Suspension, Localization and (Partial Approximation Towers for) Goodwillie Calculus

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[[Details and proofs may be found at arxiv:1108.0114]]

'Teaser'

$$\mathbb{Z}_\infty(X)$$

$$P_\infty \mathrm{Id}(X) \qquad\qquad \mathrm{holim}_\Delta(\mathrm{sk}_k\, \Delta * X)$$

e.g. $(\mathrm{sk}_0\, \Delta^* * X) \simeq CX \rightrightarrows \Sigma X \rightrightarrows \Sigma X \vee \Sigma X \cdots$

Akin to, given $A \to B$ ring hom,

$$(\mathrm{sk}_0\, \Delta^* \otimes_A B) \simeq B \rightrightarrows B \otimes_A B \rightrightarrows B \otimes_A B \otimes_A B \cdots$$

Speed Goodwillie Calc intro

Goodwillie Calc intro:

- Given an endofunctor of (nonempty) spaces F that preserves homotopy equivalences, we can construct a tower of functors approximating F.
- $\mathrm{P}_\infty F(X) := \text{holim}(\cdots \to \mathrm{P}_n F(X) \to \mathrm{P}_{n-1} F(X) \to \cdots \to \mathrm{P}_1 F(X) \to \mathrm{P}_0 F(X))$
- When F, X 'nice', $F(X) \simeq \mathrm{P}_\infty F(X)$.
  (i.e. F ρ analytic and X at least ρ-connected)
- Each $\mathrm{P}_n F(X)$ is the homotopy colimit over an iterated finite limit construction T_n:

$$\mathrm{P}_n F(X) := \text{hocolim}(\mathrm{T}_n F(X) \xrightarrow{t_n} \mathrm{T}_n^2 F(X) \xrightarrow{t_n} \cdots)$$

$\tau^k : \mathrm{T}^k_n(X)F \to \mathrm{T}^k_{n-1}F(X)$

- (E.) New model for $\mathrm{T}_n F$ which yields natural maps $\tau^k : \mathrm{T}^k_n(X)F \to \mathrm{T}^k_{n-1}F(X)$.

$$
\begin{array}{ccccccccc|c}
\mathrm{P}_\infty F := \mathrm{holim}(\cdots \longrightarrow & \mathrm{P}_n F & \longrightarrow & \mathrm{P}_{n-1}F & \longrightarrow & & \longrightarrow & \mathrm{P}_1 F) & & \\
& \| & & \| & & & & \| & & \\
& \mathrm{hocolim}\,(\uparrow & & \mathrm{hocolim}\,(\uparrow & & & & \mathrm{hocolim}\,(\uparrow & & \\
& \vdots & & \vdots & & & & \vdots & & \\
& \uparrow & & \uparrow & & & & \uparrow & & \text{row:} \\
\cdots \longrightarrow & \mathrm{T}^2_n F & \longrightarrow & \mathrm{T}^2_{n-1}F & \longrightarrow & \cdots & \longrightarrow & \mathrm{T}^2_1 F & & 1 \\
& \uparrow & & \uparrow & & & & \uparrow & & \\
\cdots \longrightarrow & \underbrace{\mathrm{T}_n F} & \longrightarrow & \underbrace{\mathrm{T}_{n-1}F} & \longrightarrow & \cdots & \longrightarrow & \underbrace{\mathrm{T}_1 F} & & 0 \\
\hline
\text{column:} & n & & (n-1) & & & & 1 & &
\end{array}
$$

Results

- Thm(E.) There is a weak equivalence for all $k \geq 0$:

$$\text{holim}(\cdots \to \mathrm{T}_n^{k+1}F(X) \to \mathrm{T}_{n-1}^{k+1}F(X) \to \cdots \to \mathrm{T}_1^{k+1}F(X)) \sim \text{holim}\, F(\mathrm{sk}_k\, \Delta^* * X)$$

- Cor(E.) If F is 'nice' (ρ-analytic) and X nonempty, then we have the following weak equivalences $\forall r \geq \rho$,

$$\mathrm{P}_\infty F(X) \sim \text{holim}_n(\cdots \mathrm{T}_n^{r+1}F(X) \overset{\tau^{r+1}}{\to} \mathrm{T}_{n-1}^{r+1}F(X) \overset{\tau^{r+1}}{\to} \cdots \mathrm{T}_1^{r+1}F(X)) \sim \text{holim}_\Delta\, F(\mathrm{sk}_r\, \Delta^* * X)$$

  If we raise the connectivity of X, we may improve this to $r \geq \rho - (conn(X))$.

- Cor^2: Within its radius of convergence (i.e. if F is ρ analytic and X at least ρ-conn), then

$$F(X) \sim \mathrm{P}_\infty F(X) \sim \text{holim}_n(\cdots \mathrm{T}_n F(X) \to \mathrm{T}_{n-1}F(X) \cdots \to \mathrm{T}_1 F(X)) \sim \text{holim}_\Delta\, F(\mathrm{sk}_0\, \Delta * X)$$

As a picture

$$
\begin{array}{rccccccc|c}
\mathrm{P}_\infty F := \text{holim}(\cdots \longrightarrow & \mathrm{P}_n F & \longrightarrow & \mathrm{P}_{n-1} F & \longrightarrow & & \longrightarrow & \mathrm{P}_1 F) & \\
 & \Vert & & \Vert & & & & \Vert & \\
 & \text{hocolim}\Big(& & \text{hocolim}\Big(& & & & \text{hocolim}\Big(& \\
 & \uparrow & & \uparrow & & & & \uparrow & \\
 & \vdots & & \vdots & & & & \vdots & \text{row:} \\
\mathrm{P}_\infty F \sim \text{holim}(\cdots \longrightarrow & \mathrm{T}_n^{k+2} F & \longrightarrow & \mathrm{T}_{n-1}^{k+2} F & \longrightarrow & \cdots & \longrightarrow & \mathrm{T}_1^{k+2} F) & (k+1) \\
 & \uparrow & & \uparrow & & & & \uparrow & \\
\mathrm{P}_\infty F \sim \text{holim}(\cdots \longrightarrow & \mathrm{T}_n^{k+1} F & \longrightarrow & \mathrm{T}_{n-1}^{k+1} F & \longrightarrow & \cdots & \longrightarrow & \mathrm{T}_1^{k+1} F) & k \\
 & \uparrow & & \uparrow & & & & \uparrow & \\
 & \vdots & & \vdots & & & & \vdots & \\
 & \uparrow & & \uparrow & & & & \uparrow & \\
\cdots \longrightarrow & \mathrm{T}_n F & \longrightarrow & \mathrm{T}_{n-1} F & \longrightarrow & \cdots & \longrightarrow & \mathrm{T}_1 F & 0 \\
 & \Big) & & \Big) & & & & \Big) & \\
\hline
\text{column:} & n & & (n-1) & & & & 1 &
\end{array}
$$

An Application

Assume X is a connected space.

- The identity functor of spaces is 1 analytic, and we have $\mathrm{P}_\infty \mathrm{Id}(X) \sim \mathrm{holim}_\Delta(\mathrm{sk}_k \Delta * X)$ for all $k \geq 0$.
- (Arone-Kankaanrinta): $\mathrm{P}_\infty \mathrm{Id}(X) \sim \mathbb{Z}_\infty(X)$
- $\Rightarrow \mathbb{Z}_\infty(X) \sim \mathrm{holim}_\Delta(\mathrm{sk}_k \Delta * X)$ for all $k \geq 0$
- For $k = 0$, this equivalence was a result of Hopkins (in his thesis), and also of interest (and re-proven differently) in Goerss' (thesis) work on the Barratt desuspension spectral sequence.
- 'Structural' Result – recover a result without spectral sequence calculations.

$$\mathbb{Z}_\infty(X)$$

$\sim_{A-K}$ $\sim_{H,G}$

$$P_\infty \mathrm{Id}(X) \quad \sim_E \quad \mathrm{holim}_\Delta(\mathrm{sk}_0\,\Delta * X)$$

Present and future: This diagram generalizes.