if $h\colon B \wedge_A B \to \prod_G B$ is a weak equivalence, then Rognes shows that the canonical map

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$THH^A(B)$ is the geometric realization of the simplicial spectrum

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We abbreviate $\pi_*(THH^A(B))$ with $THH^A_*(B)$.

The *ko* $\to$ *ku*-case

The $ko \to ku$-case

Theorem (DLR)

- As a graded commutative augmented $\pi_*(ku)$-algebra

$$\pi_*(ku \wedge_{ko} ku) \cong \pi_*(ku)[\tilde{u}]/\tilde{u}^2 - u^2$$

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- The Tor spectral sequence

$$E^2_{*,*} = \mathrm{Tor}^{\pi_*(ku \wedge_{ko} ku)}_{*,*}(\pi_*(ku), \pi_*(ku)) \Rightarrow THH^{ko}_*(ku)$$

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- $THH^{ko}_*(ku)$ is a square zero extension of $\pi_*(ku)$:

$$THH^{ko}_*(ku) \cong \pi_*(ku) \rtimes \pi_*(ku)/2u\langle y_0, y_1, \ldots \rangle$$

  with $|y_j| = (1 + |u|)(2j + 1) = 3(2j + 1)$.

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Hence

$$HH_*^{\mathbb{Z}}(\mathbb{Z}[i]) \cong \mathbb{Z}[i] \rtimes (\mathbb{Z}[i]/2i)\langle y_j, j \geq 0\rangle$$

with $|y_j| = 2j + 1$.

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As $\pi_*(ku)$ splits off $THH_*^{ko}(ku)$ the zero column has to survive and cannot be hit by differentials and hence all differentials are trivial.

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Since the generators over $\pi_*(ku)$ are all in odd degree, and their products cannot hit the direct summand $\pi_*(ku)$ in filtration degree zero, their products are all zero.

Contrast to tame ramification

Consider and odd prime p and

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- Ausoni proved that the p-completed extension even satisfies Galois descent for THH and algebraic K-theory:

$$THH(ku_p)^{hC_{p-1}} \simeq THH(\ell_p), \quad K(ku_p)^{hC_{p-1}} \simeq K(\ell_p).$$

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$p - 1$ is a p-local unit, hence no additive integral torsion appears in $THH_*^{\ell}(ku_{(p)})$.

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We can control certain quotient maps, e.g. $tmf_1(3)_{(2)} \to ku_{(2)}$.

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