Towards an understanding of ramified extensions of structured ring spectra

Birgit Richter Joint work with Bjørn Dundas, Ayelet Lindenstrauss

Women in Homotopy Theory and Algebraic Geometry

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We are interested in commutative monoids (commutative ring spectra) and their algebraic properties.

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$$KU^0(X) = Gr(Vect_{\mathbb{C}}(X)).$$

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- The homotopy groups of *KO* are more complicated.

$$\pi_*({\sf KO})=\mathbb{Z}[\eta,y,w^{\pm 1}]/2\eta,\eta^3,\eta y,y^2-4w, \quad |\eta|=1,|w|=8.$$

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The map that assigns to a real vector bundle its complexified vector bundle induces a ring map $c \colon KO \to KU$. Its effect on homotopy groups is $\eta \mapsto 0$, $y \mapsto 2u^2$, $w \mapsto u^4$. In particular, $\pi_*(KU)$ is a graded commutative $\pi_*(KO)$ -algebra.

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This definition is a direct generalization of the definition of Galois extensions of commutative rings (due to Auslander-Goldman).

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Let $\mathbb{Q} \subset K$ be a finite *G*-Galois extension of fields and let \mathcal{O}_K denote the ring of integers in *K*. Then $\mathbb{Z} \to \mathcal{O}_K$ is never unramified, hence $H\mathbb{Z} \to H\mathcal{O}_K$ is never a *G*-Galois extension.

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 $\mathbb{Z} \to \mathbb{Z}[i]$ is wildly ramified at 2, hence $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{Z}[i]$ is not isomorphic to $\mathbb{Z}[i] \times \mathbb{Z}[i]$. $\mathbb{Z}[\frac{1}{2}] \to \mathbb{Z}[i, \frac{1}{2}]$, however, is C_2 -Galois.

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$$L_p
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is a C_{p-1} -Galois extension. Here, the C_{p-1} -action is generated by an Adams operation.

If we want to understand arithmetic properties of a commutative ring spectrum R, then we try to understand its algebraic K-theory, K(R).

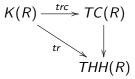
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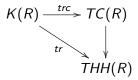
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BUT: Trace methods work for connective spectra, these are spectra with trivial negative homotopy groups.

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is certainly *not* étale. We have to live with ramification!

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Theorem (Dundas, Lindenstrauss, R) $ko \rightarrow ku$ is wildly ramified.

How do we measure ramification?

If we have a *G*-action on a commutative *A*-algebra *B* and if $h: B \wedge_A B \to \prod_G B$ is a weak equivalence, then Rognes shows that the canonical map

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 $THH^{A}(B)$ is the geometric realization of the simplicial spectrum

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whose composite is the identity on B. Thus B splits off $THH^{A}(B)$. If $THH^{A}(B)$ is larger than B, then $A \rightarrow B$ is ramified. We abbreviate $\pi_{*}(THH^{A}(B))$ with $THH_{*}^{A}(B)$.

Theorem (DLR)

• As a graded commutative augmented $\pi_*(ku)$ -algebra

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► The Tor spectral sequence

$$E_{*,*}^2 = \operatorname{Tor}_{*,*}^{\pi_*(ku \wedge_{ko} ku)}(\pi_*(ku), \pi_*(ku)) \Rightarrow THH_*^{ko}(ku)$$

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• $THH_*^{ko}(ku)$ is a square zero extension of $\pi_*(ku)$:

$$THH^{ko}_*(ku) \cong \pi_*(ku) \rtimes \pi_*(ku)/2u\langle y_0, y_1, \ldots \rangle$$

with $|y_j| = (1 + |u|)(2j + 1) = 3(2j + 1)$.

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Hence

$$HH^{\mathbb{Z}}_{*}(\mathbb{Z}[i]) \cong \mathbb{Z}[i] \rtimes (\mathbb{Z}[i]/2i) \langle y_{j}, j \geq 0 \rangle$$

with $|y_j| = 2j + 1$.

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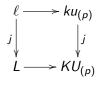
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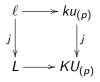
Since the generators over $\pi_*(ku)$ are all in odd degree, and their products cannot hit the direct summand $\pi_*(ku)$ in filtration degree zero, their products are all zero.

Contrast to tame ramification

Consider and odd prime p and

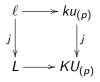


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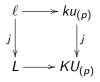
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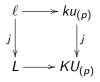
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- Sagave: The map $\ell \rightarrow ku_{(p)}$ is log-étale.

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 $\pi_*(\ell) = \mathbb{Z}_{(p)}[v_1] \to \mathbb{Z}_{(p)}[u] = \pi_*(ku_{(p)}), v_1 \mapsto u^{p-1}$ already looks much nicer.

- ▶ Rognes: $ku_{(p)} \rightarrow THH^{\ell}(ku_{(p)})$ is a K(1)-local equivalence.
- Sagave: The map $\ell \to k u_{(p)}$ is log-étale.
- Ausoni proved that the *p*-completed extension even satisfies Galois descent for *THH* and algebraic K-theory:

$$THH(ku_p)^{hC_{p-1}} \simeq THH(\ell_p), \quad K(ku_p)^{hC_{p-1}} \simeq K(\ell_p).$$

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p-1 is a *p*-local unit, hence no additive integral torsion appears in $THH_*^{\ell}(ku_{(p)})$.

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We can control certain quotient maps, e.g. $tmf_1(3)_{(2)} \rightarrow ku_{(2)}$.

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