Modules over *TMF*

Lennart Meier

Mathematisches Institut, Universität Bonn

Structured Ring Spectra TNG

TMF-modules

Our aim is to understand *TMF*-modules.

Let R be a ring spectrum and M and N be R-modules. Recall from Mark Hovey's talk, we have the following spectral sequence:

Universal coefficient spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{R_*}^{s,t}(M_*, N_*) \;\Rightarrow\; [M, N]_{Ho(R-mod)}^{s+t}$$

Problem: The global dimension of TMF_* is infinite.

The same problem already occurs for KO_*
– but $\mathrm{gldim}(KU_*) = \mathrm{gldim}(\mathbb{Z}[u^{\pm 1}]) = 1$.

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Bousfield's Work

Let $\mathcal{C}$ be the full additive subcategory of $KO - \text{mod}$ of all (finite) modules M such that $M \wedge_{KO} KU$ is a free KU-module – i.e. of the *relatively free* modules. We get a functor $\pi_*^{\mathcal{C}}$:

$$\begin{array}{rcl} KO - mod & \longrightarrow & \mathcal{C} - \text{mod} \\ X & \mapsto & (\pi_*(X \wedge_{KO} M))_{M \in \mathcal{C}} \end{array}$$

For formal reasons, prdim $\pi_*^{\mathcal{C}}(X) \leq 1$ for every finite KO-module N.

UCSS - modified version

For two R-modules M and N, we have a spectral sequence

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Still Bousfield

Theorem (Bousfield)

Every indecomposable relatively free KO-module is isomorphic to a suspension of KO, KU or KT. So we can choose

$$\mathcal{C} = CRT = \{KO, KU, KT\}.$$

UCSS
modified by CRT

Corollary

Every map between $\pi_*^{CRT}(X) \longrightarrow \pi_*^{CRT}(Y)$ can be realized.

$\Rightarrow \pi_*^{CRT}$ *classifies KO-modules.*

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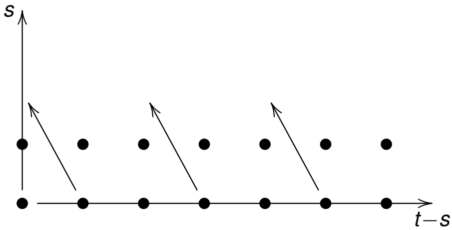

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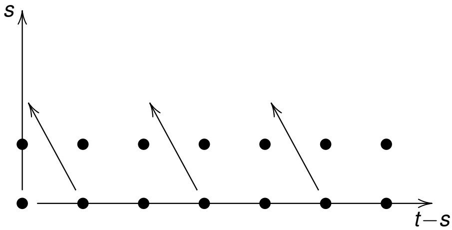

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Back to *TMF*

Recall that there is a sheaf of E_∞-ring spectra $\mathcal{O}^{top}$ on the moduli stack $\mathcal{M}$ of elliptic curves. We define $TMF = \mathcal{O}^{top}(\mathcal{M})$.

From now on, we will invert everywhere 2. There is then an etale cover

$$\begin{array}{c} \mathcal{M}(2) \\ \downarrow \\ \mathcal{M} \end{array}$$

Define $TMF(2) = \mathcal{O}^{top}(\mathcal{M}(2))$. We have

$$TMF(2)_* \cong \mathbb{Z}[x_2, y_2, \Delta^{-1}],$$

hence gldim $TMF(2)_* = 2$.

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Relatively free *TMF*-modules

Goal: We need to classify (finite) *TMF*-modules M such that $M \wedge_{TMF} TMF(2)$ is $TMF(2)$-free, called again *relatively free modules*.

Let M be a (finite) *TMF*-module. Then we can associate to it an $\mathcal{O}^{top}$-module $\mathcal{F}_M$ by

$$U \longrightarrow \mathcal{M} \text{ etale } \mapsto \mathcal{F}_M(U) = \mathcal{O}^{top}(U) \wedge_{TMF} M.$$

If M is relatively free, $\pi_*\mathcal{F}_M$ is a vector bundle on $\mathcal{M}$.

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Vector Bundles

Theorem (Mumford, Fulton-Olsson)

Every line bundle on $\mathcal{M}$ is a power of ω and $\omega^{12} \cong \mathcal{O}$.

Theorem (Extension Theorem)

Every vector bundle E allows a short exact sequence

$$0 \longrightarrow \omega^k \longrightarrow E \longrightarrow E' \longrightarrow 0,$$

where $\operatorname{rk} E' = \operatorname{rk} E - 1$.

Theorem (Classification Theorem)

There are up to tensoring with powers of ω only 3 indecomposable vector bundles: $\mathcal{O}$, E_α and $E_{\alpha,\widetilde{\alpha}}$.
These are of ranks 1, 2 and 3 respectively and $H^i(\mathcal{M}; E_{\alpha,\widetilde{\alpha}}) = 0$ for $i > 0$.

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Standard Modules

If M is a relatively free module and $a \in \pi_k M$ a torsion class, we can form a cofiber sequence

$$\Sigma^k TMF \xrightarrow{a} M \longrightarrow \mathrm{Cone}(a) \longrightarrow \Sigma^{k+1} TMF.$$

Since $TMF(2)_*$ is torsionfree, the sequence splits after $\wedge_{TMF} TMF(2)$. Therefore, $\mathrm{Cone}(a)$ is again a relatively free module. Modules built up from TMF in this way, are called standard modules.

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Every relatively free module is standard.

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$$K_0(TMF) \cong \mathbb{Z}$$

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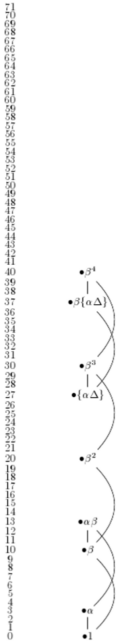
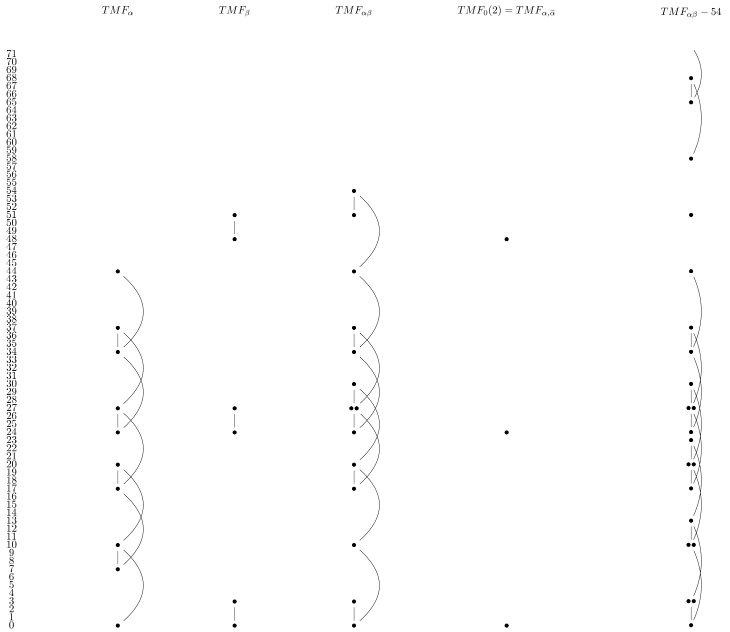


$$TMF_* \subset \mathbb{Z}[\tfrac{1}{2}][c_4, c_6, \Delta^{\pm 1}][\alpha, \beta]$$

For the picture we divide out:
3, c_4 and c_6


TMF_α
TMF_β
$TMF_{\alpha\beta}$
$TMF_0(2) = TMF_{\alpha,\tilde{\alpha}}$
$TMF_{\alpha\beta} - 54$
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