An involution on the K-theory of (some) bimonoidal categories

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Birgit Richter

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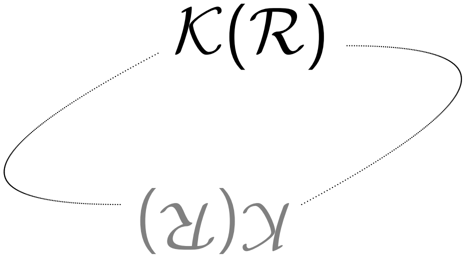
Examples of K-theory

Extra structure

K-theory of bimonoidal categories

Anti-involutions on bimonoidal categories

Involutions on K-theory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In general, K of a ring spectrum should tell us something about its 'arithmetic'.

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- Steiner and others constructed an involution on $A(X)$.

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- For any bimonoidal category $\mathcal{R}$ its *K-theory* (of the 2-category of finitely generated free modules over $\mathcal{R}$) is

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$$\begin{tikzcd}
GL_n(\pi_0 R) \arrow[r] \arrow[d, tail] & GL_n(Gr(\pi_0 R)) \arrow[d, tail] \\
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- $BGL_n(\mathcal{R})$ is a simplicial category with q-simplices of the form

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  plus isos $\phi^{i,j,k} \colon A^{i,j} \cdot A^{j,k} \to A^{i,k}$ as objects.

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- Theorem(Baas-Dundas-Richter-Rognes)

  For nice $\mathcal{R}$

$$\mathcal{K}(\mathcal{R}) \simeq K(H\mathcal{R})$$

  where $H\mathcal{R}$ is the spectrum associated to $\mathcal{R}$.

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- $\mathcal{V}_{\mathbb{R}}$ the category of real vector spaces and orthogonal morphisms:
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An anti-involution in a strictly bimonoidal category $\mathcal{R}$ consists of a functor $\zeta\colon \mathcal{R} \to \mathcal{R}$ with $\zeta \circ \zeta = \mathrm{id}$ and such that there are natural isomorphisms

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- $\zeta(1_{\mathcal{R}}) = 1_{\mathcal{R}}$ and $\mu_{1_{\mathcal{R}},A} = \mathrm{id}_{\zeta(A)} = \mu(A, 1_{\mathcal{R}})$.
- The μ are 'associative':

$$\begin{tikzcd}
\zeta(A \otimes B \otimes C) \arrow[r, "\mu_{A\otimes B,C}"] \arrow[d, "\mu_{A,B\otimes C}"'] & \zeta(C) \otimes \zeta(A \otimes B) \arrow[d, "\mathrm{id}\otimes\mu_{A,B}"] \\
\zeta(B \otimes C) \otimes \zeta(A) \arrow[r, "\mu_{B,C}\otimes\mathrm{id}"] & \zeta(C) \otimes \zeta(B) \otimes \zeta(A)
\end{tikzcd}$$

commutes for all $A, B, C \in \mathcal{R}$.

- The distributivity isomorphisms d_ℓ and d_r and the isomorphisms μ render the following diagrams commutative

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\zeta(A \otimes B \oplus A \otimes C) \arrow[r, "\zeta(d_r)"] \arrow[d, "\mu_{A\otimes B} \oplus \mu_{A\otimes C}"'] & \zeta(A \otimes (B \oplus C)) \arrow[d, "\mu_{A,B\oplus C}"] \\
\zeta(B) \otimes \zeta(A) \oplus \zeta(C) \otimes \zeta(A) \arrow[r, "d_\ell"] & (\zeta(B) \oplus \zeta(C)) \otimes \zeta(A),
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For instance $G = \pi_1(M)$, M a smooth manifold, then $w_1(M) \in H^1(M; \mathbb{Z}/2\mathbb{Z}) = [M, \mathbb{R}P^\infty]$ gives $\pi_1(w_1(M))\colon \pi_1(M) \to \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}^\times$.

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  Categories of Hopf-bimodules provide a class of examples of (non-strict) braided bimonoidal categories. Consider a Hopf algebra H in a symmetric monoidal category. An object M is an H Hopf-bimodule if it is a bimodule over H and simultaneously a H right- and left-comodule such that the comodule structure maps are morphisms of H-bimodules.

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- For a group G we define the category $\mathcal{E}G$ whose objects are the finite sets $\mathbf{n} = \{1, \dots, n\}$ for $n \in \mathbb{N}_0$ with $\mathbf{0} = \varnothing$ and
$$\mathcal{E}G(\mathbf{n}, \mathbf{m}) = \begin{cases} \varnothing & n \neq m \\ \Sigma_n \times G & n = m > 0 \\ \Sigma_0 & n = m = 0. \end{cases}$$

$\mathcal{E}G$ is bimonoidal. On objects, we take the bipermutative structure from $\mathcal{E}$, and on morphisms we define

$$(\sigma, g) \oplus (\sigma', g') = \begin{cases} (\sigma \oplus \sigma', e) & g \neq g' \\ (\sigma \oplus \sigma', g) & g = g' \end{cases}$$

for $\sigma \in \Sigma_n, \sigma' \in \Sigma_m$, $g, g' \in G$ and e the neutral element in the group G. Similarly,

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Let G be abelian. For the anti-involution, take ζ to be the identity on objects and on morphisms we define $\zeta(\sigma, g) = (\sigma, g^{-1})$ for all $g \in G$ and permutations σ.

The classifying space $B\mathcal{E}G$ has as group completion

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This is the zeroth space of the spherical group ring $S[BG] \simeq S[\Omega BBG]$ whose algebraic K-theory is $A(BBG)$.

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- For a matrix of objects $A \in M_n(\mathcal{R})$ the transpose of A, A^t, has $A^t_{i,j} = A_{j,i}$ as entries. For a morphism $\phi\colon A \to C$ in $M_n(\mathcal{R})$ we define ϕ^t as

$$\phi^t_{i,j} := \phi_{j,i}\colon A_{j,i} = A^t_{i,j} \to C^t_{i,j} = C_{j,i}.$$

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- Note that

$$(A \cdot B)^t_{i,j} = (A \cdot B)_{j,i} = \bigoplus_{k=1}^{n} A_{j,k} \otimes B_{k,i}$$

whereas

$$(B^t \cdot A^t)_{i,j} = \bigoplus_{k=1}^{n} B^t_{i,k} \otimes A^t_{k,j} = \bigoplus_{k=1}^{n} B_{k,i} \otimes A_{j,k}.$$

- We define the involution $\tau\colon B_q GL_n(\mathcal{R}) \to B_q GL_n(\mathcal{R})$ via

$$\tau\colon \begin{matrix} A^{0,1} & \dots & A^{0,q} \\ & \ddots & \vdots \\ & & A^{q-1,q} \end{matrix} \mapsto \begin{matrix} (\zeta(A^{q-1,q}))^t & \dots & (\zeta(A^{0,q}))^t \\ & \ddots & \vdots \\ & & (\zeta(A^{0,1}))^t. \end{matrix}$$

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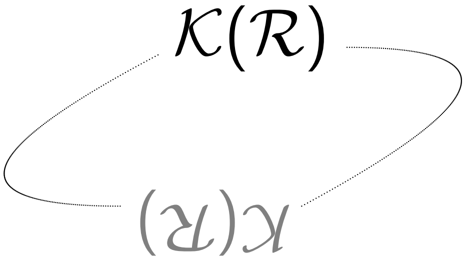
- The corresponding isomorphisms $\tau(\phi)^{i,j,k}$ are given by

$$\begin{array}{c} \tau(\phi)^{i,j,k}\colon \zeta(A^{q-j,q-i}))^t \cdot (\zeta(A^{q-k,q-j}))^t \\ \Big\downarrow{\scriptstyle \mu^{-1}} \\ (\zeta(A^{q-k,q-j} \cdot A^{q-j,q-i}))^t \\ \Big\downarrow{\scriptstyle (\zeta(\phi^{q-k,q-j,q-i}))^t} \\ (\zeta(A^{q-k,q-i}))^t. \end{array}$$

Theorem

The involution τ gives rise to an involution on $\mathcal{K}(\mathcal{R})$ for every bimonoidal category with anti-involution $(\mathcal{R}, \zeta, \mu)$.



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- For a discrete ring with anti-involution our involution on $\mathcal{K}(R) = K(R)$ agrees with Burghelea-Fiedorowicz's involution.

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- For $\mathcal{E}G$ we get an involution on $A(BBG)$ that agrees with Steiner's involution on $A(X)$ for $X = BBG$.

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$$\mathcal{K}(\mathcal{R}) \sim \mathcal{K}(\mathcal{R})^a \times \mathcal{K}(\mathcal{R})^s$$

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  There is no direct way to transfer the concept of hermitian K-theory to the K-theory of bimonoidal categories with anti-involution.

References

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