MASTER THESIS

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The p-local Splitting of the Infinite Projective Space and Co-H-structures

by

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Abstract

The present work describes, for all odd prime p, a p-local splitting of the infinite complex projective space in a wedge of $p-1$ spaces $\Sigma\mathbb{C}P^{\infty}_{(p)} \simeq K_1 \vee \cdots \vee K_{p-1}$. Each of these spaces can be endowed with a co-H-space structure inherited from the one on $\Sigma\mathbb{C}P^{\infty}$, but they are not particularly nice though, except for the case $j = p-1$ since K_{p-1} has the homotopy type of a suspension. The aim is, more precisely, to study possible co-H-stuctures on the other spaces K_1 to K_{p-2}. It shall be shown that none of them have the homotopy type of a suspension but furthermore that they can't even possess coassociative coproducts.

Übersicht

Diese Arbeit stellt eine p-lokale Zerlegung des unendlichen komplexen projektiven Raumes in ein Bouquet von $p-1$ topologischen Räumen $\Sigma\mathbb{C}P^{\infty}_{(p)} \simeq K_1 \vee \cdots \vee K_{p-1}$ vor. Auf jedem dieser Räume K_j kann durch die Co-H-Struktur auf der Einhängung $\Sigma\mathbb{C}P^{\infty}$ eine Co-H-Struktur definiert werden. Abgesehen von K_{p-1}, der den Homotopietyp einer Einhängung hat, haben diese Räume keine besonders schöne Struktur. Das Hauptziel dieser Arbeit ist das Studium möglicher Co-H-Strukturen auf den topologischen Räumen K_1 bis K_{p-2}. Wir werden erst zeigen, dass sie nicht den Homotopietyp einer Einhängung haben und dann, dass sie kein co-assoziatives Co-Produkt haben können.

Résumé

Où l'on présente, pour tout premier impair p, une décomposition p-locale de l'espace projectif complexe infini en un wedge de $p-1$ espaces topologiques $\Sigma\mathbb{C}P^{\infty}_{(p)} \simeq K_1 \vee \cdots \vee K_{p-1}$. Chacun de ces espaces peut être muni d'une structure de co-H-espace héritée de celle de $\Sigma\mathbb{C}P^{\infty}$. Cependant ces dernières ne sont pas particulièrement remarquables, à l'exception du cas $j = p-1$, puisque K_{p-1} a le type d'homotopie d'une suspension. Le but principal de ce travail est d'étudier d'éventuelles co-H-structures sur les $p-2$ autres espaces K_1 à K_{p-2}. En particulier, il y est démontré que, non seulement, aucun d'entre eux n'ont le type d'homotopie d'une suspension, mais, de mal en pis, qu'ils ne possèdent aucun co-produit co-associatif.

To Pinky and The Brain,
Whom I share the plans of!

Brain: "Are you pondering what I'm pondering?"

Pinky: " I think so, Brain. But if I put on two tutu's,
would I really be wearing a four-by-four?"

Brain: "Why do I even bother asking?"

Pinky: "I dunno, Brain. Maybe it's all part
of some huge, cosmic plot formula!"

Contents

Notation

$\mathbb{C}P^\infty$	The infinite complexe projective space
cat	Category
$\mathbf{Cat}$	Strong category
$\mathrm{cat}^{\mathrm{Wh}}$	Whitehead category
$\mathrm{cat}^{\mathrm{G}}$	Ganea category
$E_n^{s,t}$	n-th term of a spectral sequence of cohomological type
$\mathbb{F}_p$	Finite field with p elements
G_n	n-th Ganea space
$H_*(X;G)$	Homology of the space X with coefficients in the group G
$\widetilde{H}_*(X;G)$	Reduced homology of the space X with coefficients in the group G
$H^*(X;G)$	Cohomology of the space X with coefficients in the group G
I	Closed unit interval $[0,1]$
$\mathbb{K}$	Commutative ring
$\mathbb{N}$	Naturel numbers, 0 included
$\mathbb{N}_n$	$\{1,2,3,\ldots,n\}$
$\mathcal{P}^i$	i-th Steenrod reduced power
PX	Pathspace of the space X
$P(A)$	Primitive elements of the Hopf algebra A
$\mathbb{Q}$	Rational numbers
$Q(A)$	Indecomposable elements of the Hopf algebra A
$\mathbb{S}^n$	n-phere
$S^{-1}M$	Localization of the module M at the multiplicative set S
$T^m(X)$	m-fold fat wadge of the space X
$\mathcal{T}op$	Category of topological spaces
$\mathcal{T}op_*$	Categoryof pointed topological spaces
$X_{\mathcal{P}}$	$\mathcal{P}$-localization of the space X
V^*	Dual of the module V
$\mathbb{Z}$	Integers
$\mathbb{Z}_{\mathcal{P}}$	$\mathcal{P}$-local integers
$\mathbb{Z}_{(p)}$	p-local integers
ΣX	Reduced suspension of pointed space X
ΩX	Loop space of a pointed space X
$[X,Y]$	Set of homotopy classes of continuous maps from X into Y
f_*	Induced homomorphism in homology
f^*	Induced homomorphism in cohomology
$*$	Base point of a topolocical space
$\oplus$	Direct sum
$\otimes$	Tensor product
$\vee$	Wedge sum
$\wedge$	Smash product
$\cong$	Isomorphism, homeomorphism
$\simeq$	Homotopy
$\hookrightarrow$	Injective morphism
$\twoheadrightarrow$	Surjective morphism
$\forall$	Universel symbol “for all”
$\exists$	Universel symbol “there exists”
$\exists!$	Universel symbol “there exists a unique”

Introduction

Let p be an odd prime. In [McG81, Proposition 2.2] C.A. McGibbon describes a p-local splitting of the infinite complex projective space

$$\Sigma\mathbb{C}P^{\infty}_{(p)} \simeq \bigvee_{j=1}^{p-1} K_j$$

such that the homology of each piece is given by

$$\widetilde{H}_q(K_j, \mathbb{Z}) = \begin{cases} \mathbb{Z}_{(p)} & \text{if } q = 2n+1 \text{ and } n(\geq 1) \equiv j \pmod{p-1} \\ 0 & \text{otherwise.} \end{cases}$$

Using the canonical inclusions and retractions $K_j \hookrightarrow \Sigma\mathbb{C}P^{\infty}_{(p)}$ and $\Sigma\mathbb{C}P^{\infty}_{(p)} \longrightarrow K_j$, each space K_j can easily be endowed with a coproduct, which does not necessarily have nice properties though. It follows from work of D. P. Sullivan [Sul74] that the space K_{p-1} has the homotopy type of a suspension. We wish to investigate possible co-H-structures on the other spaces K_1 to K_{p-2}. The first result which gives a hint that these spaces do not bear nice co-H-stuctrures is the following.

Theoreme (4.4).
For all $j \in \mathbb{N}_{p-2}$ the spaces K_j do not have the homotopy type of a suspension.

But our principal aim is to show that coassociative co-H-structures can't exist on these spaces:

Theoreme (4.8).
Let $j \in \mathbb{N}_{p-2}$, then the space K_j does not possess any coassociative coproduct.

We shall also figure out that the difference between K_{p-1} and the other spaces is entirely related to the degrees in which homology is concentrated.

The proofs of these results shall in fact need many algebraic and topological tools. We start in chapter 1 with an introduction to co-H-spaces and present, in chapter 2, the basics of topological localizations , both of which are necessary to acquire a good understanding of the proof of the p-local splitting and the construction of the spaces $K_1, \ldots, K_{p-1}$. Chapter 3 introduces mathematical tools that we shall need in the proofs of chapter 4. The treated subjects are Hopf algebras and their duals, the elemantary properties of the Steenrod reduced powers, the Serre spectral sequences and the cohomology suspension morphism. Of course, this document is far from being self-contained and requires previous knowledge

in algebraic topology and algebra. Nonetheless, I hope I have managed to explain the main necessary steps to carry out the proofs of the theorems cited above. This work has taught me how difficult it is to work with concrete examples and how much more advanced material can be needed to treat a problem which actually does not require much knowledge in algebraic topology to be understood.

I also ought to apologize to native speakers for the style in which this text is written. English is a beautiful language which certainly does not deserve to be slaughtered by French speakers. Finally, in order to recover the atmosphere in which this work was carried through, I would advise the reader either to paint his office walls in green or more simply to print it on green paper.

Acknowledgments

I wish to thank Prof. Birgit Richter for the nice and interesting subject she has found for me to study as well as for welcoming me at Hamburg University, Fridolin Roth for all his explanations on LS-category and the Swiss Federal Institute of Technology of Lausanne for its financial support to academic exchange students.

CHAPTER 1

Introduction to Co-H-spaces

To start with we give a detailed introduction to the notion of co-H-space structure on a pointed space. This is the central notion we wish to investigate through the example of the splitting of $\Sigma\mathbb{C}P^{\infty}_{(p)}$. This chapter is based on the books by G.W. Whitehaed and Switzer ([Whi78] and [Swi02]).

1. Notation

To begin with we set up the various standard notations which shall be used throughout this work. Let $X, Y \in \mathcal{T}op_*$ be pointed topological spaces. When no confusion is to be made, the base point of a pointed space will simply be denoted by the symbol $*$. The wedge $X \vee Y$ is considered as embedded into the product $X \times Y$. The arrow

$$\begin{array}{rccc} *\colon & X & \longrightarrow & Y \\ & x & \longmapsto & * \end{array}$$

will denote the constant map (on the based point);

$$\begin{array}{rccc} j_1\colon & X & \longrightarrow & X \vee Y \\ & x & \longmapsto & (x,*) \end{array} \qquad \begin{array}{rccc} j_2\colon & Y & \longrightarrow & X \vee Y \\ & y & \longmapsto & (*,y) \end{array}$$

the inclusions in the wedge;

$$\begin{array}{rccc} i_1\colon & X & \longrightarrow & X \times Y \\ & x & \longmapsto & (x,*) \end{array} \qquad \begin{array}{rccc} i_2\colon & Y & \longrightarrow & X \times Y \\ & y & \longmapsto & (*,y) \end{array}$$

the inclusions in the cartesian product;

$$\begin{array}{rccc} q_1\colon & X \vee Y & \longrightarrow & X \\ & (x,y) & \longmapsto & x \end{array} \qquad \begin{array}{rccc} q_2\colon & X \vee Y & \longrightarrow & Y \\ & (x,y) & \longmapsto & y \end{array}$$

the projections from the wedge on each factor;

$$\begin{array}{rccc} p_1\colon & X \times Y & \longrightarrow & X \\ & (x,y) & \longmapsto & x \end{array} \qquad \begin{array}{rccc} p_2\colon & X \times Y & \longrightarrow & Y \\ & (x,y) & \longmapsto & y \end{array}$$

the projections from the cartesian product on each factor;

$$\begin{array}{rccc} k\colon & X \vee Y & \longrightarrow & X \times Y \end{array}$$

the canonical inclusion of the wedge in the cartesian product;

$$\begin{array}{rccc} \Delta\colon & X & \longrightarrow & X \times X \\ & x & \longmapsto & (x,x) \end{array} \qquad \begin{array}{rccc} \nabla\colon & X \vee X & \longrightarrow & X \\ & (x,*) & \longmapsto & x \\ & (*,x) & \longmapsto & x \end{array}$$

the diagonal and the folding maps, which can be iterated into n-fold versions.

Remark 1.1.
There are many relations between the maps above, some of which shall be indispensable in the next sections. By way of illustration, here is a non-comprehensive list :

(1) $p_1 i_1 = \mathrm{Id}_X$ and $p_2 i_2 = \mathrm{Id}_Y$;
(2) $p_1 i_2 = * = p_2 i_1$;
(3) $q_1 j_1 = \mathrm{Id}_X$ and $q_2 j_2 = \mathrm{Id}_Y$;
(4) $q_1 j_2 = * = q_2 j_1$;
(5) $p_1 \Delta = \mathrm{id}_X = p_2 \Delta$;
(6) $\nabla j_1 = \mathrm{Id}_X = \nabla j_2$;
(7) $p_1 k = q_1$ and $p_2 k = q_2$;
(8) $\nabla(j_1 \vee j_2) = \mathrm{id}_{X \vee X}$.

The proofs of these formulae are straight forward calculations.

2. Equivalent Definitions

Definition 1.2.
A pointed topological space $(X, *)$ is a *co-H-space*[1] if for all $Y \in \mathcal{T}op_*$ the set $[X, Y]$ is endowed with a product $+$ such that

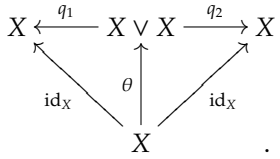
(1) the identity element is the class of the constant map $[*]$.
(2) For all $Y_1, Y_2 \in \mathcal{T}op_*$ and for all $f \in \mathcal{T}op_*(Y_1, Y_2)$ the induced map

$$\begin{array}{rccc} f_* : & [X, Y_1] & \longrightarrow & [X, Y_2] \\ & [g] & \longmapsto & [f \circ g] \end{array}$$

is a homomorphism, that is for all $[g_1], [g_2] \in [X, Y_1]$ we have

$$f_*([g_1] + [g_2]) = f_*([g_1]) + f_*([g_2]) .$$

Definition 1.3.
Let $X \in \mathcal{T}op_*$. A *coproduct* (or *comultiplication*) on X is a pointed continuous map $\theta : X \longrightarrow X \vee X$ such that the following diagram commutes up to homotopy



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Proposition 1.4.
A space $X \in \mathcal{T}op_*$ is a co-H-space if and only if there exists a coproduct θ on X. Moreover, in this case the product $+$ on $[X, Y]$ is defined for all $Y \in \mathcal{T}op_*$ and all $f_1, f_2 \in [X, Y]$ by

$$[f_1] + [f_2] = [\nabla(f_1 \vee f_2)\theta] .$$

[1] H stands for Heinz Hopf.

Proof. First assume that X is a co-H-space. Up to homotopy we can define a pointed continuous map from X to $X \vee X$ by the formula

$$[\theta] := [j_1] + [j_2]\,.$$

It is then an easy calculation to check that any element of this class is a coproduct:

$$\begin{aligned}[q_1\theta] = (q_1)_*([\theta]) = (q_1)_*([j_1] + [j_2]) = (q_1)_*([j_1]) + (q_1)_*([j_2])\\ = [q_1 j_1] + [q_1 j_2] = [q_1 j_1] + [q_1 j_2] = [\mathrm{Id}_X] + [*] = [\mathrm{Id}_X]\end{aligned}$$

And similarly $[q_2\theta] = [\mathrm{Id}_X]$. Thus X has a co-product. In addition, for all $Y \in \mathcal{T}op_*$ and all $f_1, f_2 \in [X, Y]$ we have

$$\begin{aligned}[\nabla(f_1 \vee f_2)\theta] &= (\nabla(f_1 \vee f_2))_*([\theta]) = (\nabla(f_1 \vee f_2))_*([j_1] + [j_2])\\ &= (\nabla(f_1 \vee f_2))_*([j_1]) + (\nabla(f_1 \vee f_2))_*([j_2])\\ &= \underbrace{[\nabla(f_1 \vee f_2)j_1]}_{f_1} + \underbrace{[\nabla(f_1 \vee f_2)j_2]}_{f_2} = [f_1] + [f_2]\,.\end{aligned}$$

Conversely, assume that X is endowed with a coproduct θ. Then for all $Y \in \mathcal{T}op_*$ and all $f_1, f_2 \in [X, Y]$ we can define a product " + " on $[X, Y]$ by the formula $[f_1] + [f_2] = [\nabla(f_1 \vee f_2)\theta]$.
Since the composition $(j_1 \vee j_2)\nabla$ is the identity on $X \vee X$:

$$\begin{aligned}X \vee X &\xrightarrow{j_1 \vee j_2} (X \vee X) \vee (X \vee X) \xrightarrow{\nabla} X \vee X\\ (x, *) &\longmapsto ((x, *); (*, *)) \longmapsto (x, *)\\ (*, x) &\longmapsto ((*, *); (*, x)) \longmapsto (*, x)\end{aligned}$$

we have in particular $[j_1] + [j_2] = [\nabla(j_1 \vee j_2)\theta] = [\theta]$.
Moreover, the hypothesis $q_1\theta \simeq \mathrm{Id}_X \simeq q_2\theta$ implies that for all $f \in [X, Y]$ the two following diagrams

$$\begin{array}{ccccc} X & \xrightarrow{\theta} & X \vee X & \xrightarrow{f \vee *} & Y \vee Y \\ & \searrow^{\mathrm{Id}_X} & \downarrow{\scriptstyle q_1} & \circlearrowleft & \downarrow{\scriptstyle \nabla} \\ & & X & \xrightarrow[f]{} & Y \end{array} \qquad \begin{array}{ccccc} X & \xrightarrow{\theta} & X \vee X & \xrightarrow{* \vee f} & Y \vee Y \\ & \searrow^{\mathrm{Id}_X} & \downarrow{\scriptstyle q_2} & \circlearrowleft & \downarrow{\scriptstyle \nabla} \\ & & X & \xrightarrow[f]{} & Y \end{array}$$

homotopy commute. (The commutativity of the two squares is quite obvious.) In other words:

$$[f] + [*] = [\nabla(f \vee *)\theta] = [f\,\mathrm{Id}_X] = [f]$$

and

$$[*] + [f] = [\nabla(* \vee f)\theta] = [f\,\mathrm{Id}_X] = [f]\,.$$

Hence $[*]$ is an identity element for $+$. Finally, given $f \in \mathcal{T}op_*(Y_1, Y_2)$, the fact that the induced map is an homomorphism follows from the easily-checked commutativity of the diagram

$$\begin{array}{ccc} Y_1 \vee Y_1 & \xrightarrow{f \vee f} & Y_2 \vee Y_2 \\ \downarrow{\scriptstyle \nabla} & \circlearrowleft & \downarrow{\scriptstyle \nabla} \\ Y_1 & \xrightarrow[f]{} & Y_2\,. \end{array}$$

For then for all $[g_1], [g_2] \in [X, Y_1]$ we have

$$\begin{aligned} f_*([g_1] + [g_2]) &= f_*([\nabla(g_1 \vee g_2)\theta] = [f\nabla(g_1 \vee g_2)\theta] \\ &= [\nabla(f \vee f)(g_1 \vee g_2)\theta] = [\nabla(fg_1 \vee fg_2)\theta] \\ &= [fg_1] + [fg_2] = f_*([g_1]) + f_*([g_2])\,. \end{aligned}$$

□

There still is another characterization for co-H-spaces which can be useful to bear in mind.

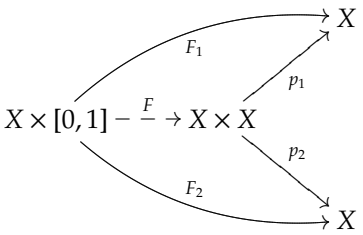
Proposition 1.5.
Let $X \in \mathcal{T}op_*$ be a pointed topological space with a pointed continuous map $\theta : X \longrightarrow X \vee X$. Then θ is a coproduct if and only if the diagonal map $\Delta : X \longrightarrow X \times X$ factors through the wedge via θ, i.e. the following diagram commutes up to homotopy

$$\begin{array}{ccc} X & \xrightarrow{\theta} & X \vee X \\ & \searrow^{\Delta} & \downarrow{\scriptstyle k} \\ & & X \times X\,. \end{array}$$

Proof. If θ is a coproduct, then $q_1\theta \simeq \mathrm{Id}_X \simeq q_2\theta$ and in other words there exist two homotopies $F_1, F_2 : X \times [0,1] \longrightarrow X$ such that, respectively,

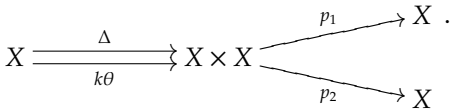
$$\begin{cases} F_1(x,0) = q_1\theta(x) = p_1k\theta(x) \\ F_1(x,1) = \mathrm{Id}_X(x) = p_1\Delta(x) \end{cases} \quad \text{and} \quad \begin{cases} F_2(x,0) = q_2\theta(x) = p_2k\theta(x) \\ F_2(x,1) = \mathrm{Id}_X(x) = p_2\Delta(x) \end{cases}$$

Applying the universal property of the product, we get the existence of a continuous map $F : X \times [0,1] \longrightarrow X \times X$ such that $p_1F = F_1$ and $p_2F = F_2$:



Therefore F is a homotopy between $k\theta$ and Δ.

Conversely, assuming that $\Delta \simeq k\theta$, it suffices to use the canonical projections on each factor of the cartesian product $X \times X$ to obtain the required result.



For, remembering that $p_1k = q_1$, $p_2k = q_2$ and $p_1\Delta = \mathrm{Id}_X = p_2\Delta$ we easily get

$$q_1\theta = p_1 \underbrace{k\theta}_{\simeq\Delta} \simeq p_1\Delta = \mathrm{Id}_X$$

and

$$q_2\theta = p_2 k\theta \simeq p_2\Delta = \mathrm{Id}_X \ .$$

Thus θ fulfills definition 1.3. □

To sum up, a co-H-space can be thought of either in terms of definition 1.2, or equivalently as a space with a comultiplication θ satisfying the condition $q_1\theta \simeq \mathrm{Id}_X \simeq q_2\theta$ of definition 1.3, or the condition on the diagonal map of proposition 1.5. Definition 1.2 gives a straight forward idea of the kind of topological properties aimed at with the notion of co-H-space and will also quite nicely explain the terminology of coassociativity and co-H-group introduced in the next section. Nonetheless, thinking of a co-H-space as a space with a comultiplication is slightly more interesting in the sense that this definition easily generalizes to a categorical definition. To understand it better, let us recall that in the category $\mathcal{T}op_*$, the wedge sum corresponds to the categorical coproduct.

Definition 1.6.
Let C be a category with zero morphisms and coproducts. A *comultiplication* on an object $A \in C$ is a morphism $\Phi : A \longrightarrow A*A$ such that $< \mathrm{Id}_A, 0 > \Phi = \mathrm{Id}_A =< 0, \mathrm{Id}_A > \Phi$, where $*$ denotes the coproduct, 0 the zero morphism from A to A, $< \mathrm{Id}_A, 0 >$ and $< 0, \mathrm{Id}_A >$ the unique morphims obtained by applying the universal property of the coproduct to the pairs of morphisms $(\mathrm{Id}_A, 0)$ and $(0, \mathrm{Id}_A)$, respectively.
A pair (A, Φ) is called a *comultiplicative object* in C and Φ is called *coassociative* if $(\mathrm{Id}_A * \Phi) * \Phi = (\Phi * \mathrm{Id}_A) * \Phi : A \longrightarrow A * A * A$.

3. Further Elementary Properties of Co-H-spaces

If X is a co-H-space, the set $[X, Y]$ is endowed with a product for all $Y \in \mathcal{T}op_*$, but it need not be a group, neither is the product necessarily associative. Let us investigate these two properties in terms of coproducts.

Associativity.
The product + is associative when for all $f_1, f_2, f_3 \in \mathcal{T}op_*(X, Y)$ we have $[f_1] + ([f_2] + [f_3]) = [f_1] + ([f_2] + [f_3])$, where the first member of the equation is represented by the composition

$$X \xrightarrow{\theta} X \vee X \xrightarrow{\theta \vee \mathrm{Id}_X} X \vee X \vee X \xrightarrow{f_1 \vee f_2 \vee f_3} Y \vee Y \vee Y \xrightarrow{\nabla} Y$$

and the second member by the composition

$$X \xrightarrow{\theta} X \vee X \xrightarrow{\mathrm{Id}_X \vee \theta} X \vee X \vee X \xrightarrow{f_1 \vee f_2 \vee f_3} Y \vee Y \vee Y \xrightarrow{\nabla} Y \, .$$

(Where ∇ obviously denotes the 3-fold folding map.) Hence + is associative if $(\theta \vee \mathrm{Id}_X)\theta \simeq (\mathrm{Id}_X \vee \theta)\theta$.

Conversely, if + is associative, the particular case $Y := X \vee X \vee X$, $f_1 = j_1$, $f_2 = j_2$ and $f_3 = j_3$ yields $\nabla(f_1 \vee f_2 \vee f_3) = \nabla(j_1 \vee j_2 \vee j_3) = \mathrm{Id}_{X\vee X\vee X}$ and

$$\begin{aligned}[(\theta \vee \mathrm{Id}_X)\theta] &= [\mathrm{id}_{X\vee X\vee X}(\theta \vee \mathrm{Id}_X)\theta] = [\nabla(j_1 \vee j_2 \vee j_3)(\theta \vee \mathrm{Id}_X)\theta] \\ &= ([j_1] + [j_2]) + [j_3] \\ &= [j_1] + ([j_2] + [j_3]) \\ &= [\nabla(j_1 \vee j_2 \vee j_3)(\mathrm{Id}_X \vee\theta)\theta] = [\mathrm{id}_{X\vee X\vee X}(\mathrm{Id}_X \vee\theta)\theta] \\ &= [(\mathrm{Id}_X \vee\theta)\theta]\,.\end{aligned}$$

This leads to the so-called notion of coassociativity for a coproduct.

Definition 1.7.
A co-H-space (X, θ) is called *coassociative* if its coproduct is *coassociative* that is if the diagram

$$\begin{array}{ccc} X & \xrightarrow{\theta} & X \vee X \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta\vee\mathrm{Id}_X} \\ X \vee X & \xrightarrow[\mathrm{Id}_X \vee\theta]{} & X \vee X \vee X\,. \end{array}$$

commutes up to homotopy.

Remark 1.8.
Coassociativity is no automatic property for co-H-spaces. We shall show that the spaces $K_1, \ldots K_{p-2}$ built in theorem 4.1 provide an example of infinitely many spaces which are co-H-spaces but fail to be coassociative.

Inverses.
We now describe a necessary and sufficient condition for inverses to exist.

If every $[f] \in [X, Y]$ has an inverse, so does, in particular, $[\mathrm{Id}_X] \in [X, X]$: $[\mathrm{Id}_X]+[\nu] = [*] = [\nu] + [\mathrm{Id}_X]$. Conversely, if $[\nu]$ exists, we easily see that any class $[f] \in [X, Y]$ admits $[f\nu]$ as inverse. For the class of the constant map $[*] = [\nu] + [\mathrm{Id}_X]$ is represented by

$$X \xrightarrow{\theta} X \vee X \xrightarrow{\nu\vee\mathrm{Id}_X} X \vee X \xrightarrow{\nabla} X\,;$$

$[f\nu] + [f]$ is represented by

$$X \xrightarrow{\theta} X \vee X \xrightarrow{f\nu\vee f} Y \vee Y \xrightarrow{\nabla} Y$$

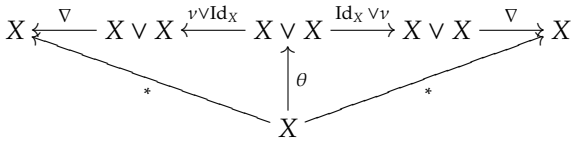
and the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\theta} & X \vee X & \xrightarrow{f\nu\vee f} & Y \vee Y & \xrightarrow{\nabla} & Y \\ & & & \searrow^{\nu\vee\mathrm{Id}_X} \circlearrowleft & \uparrow{\scriptstyle f\vee f} \circlearrowleft & & \uparrow{\scriptstyle f} \\ & & & & X \vee X & \xrightarrow[\nabla]{} & X\,. \end{array}$$

commutes. Thus $[fv] + [f] = [f*] = [*]$. Similarly, we get $[f] + [fv] = [*]$. It leads to the following definitions.

Definition 1.9.
Let (X, θ) be a co-H-space.

(1) A *homotopy inverse* for X is a pointed continuous map $v : X \longrightarrow X$ such that the diagram



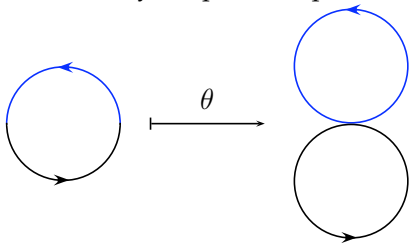
commutes up to homotopy.

(2) A co-H-space (X, θ) with homotopy inverse v and which is coassociative is called *co-H-group*.

Note that it follows from the above development that this definition of a co-H-group is equivalent to saying that the set $[X, Y]$ is endowed with a group structure for all $Y \in \mathcal{T}op_*$.

The first example that springs to mind is the honnest 1-sphere $\mathbb{S}^1$, which is a co-H-group as $[\mathbb{S}^1, Y] = \pi_1(Y)$ is the fundamental group of Y for all pointed space Y. Let us denote by $\bar{t}$ the image of $t \in [0,1] =: I$ under the canonical map $I \longrightarrow I/0 \sim 1 \cong \mathbb{S}^1$. Then the coproduct on $\mathbb{S}^1$ is described by the pinch map:

$$\theta \; : \; \bar{t} \longmapsto \begin{cases} (\overline{2t}, *) & \text{if } 0 \leq t \leq \frac{1}{2} \\ (*, \overline{2t-1}) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$



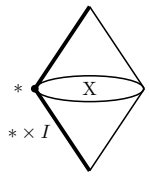
4. Suspensions

Let $(X, *)$ be a pointed topological space. We recall that the *reduced suspension* ΣX of X is often defined to be the smash product $\Sigma X := \mathbb{S}^1 \wedge X$. Since $(I; \{0,1\})$ is an NDR-pair it can be showed that there is a natural homeomorphism

$$\Sigma X \cong X \times I / X \times \{0\} \cup X \times \{1\} \cup \{*\} \times I$$

where the base point is then $\{*\} \times I$. (See [Whi78, Chapter III].) Hence the reduced suspension can also be thought of as the union of the upper and lower cones on X, described, as subset of the above quotient, by $C_-X = \{[x,t] \,|\, x \in X,\, 0 \leq t \leq \frac{1}{2}\}$ and $C_+X = \{[x,t] \,|\, x \in X,\, \frac{1}{2} \leq t \leq 1\}$. Furthermore, $C_-X \cap C_+X$ is homeomorphic to X

and we often find the following representation:



The operation of forming the reduced suspension determines a functor Σ, whose effect on morphisms is defined for all $f \in \mathcal{T}op_*(X, Y)$ by

$$\begin{array}{rccl} \Sigma f := \mathrm{Id} \wedge f\colon & \Sigma X & \longrightarrow & \Sigma Y \\ & \bar{t} \wedge x & \longmapsto & \Sigma f(\bar{t} \wedge x) := \bar{t} \wedge f(x)\ . \end{array}$$

By means of the following lemma we easily show that reduced suspensions are co-H-spaces and even better, co-H-groups. See [Whi78] for details of the proof.

LEMMA 1.10.
Let (X, θ) be a co-H-space.

(1) *Then so is $X \wedge Y$ for all $Y \in \mathcal{T}op_*$.*
(2) *If X is coassociative, then so is $X \wedge Y$ for all $Y \in \mathcal{T}op_*$.*
(3) *If X is a co-H-group, then so is $X \wedge Y$ for all $Y \in \mathcal{T}op_*$.*

SKETCH OF PROOF. Let $F(X, Y)$ denotes the set of all pointed continuous maps from X to Y with the compact-open topology. The idea of the proof is to consider the natural homeomorphism $F(X \wedge Y, Z) \cong F(X, F(Y, Z))$ which induces a bijection $[X \wedge Y, Z] \approx [X, F(Y; Z)]$. Therefore we can use this bijection to transfer the product on $[X, F(Y; Z)]$, given by the co-H-space structure of X, to one on $[X \wedge Y, Z]$, which is coassociative or gives a group structure if the first one does. □

REMARK 1.11.
In order to use the homeomorphism $F(X \wedge Y, Z) \cong F(X, F(Y, Z))$, more hypotheses on the spaces are necessary. For example, Whitehead assumes that they are compactly generated spaces[2], of which locally compact Hausdorff spaces are particular cases. Therefore, to be able to use the results of the lemma above, we shall also assume that the spaces considered in this section satisfy this supplementary condition. But we don't need to worry too much about it, because the space, which we are eventually interested in, $\mathbb{C}P^\infty$, is certainly compactly generated (as a pointed CW-complex).

COROLLARY 1.12.
Suspensions are co-H-groups.

PROOF. Since it has been seen in the previous section that $\mathbb{S}^1$ is a co-H-group, so is the smash product $\mathbb{S}^1 \wedge X = \Sigma X$. □

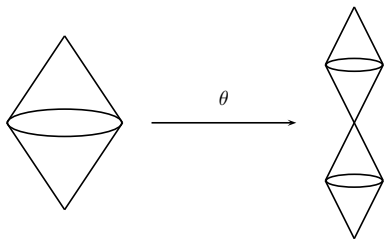
[2]Here a compactly generated space X is a Hausdorff space such that each subset A of X with the property that $A \cap C$ is closed for every compact subset C of X is itself closed.

By dissecting more deeply the bijection used in the proof of the lemma it can be seen that the coproduct on a suspension ΣX is induced from the one of $\mathbb{S}^1$ as follows:

$$\theta\colon \Sigma X = \mathbb{S}^1 \wedge X \longrightarrow (\mathbb{S}^1 \vee \mathbb{S}^1) \wedge X \xrightarrow{\cong} (\mathbb{S}^1 \wedge X) \vee (\mathbb{S}^1 \wedge X)$$

$$\bar{t} \wedge x \longmapsto \begin{cases} (\overline{2t}, *) \wedge x & \text{if } 0 \le t \le \frac{1}{2} \\ (*, \overline{2t-1}) \wedge x & \text{if } \frac{1}{2} \le t \le 1 \end{cases} \longmapsto \begin{cases} (\overline{2t} \wedge x, *) & \text{if } 0 \le t \le \frac{1}{2} \\ (*, \overline{2t-1} \wedge x) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

This map is often described as the "pinching" and represented by the naive picture:



Consequence 1.13.
In particular, it follows by induction on n that all the n-spheres are co-H-groups: $\mathbb{S}^n \cong \Sigma\mathbb{S}^{n-1}$ for all $n > 1$. As a matter of fact, given a pointed space Y, the co-H-group structure of $\mathbb{S}^n$ can be used to define the n-th homotopy group $\pi_n(Y) := [\mathbb{S}^n, Y]$ of Y for all $n \ge 1$.

5. Co-H-spaces and Homology

In this section, we assume that the spaces are such that their base point is closed and is a deformation retract of some neighbourhood in the space. CW-complexes, for instance, verify this condition. It enables the use of the isomorphism

$$(j_{\alpha *})_\alpha : \bigoplus_\alpha \widetilde{H}_q(X_\alpha) \xrightarrow{\cong} \widetilde{H}_q(\bigvee_\alpha X_\alpha)$$

induced by the canonical inclusions $j_\alpha : X_\alpha \longrightarrow \bigvee_\alpha X_\alpha$.

Proposition 1.14.
Let (X, θ) be a co-H-space satisfying the assumption above.

(1) *For all $x \in \widetilde{H}_q(X)$ the composite map*

$$\alpha : \widetilde{H}_q(X) \xrightarrow{\theta_*} \widetilde{H}_q(X \vee X) \xleftarrow[(j_{1*}, j_{2*})]{\cong} \widetilde{H}_q(X) \oplus \widetilde{H}_q(X)$$

is given by $x \mapsto (x, x)$ for all q.

(2) *If X has a homotopy inverse $\nu : X \longrightarrow X$, then the induced map in homology $\nu_* : \widetilde{H}_q(X) \longrightarrow \widetilde{H}_q(X)$ is given by $x \mapsto -x$ for all q.*

Proof.

(1) The relations $q_1\theta \simeq \mathrm{Id}_X \simeq q_2\theta$, $q_1 j_1 = \mathrm{Id}_X = q_2 j_2$ and $q_1 j_2 = * = q_2 j_1$ yield, when passing to homology, the commutative diagrams

$$\begin{array}{ccccc} \widetilde{H}_q(X) & \xrightarrow{\theta_*} & \widetilde{H}_q(X \vee X) & \xleftarrow{(j_{1\,*}, j_{2\,*})} & \widetilde{H}_q(X) \oplus \widetilde{H}_q(X) \\ & \searrow^{\mathrm{Id}} & \downarrow{\scriptstyle q_{1\,*}} & \swarrow_{(\mathrm{Id},0)} & \\ & & \widetilde{H}_q(X) & & \end{array}$$

$$\begin{array}{ccccc} \widetilde{H}_q(X) & \xrightarrow{\theta_*} & \widetilde{H}_q(X \vee X) & \xleftarrow{(j_{1\,*}, j_{2\,*})} & \widetilde{H}_q(X) \oplus \widetilde{H}_q(X) \\ & \searrow^{\mathrm{Id}} & \downarrow{\scriptstyle q_{2\,*}} & \swarrow_{(0,\mathrm{Id})} & \\ & & \widetilde{H}_q(X) & & \end{array} .$$

Therefore for all for all $x \in \widetilde{H}_q(X)$ and all q, the first diagram says that $\alpha(x) = (x, z)$ for some z and the second one implies that $\alpha(x) = (y, x)$ for some y. Hence $\alpha(x) = (x, x)$.

(2) Recall that, by definition of ν, the following diagram commutes up to homotopy:

$$\begin{array}{ccccc} X & \xleftarrow{\nabla} & X \vee X & \xleftarrow{\nu \vee \mathrm{Id}} & X \vee X \\ & \nwarrow_{*} & & & \uparrow{\scriptstyle \theta} \\ & & & & X \end{array}$$

Thus passing to homology we have $0 = (\nabla(\nu \vee \mathrm{Id})\theta)_*$ and the computation of remark (2) below yields, for all x, $0 = (\nabla(\nu \vee \mathrm{Id})\theta)_*(x) = \nu_*(x) + x$, whence $\nu_*(x) = -x$.

□

Consequence 1.15.
As a result, the expression of the induced map in homology by the coproduct is

$$\theta_*(x) = (j_{1\,*}, j_{2\,*})\alpha(x) = (j_{1\,*}, j_{2\,*})(x, x) = j_{1\,*}(x) + j_{2\,*}(x)\,.$$

Remarks 1.16.

(1) Since $\nabla j_1 = \mathrm{Id}_X = \nabla j_2$ the map $\widetilde{H}_q(X) \oplus \widetilde{H}_q(X) \xrightarrow{(j_{1\,*}, j_{2\,*})} \widetilde{H}_q(X \vee X) \xrightarrow{\nabla_*} \widetilde{H}_q(X)$ is given by $(x, y) \mapsto j_{1\,*}(x) + j_{2\,*}(y) \mapsto \nabla_* j_{1\,*}(x) + \nabla_* j_{2\,*}(x) = x + y$.

(2) Let $Y \in \mathcal{T}op_*$ and $f, g \in \mathcal{T}op_*(X, Y)$, then for $[f] + [g] \in [X, Y]$ there is a nice expression for the induced map in homology. For all $x \in \widetilde{H}_q(X)$:

$$\begin{aligned}([f]+[g])_*(x) &= (\nabla(f\vee g)\theta)_*(x) = \nabla_*(f\vee g)_*\theta_*(x)\\ &= \nabla_*(f\vee g)_*(j_{1\,*}, j_{2\,*})(j_{1\,*}, j_{2\,*})^{-1}\theta_*(x)\\ &= \nabla_*(f\vee g)_*(j_{1\,*}, j_{2\,*})\alpha(x) = \nabla_*(f\vee g)_*(j_{1\,*}, j_{2\,*})(x,x)\\ &= \nabla_*(j_{1\,*}, j_{2\,*})(f_*\oplus g_*)(x,x) = \nabla_*(j_{1\,*}, j_{2\,*})(f_*(x), g_*(x))\\ &= f_*(x)+g_*(x)\end{aligned}$$

where the last equality comes from remark (1) above and the only other non-obvious step is the equality $(f\vee g)_*(j_{1\,*}, j_{2\,*}) = (j_{1\,*}, j_{2\,*})(f_*\oplus g_*)$, which actually comes from the commutative diagrams:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle j_1}\downarrow & \circlearrowleft & \downarrow{\scriptstyle j_1} \\ X\vee X & \xrightarrow[f\vee g]{} & Y\vee Y \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{g} & Y \\ {\scriptstyle j_2}\downarrow & \circlearrowleft & \downarrow{\scriptstyle j_2} \\ X\vee X & \xrightarrow[f\vee g]{} & Y\vee Y \end{array}$$

For this yields

$$\begin{aligned}(f\vee g)_*(j_{1\,*}, j_{2\,*})(x,z) &= ((f\vee g)j_1)_*(x) + ((f\vee g)j_2)_*(z)\\ &= (j_1 f)_*(x) + (j_2 g)_*(z) = (j_{1\,*}, j_{2\,*})(f_*(x)+g_*(z))\\ &= (j_{1\,*}, j_{2\,*})(f_*\oplus g_*)(x,z)\end{aligned}$$

for all $x, z \in \widetilde{H}_q(X)$.

6. H-spaces

The dual notion to the notion of co-H-space is the notion of H-space.

An *H-space* is by definition a pointed topological space $(Y,*)$ such that for all $Y \in \mathcal{T}op_*$ the set $[X,Y]$ is endowed with a product "$\cdot$" such that the identity element is the class of the constant map $[*]$ and for all $X_1, X_2 \in \mathcal{T}op_*$ and for all $f \in \mathcal{T}op_*(X_1, X_2)$ the induced map f^* is a homomorphism.
Equivalently, Y is an H-space if and only if it possesses a pointed continuous map $\mu : Y\times Y \longrightarrow Y$, called *product*, such that $\mu i_1 \simeq \mathrm{Id}_Y \simeq \mu i_2$ or $\mu k \simeq \nabla$.

Given a co-H-space (X,θ) and an H-space (Y,μ), the set $[X,Y]$ possesses two algebraic structures:

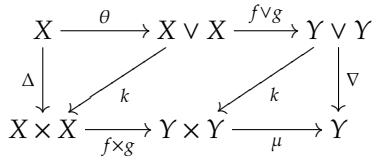
- $+$ given by the co-H-space structure;
- $\cdot$ given by the H-space structure.

They are linked by the following proposition.

Proposition 1.17.

(1) *The products $+$ and $\cdot$ are the same.*
(2) *This unique composition law is associative and commutative.*

Proof. The first claim follows from the commutativity up to homotopy of the diagram



for all $[f], [g] \in [X, Y]$. The second claim is a straight forward computation. See e.g. [Swi02]. □

Assuming again that we work with compactly generated spaces, there is a proposition similar to lemma 1.10 which links co-H-structures and H-structures:

Lemma 1.18.

(1) *If X is a co-H-space, then $F(X, Y)$ is an H-space for all Y.*
(2) *If X is a coassociative co-H-space, then $F(X, Y)$ is an associative H-space for all Y, that is the product $+$ on $[X, F(X, Y)]$ is associative.*
(3) *If X is a co-H-group, then $F(X, Y)$ is an H-group for all Y, that is $[X, F(X, Y)]$ is endowed with a group structure by $+$.*

Therefore taking X equal to the co-H-space $\mathbb{S}^1$, we obtain that the loop space $\Omega Y = F(\mathbb{S}^1, Y)$ is an H-group for all pointed spaces Y.

CHAPTER 2

Localizations

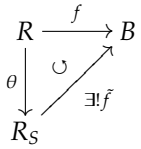
1. Localizations of Rings and Modules

In this section, unless otherwise stated, all rings are assumed to be commutative. Recall that for a ring $(R, +, \cdot)$, a subset $S \subset R$ is called *multiplicatively closed* if $(S, \cdot)$ is a monoid with identity element 1_R and such that 0_R does not belong to S. Most books dealing with an introduction to abstract algebra have a section on localization of rings and often of modules as well. Nevertheless, to my mind, the only recommendable book on the subject is [Bou61]. As usual with N. Bourbaki it is concise (and a bit old-fashioned) but comprehensive and formally done.

1.1. Localization of Rings.

Definition 2.1.
Let R be a ring and $S \subset R$ a multiplicatively closed subset. A *localization of the ring R at S* is a ring R_S together with a ring homomorphism $\theta : R \longrightarrow R_S$, which satisfy the following conditions:

(1) The elements of $\theta(S)$ are invertible in R_S.
(2) For all ring homomorphism $f : R \longrightarrow B$ such that $f(S) \subset B^*$, there exist a unique ring homomorphism $\tilde{f} : R_S \longrightarrow B$ such that $\tilde{f} \circ \theta = f$, that is such that the following diagram commutes:



As usual, a solution (R_S, θ) to such a universal mapping problem is unique up to a unique ring isomorphism.

The existence is established through the following construction. We first define an equivalence relation $\sim$ on the cartesian product $R \times S$ by

$$(r, s) \sim (\bar{r}, \bar{s}) \iff \exists\, t \in S \text{ such that } t(\bar{s}r - s\bar{r}) = 0\,.$$

Let $S^{-1}R := R \times S/\sim$. We denote by r/s the class of (r, s) in the quotient and we set $\theta(r) := r/1$ for all $r \in R$. Then, giving $S^{-1}R$ the ring structure (well-)defined by

$$\begin{array}{rccc} +: & S^{-1}R \times S^{-1}R & \longrightarrow & S^{-1}R \\ & (r/s, \bar{r}/\bar{s}) & \longmapsto & (\bar{s}r + s\bar{r})/s\bar{s} \end{array}$$

and

$$\begin{array}{rccc} \cdot: & S^{-1}R \times S^{-1}R & \longrightarrow & S^{-1}R \\ & (r/s, \bar{r}/\bar{s}) & \longmapsto & r\bar{r}/s\bar{s} \end{array}$$

(for which $0/1$ is the identity element with respect to $+$ and $1/1$ the unit) makes θ into a ring homomorphism. Furthermore, as $s/1 \cdot 1/s = 1/1$, we have $\theta(S) \subset (S^{-1}R)^*$.

Now, let $f : R \longrightarrow B$ be a ring homomorphism such that $f(S) \subset B^*$. If we want a ring homomorphism $\tilde{f} : S^{-1}R \longrightarrow B$ such that $\tilde{f} \circ \theta = f$, it is necessary that

$$1 = \tilde{f}(1/1) = \tilde{f}(s/1)\tilde{f}(1/s) = \tilde{f} \circ \theta(s)\tilde{f}(1/s) = f(s)\tilde{f}(1/s)\,.$$

But $f(s) \in B^*$ yields $\tilde{f}(1/s) = f(s)^{-1}$. Hence, $\tilde{f}$ is uniquely defined by:

$$\tilde{f}(r/s) = \tilde{f}(r/1) \cdot \tilde{f}(1/s) = f(r)f(s)^{-1} \cdot \quad \forall\, r \in R,\, s \in S\,.$$

Remarks 2.2.

(1) Any element in $S^{-1}R$ can be written as $r/s = \theta(r)\theta(s)^{-1}$.
(2) The kernel of θ is $\ker(\theta) = \{r \in R \mid \exists\, s \in S \text{ with } sr = 0\}$.
(3) Thus, if R is an integral domain θ is injective.
(4) If R is an integral domain and $S = R \setminus \{0\}$ then $S^{-1}R$ is $Q(R)$ the field of fractions of R. It follows that if $T \subseteq R \setminus \{0\}$ then $T^{-1}R$ can be identified with a subring of $Q(R)$.

1.2. Localization of Modules.

The basic idea of the localization of an R-module M at a multiplicatively closed subset S of R is to make multiplication by any element of S into an isomorphism. As described below, this is achieved, for instance, by tensoring M, over R, with $S^{-1}R$. As it happens, this very characterization is most of the time used as the definition of the localization of a module. It, nonetheless, shall be useful to copy what we have done above with localizations of rings and give a more formal definition using a universal property.

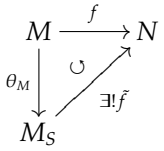
Definition 2.3.
Let R be a ring, $S \subset R$ a multiplicatively closed subset and M an R-module. A *localization of M at S* is an R-module M_S together with an R-homomorphism $\theta_M : M \longrightarrow M_S$, which satisfy the following conditions:

(1) For all $s \in S$ the multiplication by s

$$\begin{array}{rccc} h_s\colon & M_S & \longrightarrow & M_S \\ & z & \longmapsto & z \cdot s \end{array}$$

is bijective.

(2) For all R-module N such that the multiplication by s, $n \mapsto n \cdot s$, is bijective for all $s \in S$, and for all R-homomorphism $f : M \longrightarrow N$, there exist a unique R-homomorphism $\tilde{f} : M_S \longrightarrow N$ such that $\tilde{f} \circ \theta_M = f$, that is such that the following diagram commutes:



Again a solution (M_S, θ_M) to the universal mapping problem of the definition above, if ever it exists, is unique up to a unique R-isomorphism.
A concrete construction is made by setting $M_S = M \otimes_R S^{-1}R =: S^{-1}M$ and

$$\begin{array}{rccc} \theta_M\colon & M & \longrightarrow & M \otimes_R S^{-1}R \\ & m & \longmapsto & m \otimes 1\,. \end{array}$$

This construction obviously fulfills condition (1) of the definition. To see that condition (2) is satisfied as well, we observe that if an R-homomorphism $\tilde{f} : M_S \longrightarrow N$ is such that $\tilde{f} \circ \theta_M = f$, then for all $s \in S$ and $m \in M$ we have

$$f(m) = \tilde{f}(m \otimes 1) = \tilde{f}(((m \otimes 1) \cdot 1/s) \cdot s) = \tilde{f}((m \otimes 1) \cdot 1/s) \cdot s = h_s[\tilde{f}((m \otimes 1) \cdot 1/s\,.$$

But (1) says that h_s is invertible, therefore $\tilde{f}$ is uniquely defined by:

$$\tilde{f}(m \otimes 1/s) = (h_s)^{-1}[f(m)]$$

In particular, when N is an $S^{-1}R$-module, by virtue of remark (5) below:

$$\tilde{f}(m \otimes 1/s) = f(m) \cdot 1/s$$

Remarks 2.4.

(1) The construction of the localization as the tensor product $M \otimes_R S^{-1}R$ allows us to endow it with a natural $S^{-1}R$-module structure.
(2) Since the tensor product is R-balanced, any generator in $M \otimes_R S^{-1}R$ can be written as $m \otimes 1/s$ for some $m \in M$ and $s \in S$.
(3) Moreover, $m \otimes 1/s + \bar{m} \otimes 1/\bar{s} = (m\bar{s} + \bar{m}s) \otimes 1/(s\bar{s})$ implies that any element of $M \otimes_R S^{-1}R$ can be written as $m \otimes 1/s$ for some $m \in M$ and $s \in S$.
(4) In case M is an $S^{-1}R$-module, θ_M becomes an isomorphism. For the ordered pair (M, Id_M), where M is considered as an R-module, is a solution to the universal mapping problem of the definition, too. And θ_M is the unique isomorphism between M and $S^{-1}M$.
(5) In general, if P and Q are $S^{-1}R$-modules and $f : P \longrightarrow Q$ an R-homomorphism, then f can be extended to an $S^{-1}R$-homomorphism. For in this case for all $s \in S$ we have $(h_s)^{-1} = h_{1/s}$ and for all $p \in P$

$$f(p) = f((p \cdot 1/s) \cdot s) = f(n \cdot 1/s) \cdot s = h_s[f(n \cdot 1/s)]$$

implies

$$f(p \cdot 1/s) = f(p) \cdot 1/s\,.$$

(6) $\ker(\theta_M) = \{m \in M \,|\, \exists s \in S \text{ with } sm = 0\}$.
(7) Since the tensor product preserves direct sums, if M is a free R-module, then $S^{-1}M$ is a free $S^{-1}R$-module.

Proposition 2.5.
Let R and A be rings, $S \subseteq R$ and $T \subseteq A$ be multiplicatively closed subsets. Let $f : R \longrightarrow A$ be a ring homomorphism such that $f(S) \subseteq T$. Let M be an R-module, N be an A-module and $\varphi : M \longrightarrow N$ an R-linear mapping.
Then, there exists a unique $S^{-1}R$-linear mapping $\tilde{\varphi} : S^{-1}M \longrightarrow T^{-1}N$ s.t. $\tilde{\varphi}(m/1) = \varphi(m)/1$, i.e. such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ {\scriptstyle\theta_M}\downarrow & \circlearrowleft & \downarrow{\scriptstyle\theta_N} \\ S^{-1}M & \xrightarrow[\exists!\tilde{\varphi}]{} & T^{-1}N \end{array}$$

Proof. The map $\theta_N \circ \varphi$ is R-linear and for all $s \in S$ we have $\varphi(s) \in T$, thus the homothety produced by s in $T^{-1}N$ is bijective. Therefore, the existence and uniqueness of the map $\tilde{\varphi}$ follows from definition 2.3. In addition, using a similar argument as the one used in remark (5) above, we obtain $\tilde{\varphi}(m \otimes 1/s) = \varphi(m) \otimes 1/f(s)$. □

Remark 2.6.
We also notice that in case $R = A$, $S = T$ and $f = \mathrm{Id}_R$, then $\tilde{\varphi}$ is the map

$$\begin{array}{rccc} \varphi \otimes \mathrm{Id}: & M \otimes_R S^{-1}R & \longrightarrow & N \otimes_R S^{-1}R \\ & m \otimes 1/s & \longmapsto & \varphi(m) \otimes 1/s\ . \end{array}$$

It is called the *localization* of the homomorphism φ and allows to see the localization process as a functorial construction.

1.3. Localization of Abelian Groups.

The aim is for us to apply localization theory to homology groups, that is $\mathbb{Z}$-modules. To set the context, let $\mathcal{P} \subseteq \mathbb{P}$ be a set of primes and denote by $\mathbb{Z}_{\mathcal{P}}$ the localization of the ring $\mathbb{Z}$ at the multiplicative set generated by the primes *not* in $\mathcal{P}$. This is a subring of $\mathbb{Q}$ which can be described as

$$\{\frac{a}{b} \in \mathbb{Q} \,|\, a, b \in \mathbb{Z}, b \neq 0 \text{ and } p \nmid b \forall p \in \mathcal{P}\}\,.$$

We have the particular cases $\mathbb{Z}_{\mathbb{P}} = \mathbb{Z}$ and $\mathbb{Z}_{\emptyset} = \mathbb{Q}$. The more particular case we wish to treat is the case $\mathcal{P} = \{p\}$, the singleton made of a prime, then $\mathbb{Z}_{\{p\}} = \mathbb{Z}_{(p)}$ the p-local integers.

For an abelian group G, that is a $\mathbb{Z}$-module, its localization at $\mathcal{P}$ is then $G \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathcal{P}}$, that we shall simply denote by $G \otimes \mathbb{Z}_{\mathcal{P}}$. For example, the localization $G \otimes \mathbb{Z}_{(p)}$ has the effect of killing all torsion of order relatively prime to p and leaving p-torsion unchanged. We first show that localization can be seen as a colimit.

PROPOSITION 2.7.
Give an order structure to the multiplicative set S generated by the primes not in $\mathcal{P}$ by means of the divisibility relation. Form a direct system of groups and homomorphisms indexed by the directed set S by defining, for all $s \leq s' \in S$, $f_{ss'} : G_s := G \longrightarrow G = G_{s'}$ to be the multiplication by s'/s. Then

$$\varinjlim_{s \in S} G_s \;\cong\; G \otimes \mathbb{Z}_{\mathcal{P}} \,.$$

PROOF. First define a compatible system of group homomorphisms by setting for all $s \in S$

$$\begin{array}{rccc} \phi_s : & G_s & \longrightarrow & G \otimes \mathbb{Z}_{\mathcal{P}} \\ & g & \longmapsto & g \otimes \frac{1}{s} \,. \end{array}$$

The universal property of colimits then yields the existence of a group homomorphism $\Phi : \varinjlim_{s \in S} G_s \longrightarrow G \otimes \mathbb{Z}_{\mathcal{P}}$. First, for the easy case $G = \mathbb{Z}$, the maps ϕ_s are clearly injective since $\mathbb{Z}$ is torsion-free, thus Φ is an injection. In addition, any element $z \otimes \frac{1}{r} \in \mathbb{Z} \otimes \mathbb{Z}_{\mathcal{P}}$ is the image of z under ϕ_r. Therefore Φ is an isomorphism. Now, passage to colimits commutes with tensor product, therefore for all abelian group G, we have

$$G \otimes \mathbb{Z}_{\mathcal{P}} \cong G \otimes (\varinjlim_{s \in S} \mathbb{Z}_s) \cong \varinjlim_{s \in S} (G \otimes \mathbb{Z}_s) \cong \varinjlim_{s \in S} G_s \,.$$

□

REMARK 2.8.

(1) Since passage to colimits preserves exactness, the $\mathcal{P}$-localization functor $G \longrightarrow G \otimes \mathbb{Z}_{\mathcal{P}}$ takes exact sequences of abelian groups to exact sequences of abelian groups. (This can actually be deduced, in a more more rudimentary way, from the fact that $\mathbb{Z}_{\mathcal{P}}$ is torsion free.)

(2) Let $A \to B \to C \to D \to E$ be an exact sequence of abelian groups such that the groups $A, B, D; E$ are $\mathbb{Z}_{\mathcal{P}}$-modules. Let us denote by $M \xrightarrow{\otimes \mathbb{Z}_{\mathcal{P}}} M \otimes \mathbb{Z}_{\mathcal{P}}$ the natural map previously denoted by θ_M. Then there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow{\scriptstyle \otimes \mathbb{Z}_{\mathcal{P}}} & & \downarrow{\scriptstyle \otimes \mathbb{Z}_{\mathcal{P}}} & & \downarrow{\scriptstyle \otimes \mathbb{Z}_{\mathcal{P}}} & & \downarrow{\scriptstyle \otimes \mathbb{Z}_{\mathcal{P}}} & & \downarrow{\scriptstyle \otimes \mathbb{Z}_{\mathcal{P}}} \\ A \otimes \mathbb{Z}_{\mathcal{P}} & \longrightarrow & B \otimes \mathbb{Z}_{\mathcal{P}} & \longrightarrow & C \otimes \mathbb{Z}_{\mathcal{P}} & \longrightarrow & D \otimes \mathbb{Z}_{\mathcal{P}} & \longrightarrow & E \otimes \mathbb{Z}_{\mathcal{P}} \end{array}$$

where the first, second, fourth and fifth vertical arrows are isomorphisms. Thus by the 5-lemma so is the third, i.e. the third group is also a $\mathbb{Z}_{\mathcal{P}}$-module.

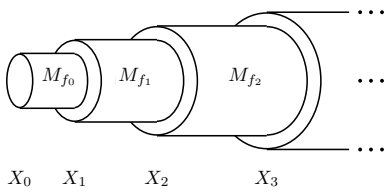
2. The Mapping Telescope Construction

Before generalizing the algebraic theory of localization to topological spaces, there is a necessary construction, which is the mapping telescope, that we need to describe.

Mapping Telescope.
The *mapping telescope* T of a sequence of topological spaces and continuous maps

$$X_0 \xhookrightarrow{f_0} X_1 \xhookrightarrow{f_1} X_2 \xhookrightarrow{f_2} \dots$$

is the union of the mapping cylinders M_{f_n} with the copies of X_n in M_{f_n} and $M_{f_{n-1}}$ identified for all $n \geq 1$.



In other words T is the quotient space

$$T := \coprod_{n\geq 0} (X_n \times [n, n+1]) \,/\sim$$

under the identifications $(x_n, n+1) \sim (f_n(x_n), n+1)$ for all $n \geq 1$, $(x_n, n+1) \in X_n \times [n, n+1]$ and $(f_n(x_n), n+1) \in X_{n+1} \times [n+1, n+2]$.

The purpose of this section is to show that the homology of T can, under good hypotheses, be described as a colimit of the homology groups of each X_n. Thus for the remainder of the section, we need to assume that the spaces considered are "nice enough", that is such that their homology satisfies the additivity axiom and the weak homotopy axiom. In particular, if the spaces X_i, $i \geq 0$ constituting the mapping telescope are CW-complexes this works well.

Theorem 2.9.
Under the hypotheses above, for all $p \geq 0$, there is an isomorphism

$$H_p(T) \cong \varinjlim_{n\in\mathbb{N}} H_p(X_n)$$

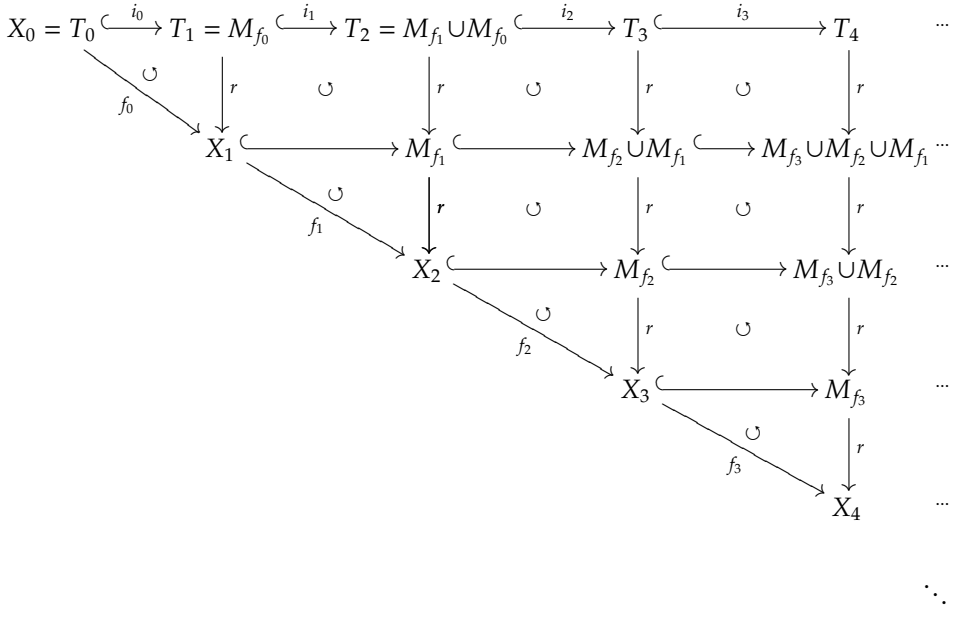
where the colimit is taken with respect to the induced maps $\{(f_i)_*\}_{i\geq 0}$.

The proof we give here is the one from [Bre97, Chapter 13] and [May99, Chapter 14]. First notice that, by additivity axiom, we may as well assume that we are working with path-connected spaces.

Define by induction

$$\begin{array}{rcl} T_0 & := & X_0 \\ T_1 & := & M_{f_0} \\ T_2 & := & T_1 \cup M_{f_1} = M_{f_0} \cup M_{f_1} \\ \cdots & & \\ \forall\, k \in \mathbb{N}\ T_k & := & T_{k-1} \cup M_{f_{k-1}} \end{array}$$

the union of the first k-th mapping cylinders in T, which is itself a finite mapping telescope. We have the following commutative diagram:

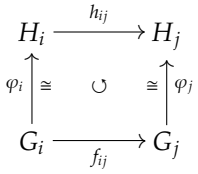


where the horizontal maps are the canonical inclusions and the vertical maps the obvious retractions of the mapping cylinders. Each r being a homotopy equivalence, it follows that the maps $r^n : T_n \longrightarrow X_n$ induce an integral homology isomorphism

$$H_q(r^n) : H_q(T_n) \xrightarrow{\cong} H_q(X_n)$$

for all $q, n \geq 0$. Recall the following general result for colimits:

Proposition 2.10.
Let $(G_i, f_{ij})_J$ and $(H_i, h_{ij})_J$ be two direct systems of groups with colimits $(\varinjlim_{i\in J} G_i, I_i)$ and $(\varinjlim_{i\in J} H_i, L_i)$ respectively. If $G_i \cong H_i$ through a group isomorphism φ_i for all $i \in J$ and the diagram



commutes for all $i \leq j \in J$, then there is a group isomorphism

$$\varinjlim_{i\in J} G_i \;\cong\; \varinjlim_{i\in J} H_i \;.$$

Proof. Applying the universal property of colimits yields the desired isomorphism.

□

It follows from the observations above and from this proposition that

$$\varinjlim_{i\geq 0} H_q(X_i) \cong \varinjlim_{i\geq 0} H_q(T_i)\,.$$

More accurately, the colimit of the direct system $(H_q(X_i), (f_i)_*)_{i\geq 0}$ is isomorphic to the colimit of the direct system $(H_q(T_i), (i_i)_*)_{i\geq 0}$.
Now T is seen as the union of the expanding sequence of subspaces T_i, $i \geq 0$ and inclusions i_i. Therefore, in order to complete the proof of theorem 2.9, it suffices to show the following result.

Proposition 2.11.
Let Y be the union of an expanding sequence of subspaces made of inclusions

$$Y_0 \xhookrightarrow{i_0} Y_1 \xhookrightarrow{i_1} Y_2 \xhookrightarrow{i_2} \cdots\;.$$

Then for all integer $p \geq 0$

$$H_p(Y) \;\cong\; \varinjlim_{i\in\mathbb{N}} H_p(Y_i)$$

Proof. Let $\widetilde{T}$ be the mapping telescope of the sequence of maps

$$Y_0 \xhookrightarrow{i_0} Y_1 \xhookrightarrow{i_1} Y_2 \xhookrightarrow{i_2} \cdots$$

that is

$$\widetilde{T} := \coprod_{n\geq 0} (Y_n \times [n, n+1])/\sim$$

under the identifications $(y_n, n) \sim (i_n(y_n), n+1)$. Let $Z_0 := Y_0$ and for all $k \geq 1$ let $Z_k := Z_{k-1} \cup M_{i_{k-1}}$ be the union of the first k-th mapping cylinders in $\widetilde{T}$.
(I) We first show that $H_*(Y) \cong H_*(\widetilde{T})$. Using again the retractions of the mapping cylinders we obtain composite retractions

$$\begin{array}{ccc} Z_k & \xhookrightarrow{} & Z_{k+1} \\ {\scriptstyle r^k}\downarrow & \circlearrowleft & \downarrow{\scriptstyle r^{k+1}} \\ Y_k & \xhookrightarrow{i_k} & Y_{k+1} \end{array}$$

such that the diagram commutes. Since the r^k's are homotopy equivalences and since homotopy groups commute with colimits (see for example [May99]) we obtain a weak equivalence

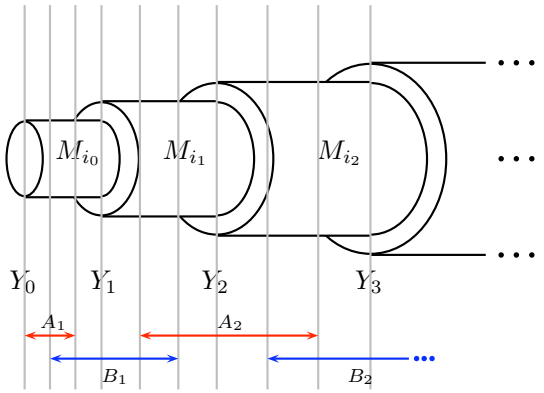
$$r : \widetilde{T} \longrightarrow Y$$

which then induces an isomorphism on homology.

(II) It remains to show that $H_q(\widetilde{T}) \cong \varinjlim_{i\geq 1} H_q(Y_i)$ for all q. The rough idea of the proof is to split $\widetilde{T}$ into a union of two spaces A and B such that $A \simeq \amalg_{i\geq 0} Y_{2i}$, $B \simeq \amalg_{i\geq 0} Y_{2i+1}$ and $A \cap B \simeq \amalg_{i\geq 0} Y_i$ so that we can apply the Mayer-Vietoris sequence to the excisive triad $(\widetilde{T}; A, B)$. Splitting the mapping cylinders M_{i_n} into thirds and taking disjoint unions of well-chosen pieces does the trick. For all $n \geq 1$ let

$$A_n := \{(y,t) \in T \,|\, 2(n-1) - \tfrac{2}{3} < t < 2(n-1) + \tfrac{2}{3}\};$$

$$B_n := \{(y,t) \in T \,|\, 2n - 1 - \tfrac{2}{3} < t < 2n - \tfrac{1}{3}\}.$$



The subspaces A_n's are disjoint open sets and each A_n is homotopy equivalent to $Y_{2(n-1)}$. Similarly, the B_n's are disjoint open sets and each B_n is homotopy equivalent to Y_{2n-1}:

$$A_n \simeq Y_{2(n-1)} \quad \text{and} \quad B_n \simeq Y_{2n-1}$$

Therefore

$$A := \coprod_{n\geq 1} A_n \simeq \coprod_{n\geq 1} Y_{2(n-1)} \quad \text{and} \quad B := \coprod_{n\geq 1} B_n \simeq \coprod_{n\geq 1} Y_{2n-1}.$$

The intersection is

$$\begin{aligned} A \cap B &= A_1 \cap B_1 \amalg B_1 \cap A_2 \amalg A_2 \cap B_2 \amalg \dots \\ &\simeq Y_0 \amalg Y_1 \amalg Y_2 \amalg \dots \end{aligned}$$

and by additivity axiom there is an isomorphism $H_q(A \cap B) \cong \bigoplus_{i\geq 0} H_q(Y_i)$ for all $q \geq 0$. So we can describe the homology groups of $A \cap B$ as:

$$H_q(A \cap B) = \{(u_i)_{i\geq 0} \,|\, u_i \in H_q(Y_i), u_i = 0 \text{ for large } i\text{'s}\}$$

And similarly:

$$H_q(A) = \{(u_i)_{i\geq 0} \,|\, u_i = 0 \text{ for all odd } i\text{'s}, u_{2j} \in H_q(Y_{2j}), u_{2j} = 0 \text{ for large } j\text{'s}\}$$

$$H_q(B) = \{(u_i)_{i\geq 0} \,|\, u_i = 0 \text{ for all even } i\text{'s}, u_{2j+1} \in H_q(Y_{2j+1}), u_{2j+1} = 0 \text{ for large } j\text{'s}\}.$$

Note that the canonical inclusions $A \cap B \xrightarrow{\imath} A$ and $A \cap B \xrightarrow{j} B$ have restrictions

$$\mathrm{Id} : Y_{2i} \longrightarrow Y_{2i} \quad \text{and} \quad \imath_{2i+1} : Y_{2i+1} \longrightarrow Y_{2i+2}$$

and

$$i_{2i} : Y_{2i} \longrightarrow Y_{2i+1} \quad \text{and} \quad \mathrm{Id} : Y_{2i+1} \longrightarrow Y_{2i+1}$$

respectively. Hence the induced maps in homology are:

$$\begin{array}{rccl} \iota_*\colon & H_q(A\cap B) & \longrightarrow & H_q(A) \\ & (u_i)_{i\geq 0} & \longmapsto & (u_0, 0, u_2 + (i_1)_*(u_1), 0, u_4 + (i_3)_*(u_3), \ldots) \end{array}$$

and

$$\begin{array}{rccl} j_*\colon & H_q(A\cap B) & \longrightarrow & H_q(A) \\ & (u_i)_{i\geq 0} & \longmapsto & (0, u_1 + (i_0)_*(u_0), 0, u_3 + (i_2)_*(u_2), 0, \ldots) \end{array}$$

Then the map

$$\begin{array}{rccl} \phi := \iota_* \oplus (-j_*)\colon & H_q(A\cap B) & \longrightarrow & H_q(A) \\ & (u_i)_{i\geq 0} & \longmapsto & (u_0, -u_1 - (i_0)_*(u_0), u_2 + (i_1)_*(u_1), -u_3 - (i_2)_*(u_2), \ldots) \end{array}$$

is one of the map in the Mayer-Vietoris sequence. (Well, the less common version of the Mayer-Vietoris sequence of the form

$$\ldots \longrightarrow H_q(U\cap V) \xrightarrow{\iota_*\oplus(-j_*)} H_q(V)\oplus H_q(U) \xrightarrow{k_*+j_*} H_q(U\cup V) \longrightarrow \ldots .)$$

We easily see by induction that ϕ is injective: $\phi((u_i)_{i\geq 0}) = 0$ implies that $u_0 = 0$, thus $(i_0)_*(u_0) = 0$ and $0 = -u_1 - (i_0)_*(u_0) = -u_1$, hence $u_1 = 0$ and so on. Therefore the Mayer-Vietoris sequence has the form:

$$\ldots \longrightarrow H_q(A\cap B) \xrightarrow{\phi} H_q(A)\oplus H_q(B) \xrightarrow{\beta} H_q(A\cup B = \widetilde{T}) \longrightarrow \ldots$$

where ϕ is injective and thus β surjective. It follows that $H_q(A\cup B) = H_q(\widetilde{T}) \cong \operatorname{Coker}\phi$. In addition, $\operatorname{Im}(\phi)$ is generated by the elements of the form

$$\phi(0, 0, \ldots, u_j, 0, 0, \ldots) = (0, 0, \ldots, \pm u_j, \pm(i_j)_*(u_j), 0, 0, \ldots) .$$

Eventually, since these relations are exactly the relations defining the colimit of the direct system $(H_q(Y_i), (i_i)_*)_{i\geq 0}$, we obtain the desired result:

$$\varinjlim_{i\in\mathbb{N}} H_q(Y_i) \cong H_q(\widetilde{T}) \cong H_q(Y) .$$

□

3. Topological Localizations

The idea of topological localization is to topologically realize the algebraic localization of an abelian group $G \longrightarrow G\otimes\mathbb{Z}_{\mathcal{P}}$, when G is taken to be homology and homotopy groups: for a given space X we are looking for a space $X_{\mathcal{P}}$ together with a map $X \longrightarrow X_{\mathcal{P}}$ such that the induced maps in homotopy and homology are just the algebraic localizations.

The reference for this section is essentially the book by D. P. Sullivan [Sul05], in which he builds his theory for spaces that he calls "simple spaces", that is topological spaces which are connected, have the homotopy type of a CW-complex, as well as an abelian fundamental group which acts trivially on the homotopy and homology of the universal covering space.

We present here only the basics of this theory, which we shall need to apply it to the case of $\Sigma\mathbb{C}P^\infty$ treated further. We present the localization construction for CW-complexes. In [Sul05], where details and further developments are to be found, Sullivan also has a construction for Postnikov towers.

Definition 2.12.

(1) A topological space $X_\mathcal{P}$ is a *$\mathcal{P}$-local space* if and only if $\pi_*(X_\mathcal{P})$ is $\mathcal{P}$-local in the algebraic sense, that is iff $\pi_*(X_\mathcal{P})$ is a $\mathbb{Z}_\mathcal{P}$-module.
(2) A *localization* of a topological space X is a pair $(X_\mathcal{P}, \ell)$, where $X_\mathcal{P}$ is a $\mathcal{P}$-local spaces and ℓ is a continuous map, satisfying the following universal property: for all $\mathcal{P}$-local space Y and for all $f \in \mathcal{T}op(X;Y)$, there exist a unique map $f_\mathcal{P} \in \mathcal{T}op(X_\mathcal{P};Y)$ such that the following diagramm commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \ell}\downarrow & \nearrow_{\exists! f_\mathcal{P}} & \\ X_\mathcal{P} & & \end{array}$$

We start with a useful result that reformulates the definition above. In particular it will enables us to consider a space X as $\mathcal{P}$-local if and only if its reduced homology $\widetilde{H}_*(X)$ is a $\mathbb{Z}_\mathcal{P}$-module.

Theorem 2.13.
Let $X \xrightarrow{\ell} X'$ be a continuous map between simple topological spaces. The following are equivalent:

(1) *The map ℓ is a localization.*
(2) *The map ℓ localizes integral reduced homology in the algebraic sense. In other words, the diagram*

$$\begin{array}{ccc} \widetilde{H}_*(X) & \xrightarrow{\ell_*} & \widetilde{H}_*(X') \\ {\scriptstyle \otimes\mathbb{Z}_\mathcal{P}}\downarrow & \overset{\cup}{\nearrow_{\cong}} & \\ \widetilde{H}_*(X) \otimes \mathbb{Z}_\mathcal{P} & & \end{array}$$

commutes, where the diagonal arrow is the one provided by definition 2.3.
(3) *The map ℓ localizes homotopy in the algebraic sense. In other words, the diagram*

$$\begin{array}{ccc} \pi_*(X) & \xrightarrow{\ell_*} & \pi_*(X') \\ {\scriptstyle \otimes\mathbb{Z}_\mathcal{P}}\downarrow & \overset{\cup}{\nearrow_{\cong}} & \\ \pi_*(X) \otimes \mathbb{Z}_\mathcal{P} & & \end{array}$$

commutes, for $* \geq 1$, where the diagonal arrow is the one provided by definition 2.3.

Proof. See [Sul05], end of chapter 2. □

As a result, taking $\ell = \mathrm{Id}$ we get that a "simple" space X is its own localization if and only if it has local integral reduced homology if and only if it has local homotopy.
Another useful result about localizations of suspensions is the following.

LEMMA 2.14.
If $f : A \longrightarrow A_{\mathcal{P}}$ localizes homology, so does $\Sigma f : \Sigma A \longrightarrow \Sigma A_{\mathcal{P}}$.

PROOF. Using the suspension isomorphism and the fact that f a localizes homology, we get the commutative diagram

$$\begin{array}{ccc}
\widetilde{H}_q(\Sigma A) & \xrightarrow{(\Sigma f)_*} & \widetilde{H}_q(\Sigma A_{\mathcal{P}}) \\
\partial \downarrow \cong & & \cong \downarrow \partial \\
\widetilde{H}_{q-1}(A) & \xrightarrow[f_*]{} & \widetilde{H}_{q-1}(A_{\mathcal{P}}) \\
& {\scriptstyle \otimes \mathbb{Z}_{\mathcal{P}}} \searrow \quad \swarrow {\scriptstyle \cong} & \\
& \widetilde{H}_{q-1}(A) \otimes \mathbb{Z}_{\mathcal{P}} & \\
& \cong \downarrow {\scriptstyle \partial^{-1} \otimes \mathrm{Id}} & \\
& \widetilde{H}_q(\Sigma A) \otimes \mathbb{Z}_{\mathcal{P}} & .
\end{array}$$

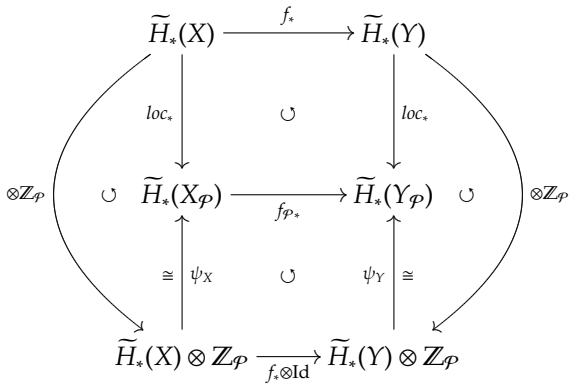
Since the map going down from $\widetilde{H}_q(\Sigma A)$ to $\widetilde{H}_q(\Sigma A) \otimes \mathbb{Z}_{\mathcal{P}}$ is the algebraic localization $\otimes \mathbb{Z}_{\mathcal{P}}$, the isomorphism going up from $\widetilde{H}_q(\Sigma A) \otimes \mathbb{Z}_{\mathcal{P}}$ to $\widetilde{H}_q(\Sigma A_{\mathcal{P}})$ is the unique homomorphism provided by the universal property 2.3, thus Σf localizes homology. □

3.1. Localization of a continuous map and induced maps in homology.

Let $f : X \longrightarrow Y$ be a continuous map between simple topological spaces and let $X \xrightarrow{loc} X_{\mathcal{P}}$ and $Y \xrightarrow{loc} Y_{\mathcal{P}}$ be localization of X and Y respectively. The universal property of definition 2.12 yields the existence of a unique continuous map $f_{\mathcal{P}} : X_{\mathcal{P}} \longrightarrow Y_{\mathcal{P}}$ making the diagram

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
{\scriptstyle loc} \downarrow & \circlearrowleft & \downarrow {\scriptstyle loc} \\
X_{\mathcal{P}} & \xrightarrow[f_{\mathcal{P}}]{} & Y_{\mathcal{P}}
\end{array}$$

commutative. We shall say that $f_{\mathcal{P}}$ is the *localization* of the map f. Passing to induced maps in homology we get the following commutative diagram:

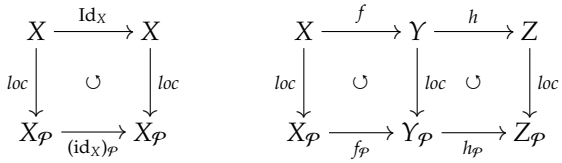


The upper square commutes because the previous diagram does. The two triangles commute by virtue of theorem 2.13. The map at the bottom is $f_*\otimes\mathrm{Id}$ given by remark 2.6 and that makes the big exterior square commute. And the commutativity of the lower square is easily computed:

$$\begin{aligned}\psi_Y^{-1}f_{\mathcal{P}*}\psi_X(x\otimes\frac{1}{z}) &= \psi_Y^{-1}f_{\mathcal{P}*}(loc_*(x)\cdot\frac{1}{z}) = \psi_Y^{-1}(f_{\mathcal{P}*}loc_*(x)\cdot\frac{1}{z})\\ &= \psi_Y^{-1}(loc_*f_*(x)\cdot\frac{1}{z}) = \psi_Y^{-1}(loc_*(f_*(x))\cdot\frac{1}{z})\\ &= f_*(x)\otimes\frac{1}{z}\end{aligned}$$

for all element $x\otimes\frac{1}{z}\in\widetilde{H}_*(X)\otimes\mathbb{Z}_{\mathcal{P}}$.

Furthermore, we see that $\mathcal{P}$-localization is a functorial construction. For let $f : X \longrightarrow Y$ and $h : Y \longrightarrow Z$ be continous maps between simple topological spaces and let $X \xrightarrow{loc} X_{\mathcal{P}}$, $Y \xrightarrow{loc} Y_{\mathcal{P}}$ and $Z \xrightarrow{loc} Z_{\mathcal{P}}$ be $\mathcal{P}$-localization, then there are commutative diagrams

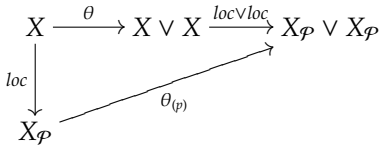


so that, by uniqueness, $(\mathrm{id}_X)_{\mathcal{P}} = \mathrm{Id}_{X_{\mathcal{P}}}$ and $(h\circ f)_{\mathcal{P}} = h_{\mathcal{P}}\circ f_{\mathcal{P}}$.

Moreover, in [Hat04] Hatcher also proves, by means of obstruction theory, that $f\simeq g$ implies $f_{\mathcal{P}}\simeq g_{\mathcal{P}}$. We use this property to show that a coproduct on a simple space induces a p-local coproduct.

Let (X,θ) be a simple co-H-space and $X \xrightarrow{loc} X_{\mathcal{P}}$ a $\mathcal{P}$-localization of X. The universal property of localizations yields the existence of a unique map $\theta_{\mathcal{P}} : X_{\mathcal{P}} \longrightarrow X_{\mathcal{P}}\vee X_{\mathcal{P}}$

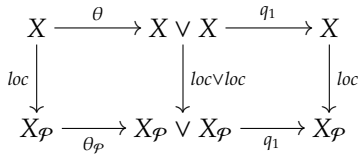
that makes the following diagram commute:



The space $X_{\mathcal{P}} \vee X_{\mathcal{P}}$ is clearly $\mathcal{P}$-local since

$$\begin{aligned}\widetilde{H}_*(X_{\mathcal{P}} \vee X_{\mathcal{P}}) &\cong \widetilde{H}_*(X_{\mathcal{P}}) \oplus \widetilde{H}(X_{\mathcal{P}}) \cong (\widetilde{H}_*(X) \otimes \mathbb{Z}_{\mathcal{P}}) \oplus (\widetilde{H}_*(X) \otimes \mathbb{Z}_{\mathcal{P}}) \\ &\cong (\widetilde{H}_*(X) \oplus \widetilde{H}_*(X)) \otimes \mathbb{Z}_{\mathcal{P}} \cong \widetilde{H}_*(X \vee X) \otimes \mathbb{Z}_{\mathcal{P}}\,.\end{aligned}$$

which is a $\mathbb{Z}_{\mathcal{P}}$-module. The diagram



commutes as well and by uniqueness we obtain $q_1\theta_{\mathcal{P}} = (q_1\theta)_{\mathcal{P}}$. Therefore it follows from the property above that $q_1\theta_{\mathcal{P}} \simeq \mathrm{Id}_{X_{\mathcal{P}}} = \mathrm{Id}_{X_{\mathcal{P}}}$ and similarly $q_2\theta_{\mathcal{P}} \simeq \mathrm{Id}_{X_{\mathcal{P}}}$, which means that $\theta_{\mathcal{P}}$ is a coproduct.

3.2. Localization of CW-complexes.

In this work, the spaces we wish to localize are CW-complexes ($\mathbb{C}P^\infty$ and $\Sigma\mathbb{C}P^\infty$ in particular), therefore we will restrict this description of local spaces to showing the existence of local CW-complexes. The first step to achieve this aim is the localization of the cells, that is the localization of the n-dimensional spheres.

Proposition 2.15.
For all $i > 0$, there exist a $\mathcal{P}$-local i-sphere $\mathbb{S}^i_{\mathcal{P}}$.

Proof. (From [Sul05])
The $\mathcal{P}$-local i-sphere $\mathbb{S}^i_{\mathcal{P}}$ is realised through the mapping telescope construction. Let $\{\ell_n\}_{n\geq 1}$ be a cofinal sequence in the multiplicative subset of $\mathbb{Z}$ generated by $\mathbb{P}\setminus\mathcal{P}$. (See remark below for a concrete example). For all n there exist a continuous map $f(\ell_n) : \mathbb{S}^i \longrightarrow \mathbb{S}^i$ with degree ℓ_n. We see that the space $\mathbb{S}^i_{\mathcal{P}}$ defined to be the infinite mapping telescope obtained from the sequence

$$\mathbb{S}^i \xrightarrow{f(\ell_1)} \mathbb{S}^i \xrightarrow{f(\ell_2)} \mathbb{S}^i \xrightarrow{f(\ell_3)} \dots$$

together with the canonical inclusion $\ell : \mathbb{S}^i \hookrightarrow \mathbb{S}^i_{\mathcal{P}}$ as the "left-hand" i-sphere in the mapping telescope is a localization of $\mathbb{S}^i$.

Since the colimits of the systems

$$\mathbb{Z} \xrightarrow{\ell_1} \mathbb{Z} \xrightarrow{\ell_2} \mathbb{Z} \xrightarrow{\ell_3} \dots$$

and

$$0 \xrightarrow{\ell_1} 0 \xrightarrow{\ell_2} 0 \xrightarrow{\ell_3} \dots$$

are, respectively, the groups $\mathbb{Z}_{\mathcal{P}}$ and 0, we get

$$\widetilde{H}_q(\mathbb{S}^i_{\mathcal{P}}) = \begin{cases} \varinjlim_{\ell_n} \mathbb{Z} = \mathbb{Z}_{\mathcal{P}} \cong \mathbb{Z} \otimes \mathbb{Z}_{\mathcal{P}} & \text{if } q = i \\ \varinjlim_{\ell_n} 0 = 0 \cong 0 \otimes \mathbb{Z}_{\mathcal{P}} & \text{otherwise.} \end{cases}$$

Since the homology of $\mathbb{S}^i$ is given by $\mathbb{Z}$ in degree i and 0 elsewhere, it follows from theorem 2.13 that ℓ is a localization, as it localizes homology. □

Remark 2.16.
Later we shall use only the case $\mathcal{P} = \{p\}$, the singleton made of a prime. In this very case we can choose ℓ_n to be the product of the first n primes excluding p. The sequence $\{\ell_n\}_{n\geq 1}$ is then obviously cofinal.

We can now localize CW-complexes.

Definition 2.17.
A *$\mathcal{P}$-local CW-complex* is a topological space built inductively from a point or a $\mathcal{P}$-local 1-sphere by attaching cones over the local sphere using maps of the local spheres $\mathbb{S}^i_{\mathcal{P}}$ into the lower local-skeletons.

Since $\mathbb{S}^0$ is not connected, there is in this theory no local 0-sphere and thus the definition above gives no local 1-cell. Both this observation and the definition lead to the following theorem of existence of local CW-complexes.

Theorem 2.18.
Let X be a CW-complex with one 0-cell and no 1-cell. Then there exist a local CW-complex $X_{\mathcal{P}}$ and a cellular map $\ell : X \longrightarrow X_{\mathcal{P}}$ such that ℓ induces a bijection between the cells of X and the local cells of $X_{\mathcal{P}}$. Furthermore, ℓ localizes homology.

The proof is the one from [Sul05]. We only try to explain it with more details.

Proof. The first step is to show that the theorem holds for finite CW-complexes. This proof is done by induction on the dimension and makes use of theorem 2.13 by assuming that a map is localization if and only if it localizes homology.
First, if X is a 2-complex with no one-cell and 0-skeleton equal to a point, then $X = \bigvee \mathbb{S}^2$ is just a wedge of 2-spheres and the map $\bigvee \mathbb{S}^2 \longrightarrow \bigvee \mathbb{S}^2_{\mathcal{P}}$, which is the wedge of the localizations of $\mathbb{S}^2$ described in the proof of proposition 2.15, satisfies,

by definition, the one-to-one correspondence condition and also localizes integral reduced homology since

$$\begin{aligned}\widetilde{H}_*(\bigvee \mathbb{S}^2_{\mathcal{P}}) &\cong \bigoplus \widetilde{H}_*(\mathbb{S}^2_{\mathcal{P}}) \cong \bigoplus (\widetilde{H}_*(\mathbb{S}^2) \otimes \mathbb{Z}_{\mathcal{P}}) \\ &\cong (\bigoplus \widetilde{H}_*(\mathbb{S}^2)) \otimes \mathbb{Z}_{\mathcal{P}} \cong \widetilde{H}_*(\bigvee \mathbb{S}^2) \otimes \mathbb{Z}_{\mathcal{P}}\,.\end{aligned}$$

For the general step, assume that the theorem is true for all complexes of dimension less or equal to $n-1$ ($n \geq 3$). Let X be an n-dimensional CW-complex with one 0-cell and no 1-cell and let $f : \bigvee \mathbb{S}^{n-1} \longrightarrow X^{(n-1)}$ be the map which attaches the n-cells to the $(n-1)$-skeleton in X, and let $\ell_{n-1} : X^{(n-1)} \longrightarrow X^{(n-1)}_{\mathcal{P}}$ be the localization of $X^{(n-1)}$ given by the induction hypothesis. The universal property of definition 2.12 yields the commutative diagram

$$\begin{array}{ccc} \bigvee \mathbb{S}^{n-1} & \xrightarrow{f} & X^{(n-1)} \\ {\scriptstyle l}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \ell_{n-1}} \\ \bigvee \mathbb{S}^{n-1}_{\mathcal{P}} & \xrightarrow[f_{\mathcal{P}}]{} & X^{(n-1)}_{\mathcal{P}}\,. \end{array}$$

To recover X, we use the cofiber sequence associated with the map f:

$$\bigvee \mathbb{S}^{n-1} \xrightarrow{f} X^{(n-1)} \xrightarrow{i} C_f \xrightarrow{j} C_i \simeq \Sigma(\bigvee \mathbb{S}^{n-1}) \xrightarrow{\Sigma f} \Sigma(X^{(n-1)}) \longrightarrow \cdots$$

where $C_f = X^{(n-1)} \cup_f C(\bigvee \mathbb{S}^{n-1}) = X^{(n)} = X$, and $\Sigma(\bigvee \mathbb{S}^{n-1}) \simeq \bigvee(\Sigma \mathbb{S}^{n-1}) \simeq \bigvee \mathbb{S}^n$. Similarly, we can build the cofiber sequence associated to the map $f_{\mathcal{P}}$:

$$\bigvee \mathbb{S}^{n-1}_{\mathcal{P}} \xrightarrow{f_{\mathcal{P}}} X^{(n-1)}_{\mathcal{P}} \xrightarrow{i_{\mathcal{P}}} C_{f_{\mathcal{P}}} \xrightarrow{j_{\mathcal{P}}} \Sigma(\bigvee \mathbb{S}^{n-1}_{\mathcal{P}}) \xrightarrow{\Sigma f_{\mathcal{P}}} \Sigma(X^{(n-1)}_{\mathcal{P}}) \longrightarrow \cdots$$

where $\Sigma(\bigvee \mathbb{S}^{n-1}_{\mathcal{P}}) \simeq \bigvee \mathbb{S}^n_{\mathcal{P}}$. Then let $X_{\mathcal{P}}$ be the cofiber of $f_{\mathcal{P}}$, $C(f_{\mathcal{P}}) = X^{(n-1)}_{\mathcal{P}} \cup_{f_{\mathcal{P}}} C(\bigvee \mathbb{S}^{n-1}_{\mathcal{P}})$ and define a map $\ell : X \longrightarrow X_{\mathcal{P}}$ to be the map

$$\ell_{n-1} \cup c(l) : X^{(n-1)} \cup_f C(\bigvee \mathbb{S}^{n-1}) \longrightarrow X^{(n-1)}_{\mathcal{P}} \cup_{f_{\mathcal{P}}} C(\bigvee \mathbb{S}^{n-1}_{\mathcal{P}})$$

so that the diagram

$$\begin{array}{ccccccccc} \bigvee \mathbb{S}^{n-1} & \xrightarrow{f} & X^{(n-1)} & \xrightarrow{i} & X & \xrightarrow{j} & \bigvee \mathbb{S}^n & \xrightarrow{\Sigma f} & \Sigma(X^{(n-1)}) \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle \ell_{n-1}} & & \downarrow{\scriptstyle \ell} & & \downarrow & & \downarrow{\scriptstyle \Sigma \ell_{n-1}} \\ \bigvee \mathbb{S}^{n-1}_{\mathcal{P}} & \xrightarrow[f_{\mathcal{P}}]{} & X^{(n-1)}_{\mathcal{P}} & \xrightarrow[i_{\mathcal{P}}]{} & X_{\mathcal{P}} & \xrightarrow[j_{\mathcal{P}}]{} & \bigvee \mathbb{S}^n_{\mathcal{P}} & \xrightarrow[\Sigma f_{\mathcal{P}}]{} & \Sigma(X^{(n-1)}_{\mathcal{P}}) \end{array}$$

commutes. For the last square apply lemma 2.14 which ensures that $\Sigma \ell_{n-1}$ is a localization and note that by uniqueness $\Sigma f_{\mathcal{P}} = (\Sigma f)_{\mathcal{P}}$.

By theorem 2.13 it suffices to show that ℓ localizes homology. But by exactness of the cofiber sequence, since the whole diagram commutes and since all the vertical maps, but ℓ, localize homology, passing to homology we can apply remark 2.8 (2) and obtain the desired result. Moreover, since ℓ_{n-1} sets a bijection between the cells and the local cells, so does ℓ by construction. This completes the proof for finite CW-complexes. For infinite CW-complexes, let $X_{\mathcal{P}} = \bigcup_{n=0}^{\infty} X^{(n)}$.

□

CHAPTER 3

Algebraic and Topological Tools

The present chapter is a toolbox containing useful mathematical objects necessary to carry out the proofs concerning the splitting of $\Sigma\mathbb{C}P^{\infty}_{(p)}$ and the co-H-structures on the spaces K_1 to K_{p-1}. Each section introduces another object and they can easily be read independently. The aim is not to start to develop these objects but to focus on some specific properties we shall need in the next chapter.

1. Hopf Algebras

Under a few good hypotheses on topological spaces, their homology and cohomology can each be endowed with Hopf algebra structures, which are dual to each other. To study these structures, it shall be easier to take a step back in the general theory of Hopf algebras. We begin by reviewing the basic definitions of algebras and coalgebras. The main purpose of the section is to develop some properties of the primitive, indecomposable elements and dual Hopf algebras.The source for the cited results is the classical reference for Hopf algebras, the article [MM65] by J.W. Milnor and J.C. Moore.
To set up notations $\mathbb{K}$ shall always denote a commutative ring, the tensor products shall be balanced over $\mathbb{K}$, the gradings shall be taken over the non-negative integers and in a graded module $A_\bullet$ the notation $|a| = m$ shall mean that $a \in A_m$, i.e. denote the degree of a. Morever $\mathbb{K}$ shall be considered as a graded $\mathbb{K}$-module concentrated in degree 0.

Definition 3.1.

(1) A $\mathbb{K}$-*algebra* is a graded $\mathbb{K}$-module A endowed with two morphisms of graded $\mathbb{K}$-modules $\varphi : A \otimes A \longrightarrow A$ and $\eta : \mathbb{K} \longrightarrow A$, called the *multiplication* and the *unit* of the algebra, and making the diagrams

$$\begin{array}{ccc} A\otimes A\otimes A & \xrightarrow{\mathrm{Id}_A\otimes\varphi} & A\otimes A \\ {\scriptstyle \varphi\otimes\mathrm{Id}_A}\downarrow & & \downarrow{\scriptstyle \varphi} \\ A\otimes A & \xrightarrow[\varphi]{} & A \end{array} \qquad \begin{array}{ccccc} \mathbb{K}\otimes A & \xrightarrow{\eta\otimes\mathrm{Id}_A} & A\otimes A & \xleftarrow{\mathrm{Id}_A\otimes\eta} & A\otimes\mathbb{K} \\ & \searrow^{\cong} & \downarrow{\scriptstyle \varphi} & \swarrow_{\cong} & \\ & & A & & \end{array}$$

commute. The commutativity of the first diagram is called *associativity*. Moreover, A is called *(graded) commutative* if $\varphi T = \varphi$, T being the *twisting morphism* defined by $T(a \otimes a') = (-1)^{|a||a'|} a' \otimes a$. A is called *connected* if η maps $\mathbb{K}$ isomorphically to A_0. A *morphism of algebras* $f : A \longrightarrow B$ is a morphism of graded $\mathbb{K}$-modules that preserves the multiplication and the unit, that is such that $f\varphi_A = \varphi_B(f \otimes f)$ and $f\eta_A = \eta_B$.

(2) A $\mathbb{K}$-*coalgebra* is a graded $\mathbb{K}$-module C endowed with two morphisms of graded $\mathbb{K}$-modules $\Delta : C \longrightarrow C \otimes C$ and $\varepsilon : C \longrightarrow \mathbb{K}$, called the *comultiplication* and the *counit* of the coalgebra, and making the diagrams

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle 1\otimes\Delta} \\ C \otimes C & \xrightarrow[\Delta\otimes 1]{} & C \otimes C \otimes C \end{array} \qquad \begin{array}{ccccc} \mathbb{K} \otimes C & \xleftarrow{\varepsilon\otimes \mathrm{Id}_C} & C \otimes C & \xrightarrow{\mathrm{Id}_C \otimes \varepsilon} & C \otimes \mathbb{K} \\ & {\scriptstyle\cong}\nwarrow & {\scriptstyle\Delta}\uparrow & \nearrow{\scriptstyle\cong} & \\ & & C & & \end{array}$$

commute. The commutativity of the first diagram is called *coassociativity*. Moreover, C is called *cocommutative* if $T\Delta = \Delta$. C is called *connected* if ε maps C_0 isomorphically to $\mathbb{K}$. A *morphism of coalgebras* $f : C \longrightarrow D$ is a morphism of graded $\mathbb{K}$-modules that preserves the comultiplication and the counit, that is such that $(f \otimes f)\Delta_C = \Delta_D f$ and $\varepsilon_C = \varepsilon_D f$.

Definition 3.2.
A *Hopf algebra* over $\mathbb{K}$ is a graded $\mathbb{K}$-module A endowed with graded $\mathbb{K}$-module morphisms

$$\begin{array}{ll} \varphi : A \otimes A \longrightarrow A & \eta : \mathbb{K} \longrightarrow A \\ \Delta : A \longrightarrow A \otimes A & \varepsilon : A \longrightarrow \mathbb{K} \end{array}$$

such that

(1) (A, φ, η) is a $\mathbb{K}$-algebra with augmentation ε;
(2) (A, Δ, ε) is a $\mathbb{K}$-coalgebra with augmentation η;
(3) Δ is a morphism of algebras or equivalently φ is a morphism of coalgebras.

In case, the multiplication φ and the comultiplication Δ are not associative and coassociative, we shall call A a *quasi-Hopf algebra*.

1.1. Primitive and Indecomposable Elements.

Proposition-Definition 3.3.
Let $(A, \varphi, \Delta, \eta, \varepsilon)$ be a Hopf algebra.

(1) In the algebra structure (A, φ, η) augmented by ε we denote by IA the kernel of ε. It is called the *augmentation ideal*. We have $(IA)_n = A_n$ for all $n \geq 1$, $(IA)_0 = \ker \varepsilon_0$ and $\varepsilon\eta = \mathrm{Id}_{\mathbb{K}}$, so that $A \cong \mathbb{K} \oplus IA$ (as a graded $\mathbb{K}$-module).

(2) In the coalgebra structure (A, Δ, ε) augmented by η we denote by JA the cokernel of η. We have $(JA)_n = A_n$ for all $n \geq 1$, $(JA)_0 = \operatorname{Coker} \eta_0$ and $A \cong \mathbb{K} \oplus JA$ (as a graded $\mathbb{K}$-module).

In fact, since $\varepsilon\eta = \mathrm{Id}_{\mathbb{K}}$, the composite $\ker \varepsilon \overset{i}{\hookrightarrow} A \overset{p}{\twoheadrightarrow} \operatorname{Coker} \eta$ is an isomorphism, so that we can identify $IA = JA =: \overline{A}$.

Primitive Elements.
For a Hopf algebra A we say that an element $a \in JA$ is *primitive* if and only if it lies in the kernel of the composite map

$$\overline{A} \xrightarrow{\ i\ } A \xrightarrow{\ \Delta\ } A \otimes A \xrightarrow{\ p\otimes p\ } \overline{A} \otimes \overline{A}\,.$$

We denote by $P(A)$ this kernel. When A is connected, JA is the positive part of A. Moreover the map Δ is a morphism of algebras, thus respects degrees. Thus if a is in the kernel of the composite map above, then $\Delta(a)$ has the form $a_i \otimes 1 + 1 \otimes a_j$ with $a_i, a_j \in A_{|a|}$. Then using the definition of the counit ε, we get $a \otimes 1 = (\varepsilon \otimes \mathrm{Id}_A)\Delta(a) = a_i \otimes 1$ and $1 \otimes a = (\mathrm{Id}_A \otimes \varepsilon)\Delta(a) = 1 \otimes a_j$ hence $a_i = a = a_j$. Therefore a is primitive if and only if

$$\Delta(a) = a \otimes 1 + 1 \otimes a\,.$$

Definition 3.4.
A Hopf algebra A is said to be *primitively generated* when the subalgebra generated by the set $P(A)$ is equal to A itself.

Indecomposable Elements.
For an augmented algebra A we say that an element $a \in IA$ is *decomposable* if and only if it lies in the image of the composite map

$$\overline{A} \otimes \overline{A} \xrightarrow{\ i\otimes i\ } A \otimes A \xrightarrow{\ \varphi\ } A \xrightarrow{\ p\ } \overline{A}\,.$$

When A is a connected, we get by a similar calculation as above that an element $a \in IA$ is decomposable if and only if it has the form $a = \sum x_i y_i$ with $x_i \in A_{m_i}$, $y_i \in A_{|a|-m_i}$, $0 < m_i < |a|$.
By abuse of language, we shall call *indecomposable* the elements of the quotient module $IA/\varphi(IA \otimes IA) =: Q(A)$.

Note that the natural map $IA \longrightarrow Q(A)$ restricts to define a map $P(A) \longrightarrow Q(A)$, which is natural with respect to morphisms of Hopf algebras.

1.2. Duality.

The *dual* of a graded $\mathbb{K}$-module A is the graded $\mathbb{K}$-module A^* such that $(A^*)_n = \operatorname{Hom}(A_n, \mathbb{K})$ and the dual of a morphism of graded $\mathbb{K}$-modules $f : A \longrightarrow B$ is the graded morphism $f^* : B^* \longrightarrow A^*$ such that $(f^*)_n = \operatorname{Hom}(f_n, \mathbb{K})$.

A graded $\mathbb{K}$-module A is said to be *of finite type* if each A_n is finitely generated as a $\mathbb{K}$-module. Similarly it is projective, if each A_n is.

Let A and B be graded $\mathbb{K}$-modules which are projective of finite type. Then, from linear algebra, we know the following properties:

(1) A and its bidual A^{**} are isomorphic through the morphism $\lambda : A \longrightarrow A^{**}$ defined by $\lambda(x)(a^*) := a^*(x)$ for all $x \in A_n$ and $a \in A_n^*$, that is A and A^* are dual to each other.
(2) The morphism $\alpha : A^* \otimes B^* \longrightarrow (A \otimes B)^*$ defined for all $x \in A_n, y \in A_m, a^* \in A_n^*, b^* \in A_m^*$ by $\alpha(a^* \otimes b^*)(x \otimes y) := a^*(x)b^*(y)$ is an isomorphism.
(3) There are canonical isomorphisms $\mu : \mathbb{K} \longrightarrow \mathbb{K}^*$ defined by $1 \longmapsto \mathrm{Id}_{\mathbb{K}}$ and its inverse $\mu^{-1} : \mathbb{K}^* \longrightarrow \mathbb{K}$ defined by $f \longmapsto f(1)$.

The next proposition describes the links between algebras, coalgebras and their duals.

Proposition 3.5.
Let A be a graded $\mathbb{K}$-module which is projective of finite type. Then

(1) *(A, φ, η) is an algebra if and only if $(A^*, \alpha^{-1}\varphi^*, \mu^{-1}\eta^*)$ is a coalgebra. In particular, $\varphi : A \longrightarrow A \otimes A$ is a multiplication on A if and only if $\alpha^{-1}\varphi^* : A^* \otimes A^* \longrightarrow A^*$ is a comultiplication on A^* and $\eta : \mathbb{K} \longrightarrow A$ is a unit for A if and only if $\mu^{-1}\eta^* : A^* \longrightarrow \mathbb{K}^* = \mathbb{K}$ is a counit for A^*;*
(2) *the morphism $\varepsilon : A \longrightarrow \mathbb{K}$ is an augmentation for the algebra (A, φ, η) if and only if $\varepsilon^*\mu : \mathbb{K} \longrightarrow A^*$ is an augmentation for the coalgebra $(A^*, \alpha^{-1}\varphi^*, \mu^{-1}\eta^*)$;*
(3) *the algebra (A, φ, η) is commutative, respectively associative, if and only if the coalgebra $(A^*, \alpha^{-1}\varphi^*, \mu^{-1}\eta^*)$ is cocommutative, respectively coassociative;*
(3) *if $(A, \varphi, \Delta, \eta, \varepsilon)$ is a Hopf algebra, then $(A, \Delta^*\alpha, \alpha^{-1}\varphi^*, \varepsilon^*\mu, \mu^{-1}\eta^*)$ is a Hopf algebra, called the dual Hopf algebra.*

Proof. The proof of this proposition is omitted in [MM65] and we leave it out as well[1]. □

Our ultimate aim, in this description of Hopf algebras, is to study the behaviour of p^{th} powers in the dual of a primitively generated Hopf algebra of finite type over the field $\mathbb{F}_p$. It will be achieved by means of the following proposition from [MM65, proposition 4.20]. Besides, as it constitutes a central argument for the proof of theorem 4.8, we recall the main ideas of proof.

Proposition 3.6.
Let A be a connected quasi-Hopf algebra over the field $\mathbb{F}_p$. Then the natural morphism $P(A) \longrightarrow Q(A)$ is injective if and only if the multiplication φ is commutative, associative and the p^{th} powers are zero.

[1] EPFL readers can nonetheless refer to a previous students joint work [DKKL06, Part III, Chapter 12 and 13], in which most of these calculations are carried through.

Sketch of proof. To prove commutativity, consider the graded commutator $[a,b] := ab - (-1)^{|a||b|}ba$ and notice that if a and b are primitive, so is $[a,b]$ and has image zero in $Q(A)$. Then, proceeding by induction we obtain that the map $[\,,\,] : A \otimes A \longrightarrow A$ is zero. The same procedure applies to the morphism $\Psi : A \otimes A \otimes A \longrightarrow A : a \otimes b \otimes c \longmapsto (ab)c - a(bc)$ to show associativity. Similarly, if $a \in P(A)$, then $a^p \in P(A)$ as well and has image zero in $Q(A)$, thus by injectivity $a^p = 0$. Then by induction, if $(A_m)^p = 0$ for $m \in \mathbb{N}_{n-1}$, then $(A_n)^p \subset P(A)_{pn}$ and $(A_n)^p$ has image zero in $Q(A)_{pn}$, that is $(A_n)^p = 0$.
The converse is proved by induction on the number of generators and is easily checked for one generator $x \in A_n$. If $p = 2$ or p, n are odd, $x^p = 0$ implies that $P(A) = I(A) = Q(A)$. If p is odd and n even, then $A_q = 0$ for $q \neq kn, k = 0, \ldots, p-1$ and $\Delta(x^k) = \sum_{j=0}^{k} \binom{k}{j} x^j \otimes x^{k-j}$, then as the characteristic is p, we have $P(A)_n = A_n = Q(A)_n$ and $P(A)_q = 0 = Q(A)_q$, whence the injection. □

In order to know what happens to the primitives and indecomposable when dualizing, we first need a technical lemma.

Lemma 3.7.
Let $(A, \varphi, \Delta, \varepsilon, \eta)$ be a primitively generated Hopf algebra which is free of finite type and let A^* be its dual Hopf algebra. Then

(1) *the dual of a generator is indecomposable in A^*;*
(2) *the natural morphism $P(A^*) \longrightarrow Q(A^*)$ is injective.*

Proof.
(1) Consider the evaluation map

$$\begin{array}{rccc} <\,,\,>: & A^* \otimes A & \longrightarrow & \mathbb{K} \\ & x^* \otimes a & \longmapsto & < x^*, a >= x^*(a)\,. \end{array}$$

Let x be a primitive generator of A. Then $< x^*, x >= 1$ and $\Delta(x) = x \otimes 1 + 1 \otimes x$. Now suppose, ab absurdo, that $x^* = a^* \cdot b^*$ with $a^*, b^* \in A^*$ and "$\cdot$" being the multiplication in A^*, that is the map $\Delta^*\alpha$. Then

$$\begin{aligned} 1 &=< x^*, x >=< a^* \cdot b^*, x >=< \Delta^*\alpha(a^* \otimes b^*), x >=< \alpha(a^* \otimes b^*) \circ \Delta, x > \\ &=< \alpha(a^* \otimes b^*), \Delta(x) >=< \alpha(a^* \otimes b^*), x \otimes 1 + 1 \otimes x > \\ &=< \alpha(a^* \otimes b^*), x \otimes 1 > + < \alpha(a^* \otimes b^*), 1 \otimes x > \\ &=< a^*, x >< b^*, 1 > + < a^*, 1 >< b^*, x > \ . \end{aligned}$$

But, A being connected, either a^* or b^* has to be in $A_0^* \cong \mathbb{K}$, otherwise the last term of the equality is 0 . Thus x^* couldn't be decomposable.

(2) In other words, (1) says that there is a natural injection $Q(A)^* \hookrightarrow Q(A^*)$. It remains to see that $P(A^*)$ injects in $Q(A)^*$, which doesn't require the primitive generation hypothesis and is easily showed as follows. The evaluation map introduced above restricts to $<\,,\,>: P(A^*) \otimes IA \longrightarrow \mathbb{K}$. Let $x^* \in P(A^*)$ and $a, b \in IA$, then

$$\begin{aligned} < x^*, \varphi(a \otimes b) > &=< x^* \circ \varphi, a \otimes b >=< \alpha\alpha^{-1}\varphi^*(x^*), a \otimes b > \\ &=< \alpha(x^* \otimes \mathrm{Id}_{\mathbb{K}} + \mathrm{Id}_{\mathbb{K}} \otimes x^*), a \otimes b > \\ &=< x^*, a >< \mathrm{Id}_{\mathbb{K}}, b > + < \mathrm{Id}_{\mathbb{K}}, a >< x^*, b >= 0 \end{aligned}$$

since $a, b \in IA = A_{>0}$. Thus we have a well-defined induced evaluation map

$$\begin{array}{rccc} <,>: & P(A^*) \otimes Q(A) & \longrightarrow & \mathbb{K} \\ & x^* \otimes a & \longmapsto & < x^*, \bar{a} >= x^*(a)\,. \end{array}$$

Then the map

$$\begin{array}{rccc} \Phi: & P(A^*) & \longrightarrow & Q(A)^* \\ & x^* & \longmapsto & < x^*, - > \end{array}$$

shall furnish us with the required monomorphism.
Let $x^* \in P(A^*)$. If $< x^*, \bar{a} >= 0$ for all $\bar{a} \in Q(A)$ then $< x^*, a >= 0$ for all $a \in IA$. But A being connected we have $|x^*| > 0$, thus $x^* = 0$, which proves that Φ is injective.
Thus we have $P(A^*) \hookrightarrow Q(A)^* \hookrightarrow Q(A^*)$.

□

Remark 3.8.

(1) First notice that the connectedness hypothesis is not fancy and decorative, but is crucial.
(2) The map Φ is actually an isomorphism. A^* being of finite type, to see that it is surjective, one can, for example show, that if $< x^*, \bar{a} >= 0$ for all $x^* \in P(A^*)$ then $\bar{a} = 0$ necessarily. But this is in fact a more general result stated in [MM65, theorem 3.10], and proved more conceptually, which says that: *if A is a connected Hopf algebra which is free of finite type and A^* is its dual Hopf algebra, then the primitive and the indecomposable elements are dual to each other: $P(A^*) = Q(A)^*$ and $P(A^*)^* = Q(A)$.*

The lemma allows us to restate, to some extent, proposition 3.6 of Milnor and Moore in terms of the dual Hopf algebra.

Corollary 3.9.
Let A be a connected Hopf algebra of finite type over the field $\mathbb{F}_p$ and let A^* be its dual Hopf algebra. Then A is primitively generated, as an algebra, if and only if A^* is commutative and has only trivial p^{th} powers.

Proof. Assume that A is primitively generated, then the lemma and proposition 3.7 yield the result.
Conversely, if A^* is commutative and has only trivial p^{th} powers, then proposition 3.7 implies that $P(A^*) \hookrightarrow Q(A^*)$. Thus, by remark 3.8 (2), going to the dual gives an injection $Q(A) \hookrightarrow P(A)$, i.e. A is primitively generated as an algebra. □

Furthermore, if the Hopf algebra is cocommutative, its dual is commutative and we obtain a nice criterion for p^{th} powers to be trivial in the dual:

Corollary 3.10.
Let A be a connected, cocomutative Hopf algebra of finite type over the field $\mathbb{F}_p$. Then A is primitively generated, as an algebra, if and only if all p^{th} powers vanish in A^* .

1.3. Hopf algebras in Homology and Cohomology.

Let us look at cohomology first. Let $\mathbb{K}$ be a commutative ring, then $H^*(X;\mathbb{K})$ is a graded commutative ring under cup product, but it may as well be regarded as an algebra over $\mathbb{K}$ rather than merely a ring. Now assume that (X,μ) is an H-space satisfying:

(1) X is path-connected, so that $H^0(X;\mathbb{K}) \cong \mathbb{K}$;
(2) $H^q(X,\mathbb{K})$ is a free $\mathbb{K}$-module of finite type for all q, so that the cross product $\gamma : H^*(X;\mathbb{K}) \otimes H^*(X;\mathbb{K}) \longrightarrow H^*(X \times X;\mathbb{K})$ is an isomophism.

In that case, the properties of the product $\mu : X \times X \longrightarrow X$ allows to endow $H^*(X;\mathbb{K})$ with a comultiplication Δ^* which is defined to be the graded morphism

$$\Delta^* : H^*(X;\mathbb{K}) \xrightarrow{\mu^*} H^*(X \times X;\mathbb{K}) \xrightarrow[\cong]{\gamma^{-1}} H^*(X;\mathbb{K}) \otimes H^*(X;\mathbb{K}) .$$

Then $H^*(X,\mathbb{K})$ together with cup product and Δ^* becomes a connected, associative, commutative quasi-Hopf algebra of finite type; the counit being the graded morphism $\varepsilon : H^*(X;\mathbb{K}) \longrightarrow H^*(*;\mathbb{K}) \cong \mathbb{K}$ induced by the inclusion $* \hookrightarrow X$. To be more accurate, the coassociativity condition is not automatic, but it is satisfied when the product μ is homotopy associative. Proofs of these facts can be found in [Hat02, Chapter 3] or in [Rot88, Chapter 12].

Let us now look at homology. Still assume that (X,μ) is an H-space satisfying conditions (1) and such that $H_q(X,\mathbb{K})$ is a free $\mathbb{K}$-module of finite type for all q, so that the homology cross product is an isomorphism. Then its homology groups also have a multiplication operation φ_*, called the *Pontrijagin product*, it is defined by the composition of the cross product with the induced homomorphism in homology by μ:

$$\varphi_* : H_*(X;\mathbb{K}) \otimes H_*(X;\mathbb{K}) \xrightarrow{\cong} H_*(X \times X;\mathbb{K}) \xrightarrow{\mu_*} H_*(X;\mathbb{K})$$

This multiplication happens to be associative when the product μ is homotopy associative. Similarly as cup products in cohomology can be obtained from the diagonal map $\Delta : X \longrightarrow X \times X$, there is a comultiplication in homology, which by abuse of notation we denote by Δ_* and which is induced by the diagonal and defined to be the compostion:

$$H_*(X;\mathbb{K}) \xrightarrow{\Delta_*} H_*(X \times X;\mathbb{K}) \xrightarrow{\cong} H_*(X,\mathbb{K}) \otimes H_*(X;\mathbb{K}) .$$

Then $H_*(X,\mathbb{K})$ together with the morphisms φ_* and Δ_* becomes a connected, coassociative Hopf algebra of finite type, not necessarily associative though.

Additionally, since homology is assumed to be free of finite type, the universal coefficient theorem yields an isomorphism $H^q(X;\mathbb{K}) \cong Hom_{\mathbb{K}}(H_q(X;\mathbb{K});\mathbb{K})$, that is $H_*(X;\mathbb{K})$ and $H^*(X;\mathbb{K})$ are dual graded $\mathbb{K}$-modules. In fact, the cup product and Δ_*, as well as the Pontrijagin product and Δ^* are dual to each other and $H_*(X;\mathbb{K})$ and $H^*(X;\mathbb{K})$ are dual as Hopf algebras.

We finally notice that, in particular, when X is an H-group, as for instance a loop-space, the associated product μ is homotopy associative, so that $H_*(X;\mathbb{K})$ and $H^*(X;\mathbb{K})$ are dual as Hopf algebras, which are associative and coassociative.

2. Cohomology Suspension

Let $(X,*)$ be pointed topological space and let us consider the pathspace fibration $\Omega X \longrightarrow PX \xrightarrow{p} X$ where the map p is the evaluation in 1, i.e. explicitly $p : u \in PX = X^{[0,1]} \longmapsto u(1)$. It is well-known that he pathspace PX is contractible. (A proof of this fact can be found, for example, in [Whi78, Chapter 2].)

In cohomology there is a homomorphism of particular interest, that connects the cohomology of the loop space ΩX to the cohomology of the space X itself. It is constructed as follows: first, the long exact sequence of the pair $(PX, \Omega X)$

$$\cdots \to H^q(PX,\Omega X;G) \to \underbrace{\widetilde{H}^q(PX;G)}_{=0} \to \widetilde{H}^q(\Omega X;G) \xrightarrow{\partial} H^{q+1}(PX,\Omega X;G) \to \underbrace{\widetilde{H}^{q+1}(PX;G)}_{=0} \to \cdots$$

gives an isomorphism $\widetilde{H}^q(\Omega X;G) \xrightarrow[\cong]{\partial} H^{q+1}(PX,\Omega X;G)$ for all q. Then looking at the map p as a map of pairs $p : (PX,\Omega X) \longrightarrow (X,*)$, we get an induced homomorphism in cohomology

$$p^* : H^{q+1}(PX,\Omega X;G) \longrightarrow H^{q+1}(X,*;G)$$

for all q. Therefore, combining both, we get the desired homomorphism:

$$\widetilde{H}^{q+1}(X;G) \cong H^{q+1}(X,*;G) \xrightarrow{p^*} H^{q+1}(PX,\Omega X;G) \xleftarrow[\cong]{\partial} \widetilde{H}^q(\Omega X;G)$$

which lowers dimension by one. In literature, it is often denominated *cohomology suspension*.

In the next section we show that for sufficiently nice co-H-spaces, cohomology suspension is an injection, that is to say, that the map p^* is an injection.

2.1. Intrusion in the world of Lusternik-Schnirelmann category.

The purpose of this section is to show that for a normal, path-connected co-H-space whose base point has an open contractible neighbourhood, the cohomology suspension homomorphism introduced above is actually injective. It can seem inappropriate to introduce a new theory but in the end we shall only have reformulated the definition of a co-H-space through an equivalent characterization that provides us with a better viewpoint to look at cohomology suspension. This viewpoint happens to be the vast theory of Lusternik-Schnirelmann category. The reference for the notions presented below is [CLOT03], where all the proofs of the cited propositions can be found. The basic definition is the following.

Definition 3.11.
The *Lusternik-Schnirelmann (or LS) category* of a topological space X, denoted $\mathrm{cat}(X)$, is the least integer $n \in \mathbb{N}$ such that there exists an open covering $\mathcal{U}_1, \ldots, \mathcal{U}_{n+1}$ of X with each $\mathcal{U}_i$ contractible to a point *in* X. In case no such integer exists, we write $\mathrm{cat}(X) = \infty$.

G.W. Whitehead has given another definition of category, which we are most interested in, because, to some extent, it generalises the notion of co-H-space. First recall that for a pointed space $(X, *)$ the *fat wedge* of X is defined to be the pointed space:

$$T^m(X) := \{(x_1, \ldots, x_n) \in X^n \mid x_j = * \text{ for at least one } j \in \mathbb{N}_n\}$$

Definition 3.12.
The *Whitehead category* of a based topological space $(X, *)$, denoted $\mathrm{cat}^{\mathrm{Wh}}(X)$, is the least integer n such that there exist a continuous map $\Delta' : X \longrightarrow T^{n+1}(X)$ which makes the diagram

$$\begin{array}{ccc} & & T^{n+1}(X) \\ & \overset{\exists \Delta'}{\nearrow} & \big\downarrow {\scriptstyle j_{n+1}} \\ X & \xrightarrow[\Delta]{} & X^{n+1} \end{array}$$

homotopy commute. The map Δ being the $(n+1)$-fold diagonal.

Note that since $T^2(X)$ is $X \vee X$, a space has Whitehead category precisely when it is a co-H-space. A link between LS-category and Whitehead category is given by the following proposition.

Proposition 3.13.
Let $(X, *)$ be a path-connected pointed topological space.

(1) *If X is normal, then* $\mathrm{cat}^{\mathrm{Wh}}(X) \leq \mathrm{cat}(X)$.
(2) *If the base point has a contractible open neighbourhood, then* $\mathrm{cat}^{\mathrm{Wh}}(X) \geq \mathrm{cat}(X)$.

There is yet a third characterisation for category, that will provide us with a better viewpoint to look at cohomology suspension

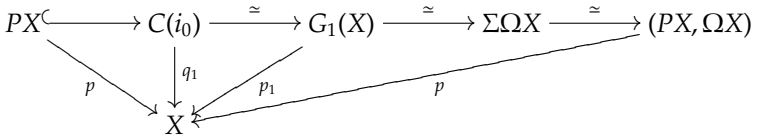
Proposition 3.14.
Let $(X, *)$ be a normal, path-connected, pointed topological space such that the base point $*$ has a contractible open neighbourhood. Consider the homotopy-pullback

$$\begin{array}{ccc} \widetilde{G}_n(X) & \xrightarrow{\delta_n} & T^{n+1}(X) \\ {\scriptstyle \exists s_n ?}\big\downarrow {\scriptstyle \widetilde{p}_n} & & \big\downarrow {\scriptstyle j_{n+1}} \\ X & \xrightarrow[\Delta]{} & X^{n+1}(X)\,. \end{array}$$

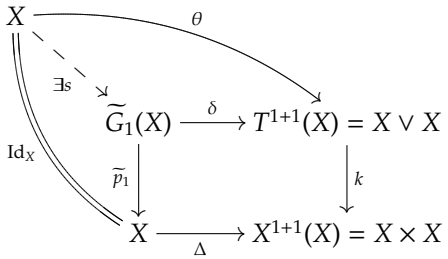
Then $\mathrm{cat}(X) = n$ *exactely when* n *is the least integer such that there exist a section* $s_n : X \longrightarrow \widetilde{G}_n(X)$ *for the* n^{th} *fibration* $\widetilde{p}_n$. *We shall denote this number by* $\mathrm{cat}^{\mathrm{G}}(X)$.

This new viewpoint was given by Tudor Ganea, who has given an explicite construction of spaces $G_n(X) \simeq \widetilde{G}_n(X)$.

Now, let $(X, *)$ be a co-H-space which is normal, path-connected and such that the base point $*$ has a contractible open neighbourhood. (For instance a path-connected CW-complexe.) From 3.13 and 3.14, we have $\mathrm{cat}(X) = \mathrm{cat}^{\mathrm{Wh}}(X) = \mathrm{cat}^{\mathrm{G}}(X)$. In order to understand why the cohomology suspension homomorphism associated to X is injective, we need to study the first steps of Ganea's construction. The zero-th *Ganea fibration* $F_0(X) \xrightarrow{i_0} G_0(X) \xrightarrow{p_0} X$ is defined to be the pathspace fibration $\Omega X \xrightarrow{i_0} PX \xrightarrow{p_0=p} X$. Then the Ganea fibration $F_{n+1}(X) \xrightarrow{i_{n+1}} G_{n+1}(X) \xrightarrow{p_{n+1}} X$, $n \geq 0$ is constructed inductively from the fibration $F_n(X) \xrightarrow{i_n} G_n(X) \xrightarrow{p_n} X$. Namely, let $C(i_n) = G_n(X) \cup_{F_n(X)} C(F_n(X))$ be the cofiber of the map $i_n : F_n(X) \longrightarrow G_n(X)$, set $q_n : C(i_n) \longrightarrow X$ to be the map defined by $q_n(x) = p_n(x)$ for $x \in G_n(X)$ and $q_n(x) = *$ for $x \in C(F_n(X))$, and finally define $G_{n+1}(X) := C(i_n) \times_X X^I \simeq C(i_n)$ and the associated fibration $p_{n+1} : G_{n+1}(X) \longrightarrow X$ is defined by the composition of q_n with the last-mentioned homotopy equivalence. In [CLOT03, Chapter 2], it is shown that for all n there is a homotopy equivalence $f_n : G_n(X) \longrightarrow \widetilde{G}_n(X)$ such that $\widetilde{p}_n f = p_n$ and as a result, $\mathrm{cat}^{\mathrm{G}}(X) = n$ when n is the least integer such that there exist a section for the n^{th} Ganea fibration $p_n : G_n(X) \longrightarrow X$. Moreover, it is also shown in [CLOT03] that $G_1(X)$ has the homotopy type of the space $\Sigma\Omega X$, which itself as a pointed space $(\Sigma\Omega X, *)$ has the same homotopy type as the pair $(PX, \Omega X)$. Then, the map of pairs $p : (PX, \Omega X) \longrightarrow (X, *)$ used to define the cohomology suspension homomorphism is the right-hand one defined by composition in the following commutative diagram of pointed spaces:



Since X is a co-H-space, we have $\mathrm{cat}(X) = \mathrm{cat}^{\mathrm{Wh}}(X) = \mathrm{cat}^{\mathrm{G}}(X) = 1$. Then theorem 3.14 yields a section for $\widetilde{p}_1$ as follows



Thus we have a homotopy section for p_1 and in consequence also one for $p : (PX, \Omega X) \longrightarrow (X, *)$. Therefore the induced map in cohomology p_* has a strict left-inverse, which means that it is injective.

Other notions of category have been defined. We shortly introduce the *strong category* and describes how the spaces K_1 to K_{p-2} built in theorem 4.1 become interesting in this theory.

Definition 3.15.

(1) The *geometric category* of a topological space X, denoted $\mathrm{gcat}(X)$, is the least integer $m \in \mathbb{N}$ such that there exists a covering of X with $m+1$ open, contractible subsets.
(2) The *strong Lusternik-Schnirelmann category* of a topological space X, denoted $\mathrm{Cat}(X)$, is the number $\min\{\mathrm{gcat}(Y) \,|\, Y \simeq X\}$. It is obviously a homotopy invariant.

An important property of strong category is that it characterizes suspensions, that is it can be shown that:

$$\mathrm{Cat}(X) = 1 \text{ if and only if } X \simeq \Sigma Z \text{ for some topological space } Z\,.$$

Another property, proved by Ganea, says that for any path-connected ANR space X, LS category and strong category differ by at most one, i.e.

$$\mathrm{cat}(X) \leq \mathrm{Cat}(X) \leq \mathrm{cat}(X) + 1\,.$$

Thus the question raised by this result is whether the strong category is in fact equal to the LS-category. In fact, it has been settled by Berstein [Ber64] that it is not the case. But it is interesting to notice that the spaces K_1 to K_{p-2} built in theorem 4.1 are other counterexamples. In theorem 4.4, we shall show that they don't have the homotopy type of a suspension, therefore $\mathrm{Cat}(K_j) \neq 1$ for all $j \in \mathbb{N}_{p-2}$, but on the other hand since they are path-connected co-H-spaces but not contractible we have $\mathrm{cat}(K_j) = \mathrm{cat}^{\mathrm{Wh}}(K_j) = 1$ for all $j \in \mathbb{N}_{p-2}$.

3. Steenrod Reduced Powers

The present section briefly introduces the Steenrod reduced powers which are cohomology operations for the special case of cohomology with coefficient taken modulo a prime $p > 2$. The definitive reference, in which all the result cited above are proved, is the lectures notes [Ste62] by N.E. Steenrod himself. Another good description is given by Hatcher in [Hat02].

Definition 3.16.
A *cohomology operation* of type (G, n, G', m) is, by definition, a natural transformation $\theta : H^n(\,;G) \longrightarrow H^m(\,;G')$. That is, for all $X, Y \in \mathcal{T}op$, $f \in \mathcal{T}op(X, Y)$ there are functions

θ_X, θ_Y such that the following diagram commutes:

$$\begin{array}{ccc} H^n(X;G) & \xrightarrow{\theta_X} & H^m(X;G') \\ \uparrow{\scriptstyle f^*} & \circlearrowleft & \uparrow{\scriptstyle f^*} \\ H^m(Y;G) & \xrightarrow[\theta_Y]{} & H^m(Y;G') \end{array}$$

Let $p > 2$ be an odd prime. The *Steenrod reduced powers* can be defined axiomatically in the following way:

(1) For all $i \geq 0$, there is a natural transformation
$$\mathcal{P}^i \,:\, H^q(X;\mathbb{F}_p) \longrightarrow H^{q+2i(p-1)}(X;\mathbb{F}_p)$$
which is a group homorphism. In other terms, the $\mathcal{P}^i$'s commute with induced homomorphisms as in the definition above.
(2) $\mathcal{P}^0$ is the identity map.
(3) if $|x| = 2i$, $i \geq 1$, then $\mathcal{P}^i x = x^p$;
(4) if $2i > |x|$, $i \geq 1$, then $\mathcal{P}^i x = 0$;
(5) for all $x, y \in H^*(X;\mathbb{F}_p)$ the Cartan formula holds:
$$\mathcal{P}^i(x \smile y) = \sum_{j=0}^{i} \mathcal{P}^j x \smile \mathcal{P}^{i-j} y$$

The existence of these operations is asserted by the explicite construction made by Steenrod in [Ste62]. Moreover, Serre and Cartan have showed that Steenrod's constructions gave all the possible stable cohomology operations over the field $\mathbb{F}_p$.

Properties 3.17.

(1) *The Bockstein homomorphism $\beta : H^q(X;\mathbb{F}_p) \longrightarrow H^{q+1}(X;\mathbb{F}_p)$, built from the short exact sequence of coefficients $0 \to \mathbb{Z}/p\mathbb{Z} \xhookrightarrow{\cdot p} \mathbb{Z}/p^2\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p\mathbb{Z} \to 0$, and the Steenrod reduced powers satisfy the following relations called the* Adem relations*:*
$$\mathcal{P}^a\mathcal{P}^b = \sum_{j=0}^{[a/p]} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} \mathcal{P}^{a+b-j}\mathcal{P}^j \qquad \text{if } a < bp$$
$$\mathcal{P}^a\beta\mathcal{P}^b = \sum_{j=0}^{[a/p]} (-1)^{a+j} \binom{(p-1)(b-j)}{a-pj} \beta\mathcal{P}^{a+b-j}\mathcal{P}^j + \sum_{j=0}^{[(a-1)/p]} (-1)^{a+j-1} \binom{(p-1)(b-j)-1}{a-pj-1} \mathcal{P}^{a+b-j}\beta\mathcal{P}^j \qquad \text{if } a \leq b$$
(2) *The $\mathcal{P}^i$'s commute with the connecting homomorphisms of the long exact sequence for pairs (X, A): $\partial : H^i(X,\mathbb{F}_p) \longrightarrow H^{i+1}(X, A;\mathbb{F}_p)$.*
(3) *The $\mathcal{P}^i$'s commute with the suspension isomorphism $\Sigma : \widetilde{H}^q(X) \longrightarrow \widetilde{H}^{q+1}(\Sigma X)$.*
(3) *The $\mathcal{P}^i$'s commute with the cohomology suspension homomorphism $\sigma^* : \widetilde{H}^q(X) \longrightarrow \widetilde{H}^{q+1}(\Omega X)$.*

Proof. As already said, these lengthy proofs can be found in [Ste62] and [Hat02]. For the third affirmation, we recall that for a nondegenerately pointed space the suspension isomorphism is obtained from the characterisation of the suspension $\Sigma X \simeq C_+X/X$. For C_+X being contractible, the long exact sequence of the pair (C_+X, X) provides an isomorphism $\partial : \widetilde{H}^q(X) \xrightarrow{\cong} H^{q+1}(C_+X, X)$ (where ∂ is the connecting homomorphism) and, in addition, $H^{q+1}(C_+X, X) \cong H^{q+1}(C_+X/X, *) \cong \widetilde{H}^{q+1}(C_+X/X) = \widetilde{H}^{q+1}(\Sigma X)$. As a result, (3) and (4) follow from (2) and axiom (1). □

Definition 3.18.
The modulo p *Steenrod algebra* $\mathcal{A}_p$ is defined to be the graded associative algebra over $\mathbb{F}_p$ generated by the reduced powers $\mathcal{P}^i$, $i \geq 0$, and the Bockstein homomorphism β modulo the Adem relations, $\beta^2 = 0$ and $\mathcal{P}^0 = 1$. The elements of degree k being those that map $H^q(X; \mathbb{F}_p)$ to $H^{q+k}(X; \mathbb{F}_p)$, $q, k \geq 0$.

It results from the axioms that for all spaces X, we can consider $H^*(X; \mathbb{F}_p)$ as a module over the Steenrod algebra.

Remark 3.19.
Using the operation $\mathcal{P} := \mathcal{P}^0 + \mathcal{P}^1 + \mathcal{P}^2 + \mathcal{P}^3 + \cdots$, which acts on $H^*(X; \mathbb{F}_p)$ since by axiom (4) only a finite number of $\mathcal{P}^j$ are nonzero, we can compute the action of $\mathcal{P}^i$ on $H^*(\mathbb{C}P^\infty; \mathbb{F}_p) \cong \mathbb{F}_p[\alpha]$ with $|\alpha| = 2$. First, axioms (2) and (4) yield $\mathcal{P}(\alpha) = \alpha + \alpha^p = \alpha(1 + \alpha^{p-1})$. Then the Cartan formula yields (by induction) $\mathcal{P}(\alpha^n) = \mathcal{P}(\alpha)^n$ for all $n \geq 1$ so that

$$\mathcal{P}(\alpha^n) = \alpha^n(1 + \alpha^{p-1})^n = \sum_i \binom{n}{i} \alpha^{n+i(p-1)}.$$

Thus comparing degrees we get for all $i, n \geq 0$

$$\mathcal{P}^i(\alpha^n) = \binom{n}{i} \alpha^{n+i(p-1)}$$

where the binomial coefficient are taken modulo p.

4. Spectral Sequences

The last tool we introduce is the Serre spectral sequence for cohomology. We shall only described the basic properties without giving any proof, but only examples of applications. The references are [McC01] and [Hat04] where all the proofs can be found.

Definition 3.20.
Let $\mathbb{K}$ be a commutative ring. A *spectral sequence of cohomological type* is a collection of differential bigraded $\mathbb{K}$-modules $\{E_r^{*,*}, d_r\}$, $r \geq 1$, where the differentials d_r are all of bidegree $(r, 1-r)$ and for all p, q, r, $E_{r+1}^{p,q}$ is isomorphic to

$$H^{p,q}(E_r^{*,*}, d_r) = \ker(d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}) / \operatorname{Im}(d_r : E_r^{p-r,q+r-1} \to E_r^{p,q}).$$

Adding multiplicative structure, a *spectral sequence of algebras* over $\mathbb{K}$ is a spectral sequence $\{E_r^{*,*}, d_r\}$ together with multiplications $\varphi_r : E_r^{*,*} \otimes_{\mathbb{K}} E_r^{*,*} \longrightarrow E_r^{*,*}$ for all $r \geq 1$, which are morphisms of bigraded $\mathbb{K}$-modules such that φ_{r+1} can be computed from φ_r as the composite map:

$$\varphi_{r+1} : E_{r+1}^{*,*} \otimes E_{r+1}^{*,*} \cong H^{*,*}(E_r) \otimes H^{*,*}(E_r) \xrightarrow{\psi} H^{*,*}(E_r \otimes E_r) \xrightarrow{H^{*,*}(\varphi_r)} H^{*,*}(E_r) \cong E_r^{*,*}$$

where ψ is the homomorphism given by $\psi([u] \otimes [v]) = [u \otimes v]$.

Spectral sequences, called the Serre spectral sequences, arise in topology when working with fibrations. Their existence is described through the next theorem. All the given hypotheses are not necessary for the existence statement, but they will provide us with the nice properties listed bellow.

A helpful image to bear in mind is to see the bigraded modules $E_r^{*,*}$ in a Serre spectral sequence as the pages of a book, each consisting of $\mathbb{K}$-modules indexed on $\mathbb{Z} \times \mathbb{Z}$, but with only the entries in the first quadrant that can be nonzero. So we shall call $E_r^{*,*}$ the *E_r-page* of the spectral sequence. The $\mathbb{K}$-modules and the diffrentials d_r on the r^{th} page form chain complexes and the cohomology groups of these chain complexes are the $\mathbb{K}$-modules appearing in the $(r+1)^{\text{st}}$ page.

In a given page, the $\mathbb{K}$-modules located in the lower left corner of the first quadrant are such that the differentials leaving them and joining them come from *and* go to trivial modules because they are outside the first quadrant. This means that the modules in these positions remain unchanged in the higher-degree pages. If one focuses on the module at the (p, q) position, there will eventually be a degree r for which the differentials entering and leaving it will both be zero, so that it remains unchanged in the pages of degree higher than r. According to this observation, there is a well-defined limiting page for the spectral sequence, that we shall call the E_∞-page.

Theorem 3.21.
Let $\mathbb{K}$ be a commutative ring. Let $F \longrightarrow X \longrightarrow B$ be a fibration such that B is path-connected, F is connected and $\pi_1(B) = 0$. Then there is a spectral sequence of algebras $\{E_r^{p,q}, d_r\}$ with $p, q \geq 0$, called the Serre spectral sequence*, which satisfies*

1. *$\{E_r^{p,q}, d_r\}$ converges as an algebra to $H^*(X; \mathbb{K})$. More accurately, the stable terms $E_\infty^{p,q}$ are isomorphic to the successive quotients F_p^n / F_{p+1}^n in a filtration $0 \subset F_n^n \subset \cdots \subset F_0^n$ of $H^{p+q}(X; \mathbb{K})$;*
2. *$E_2^{p,q} \cong H^p(B; H^q(F; \mathbb{K}))$;*
3. *each differential d_r is a graded derivation: $d_r(xy) = d_r(x)y + (-1)^{p+q} x d_r(y)$ for $x \in E_r^{p,q}$;*

(4) *the product structure on $E_2^{*,*}$ restricted to $E_2^{*,0}$ and $E_2^{0,*}$ coincides with the cup product structures on $H^*(B;\mathbb{K})$ and $H^*(F;\mathbb{K})$ respectively;*
(5) *if, for all i, $H^i(F;\mathbb{K})$ and $H^i(B;\mathbb{K})$ are free $\mathbb{K}$-modules of finite type, then*

$$E_2^{p,q} \cong H^p(B;\mathbb{K}) \otimes_{\mathbb{K}} H^q(F;\mathbb{K}) .$$

Proof. See [McC01, Chapter 5] □

Remark 3.22.
Notice that when the space X is contractible, as for instance when the fibration is the pathspace fibration $\Omega B \longrightarrow PB \longrightarrow B$, then the E_∞-page is given by

$$E_\infty^{p,q} = H^{p+q}(*;\mathbb{K}) = \begin{cases} \mathbb{K} & \text{if } p = q = 0; \\ 0 & \text{otherwise.} \end{cases}$$

In order to make all this theory a little more concrete, we go into two examples, which should illustrate the basic usage one can make of Serre spectral sequences to deduces ring structures in cohomology. We first compute the cup product structure on $H^*(\mathbb{C}P^\infty)$.

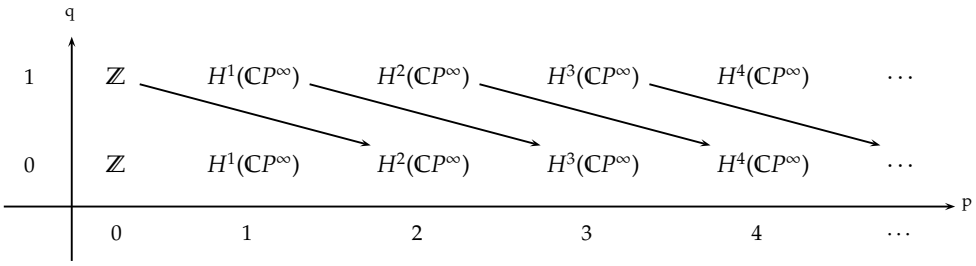
Example 3.23 (Ring structure of $H^*(\mathbb{C}P^\infty)$).
Let us use the fact that $\mathbb{C}P^\infty$ is a $K(\mathbb{Z},2)$ to compute the multiplicative structure on its cohomology ring. Since $K(\mathbb{Z},2)$ is simply connected, we can apply the Serre spectral sequence to the pathspace fibration $K(\mathbb{Z},1) \longrightarrow PK(\mathbb{Z},2) \longrightarrow K(\mathbb{Z},2)$, where $K(\mathbb{Z},1)$ can be identified with $\mathbb{S}^1$, so that we know exactly its cohomology groups, which are:

$$H^i(K(\mathbb{Z},1)) = \begin{cases} \mathbb{Z} & \text{if } i = 0,1 \\ 0 & \text{otherwise.} \end{cases}$$

By part (2) of theorem 3.21, the E_2-page is given by

$$E_2^{p,q} \cong H^p(\mathbb{C}P^\infty; H^q(K(\mathbb{Z},1)) = \begin{cases} H^p(\mathbb{C}P^\infty,\mathbb{Z}) & \text{if } q = 0,1; \\ 0 & \text{otherwise;} \end{cases}$$

i.e. only the two first rows are nonzero.

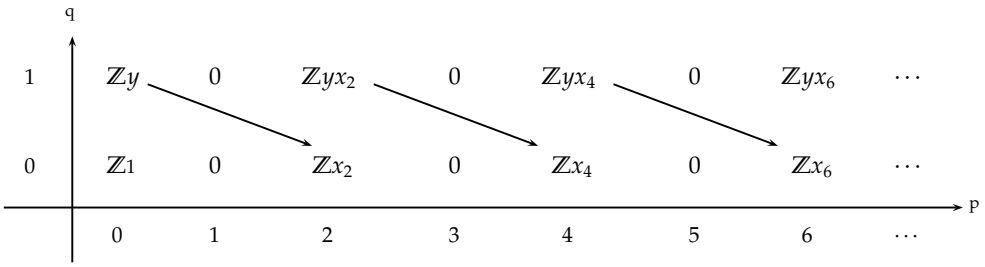


Furthermore, according to remark 3.22 the E_∞-page consists only of trivial groups, except for a $\mathbb{Z}$ in the (0,0) position.

We first determine the additive structure. The differentials $d_3, d_4, \dots$ go downward at least two rows, thus they either come from or go to a trivial group and therefore none of them can be nontrivial, which means that the E_3-page must equal the E_∞-page with just a $\mathbb{Z}$ is the $(0,0)$ position. But the E_3-page is calculated from the E_2-page, thus the groups in the zero-th row are the kernels of the differentials d_2 leaving them. This forces all the differentials d_2 coming from the first row to be isomorphisms, otherwise they would provide a nonzero entry in the zero-th row. In addition, the differential d_2 going to $H^1(\mathbb{C}P^\infty)$ in the $(1,0)$ position comes from a trivial group, thus has image zero, which means that $0 = E_3^{1,0} \cong H^1(\mathbb{C}P^\infty)$. Summing up this argument we have

$$H^i(\mathbb{C}P^\infty) \cong \begin{cases} \mathbb{Z} & \text{if } i \text{ is even;} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

Let us now look at the multiplicative structure of $H^*(\mathbb{C}P^\infty)$. Let y be a generator for $E_2^{0,1} \cong \mathbb{Z}$ and x_{2i} be a generator for $E_2^{2i,0} \cong H^{2i}(\mathbb{C}P^\infty) \cong \mathbb{Z}$, $i \geq 1$, as follows:



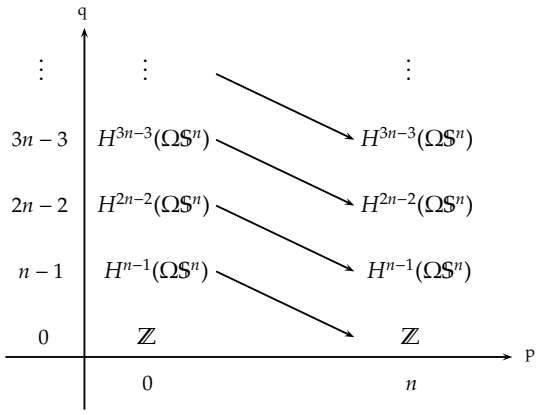
The product $E_2^{0,1} \otimes E_2^{2i,0} \longrightarrow E_2^{s,q+t}$ is multiplication of the coefficients, thus the generators for the $\mathbb{Z}$'s in the first row must be y times the generator in the lower row. The differentials d_2 shown in the figure being isomorphisms, $d_2(y)$ generates $\mathbb{Z}x_2$ so that $d_2(y) = \pm x_2$. Since the differentials are derivations, we have

$$d_2(yx_{2i}) = d_2(y)x_{2i} \pm y\underbrace{d_2(x_{2i})}_{=0} = \pm x_2 x_{2i}$$

Again because d_2 is an isomorphism between $\mathbb{Z}yx_{2i}$ and $\mathbb{Z}x_{2i+2}$, $d_2(yx_{2i})$ must be a generator of $\mathbb{Z}x_{2i+2}$, thus we may assume that $x_2x_{2i} = x_{2i+2}$, changing the sign if necessary. As a result, since the groups $\mathbb{Z}_{x_2 i}$ in the zero-th row are isomorphic to $H^{2i}(\mathbb{C}P^\infty)$, the latter relation means that $H^*(\mathbb{C}P^\infty)$ is isomorphic to the polynomial ring $\mathbb{Z}[x_2]$ with $|x_2| = 2$.
Finally notice that replacing the coefficient ring $\mathbb{Z}$ with the finite field $\mathbb{F}_p$ doesn't change the general method for computations, which can be carried out exactly the same way, to find $H^*(\mathbb{C}P^\infty; \mathbb{F}_p) \cong \mathbb{F}_p[x]$ with $|x| = 2$.

The ring structure of $H^*(\Omega\mathbb{S}^n)$ can be computed in much the same way, just the cup product structure, being a little more complex, illustrates better the power of the derivation property of the differentials.

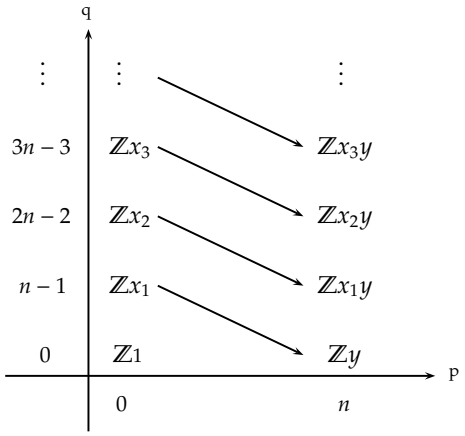
Example 3.24 (Ring structure of $H^*(\Omega\mathbb{S}^n)$).
Let $n \geq 2$, so that $\mathbb{S}^n$ is simply connected. Then, we have a Serre spectral sequence for cohomology associated to the the pathspace fibration $\Omega\mathbb{S}^n \longrightarrow P\mathbb{S}^n \longrightarrow \mathbb{S}^n$. Again the E_∞-page is trivial, except for a $\mathbb{Z}$ is the (0,0) position. From the fact that $H^i(\mathbb{S}^n) \cong \mathbb{Z}$ if $i = 0, n$ and is trivial otherwise and from part (2) of theorem 3.21 the E_2-page has the following form:



Since the differentials $d_2, d_3, \ldots d_{n-1}$ all come from or go to a trivial group we have $E_2^{*,*} \cong E_3^{*,*} \cong \cdots \cong E_n^{*,*}$. In fact, only the differentials d_n can be nontrivial, therefore the E_{n+1}-page must be equal to the E_∞-page. Thus, by a similar argument as in the preceeding example, the d_n's coming from $E_n^{0,q} \cong E_2^{0,q}$ with $q \geq n-1$ are isomorphisms and $H^1(\Omega\mathbb{S}^n) \cong \cdots \cong H^{n-2}(\Omega\mathbb{S}^n) \cong 0$. Which, by induction, leads to

$$H^i(\Omega\mathbb{S}^n) \cong \begin{cases} \mathbb{Z} & \text{if } i = kn - k, k \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The multiplicative structure can then be deduced from the fact that the differentials are derivations. Let x_i be a generator of $H^{in-i}(\Omega\mathbb{S}^n) \cong E_n^{0,in-i} \cong \mathbb{Z}$ and let y be a generator of $E_n^{n,0} \cong \mathbb{Z}$. Analogously to the previous example, the E_n-page has the following form



where the differentials d_n shown in the figure are isomorphisms. Thus we may assume that $d_n(x_1) = y$ and $d_n(x_i) = x_{i-1}y$ for $i > 1$, changing the sign of the x_i's if

necessary. We have to consider separately the cases that n is odd and n is even. First, when n is odd, we have

$$d_n(x_1^2) = d_n(x_1)x_1 + x_1 d_n(x_1) = yx_1 + x_1 y = 2x_1 y$$

since $yx_1 = (-1)^{|y||x_1|}x_1 y = x_1 y$. Thus, d_n being an isomorphism and $d_n(x_2) = x_1 y$ imply that $x_1^2 = 2x_2$. In similar fashion, for $i > 2$, $d_n(x_1^i) = ix_1^{i-1}y$. It follows by induction that $x_1^i = i!x_i$. Therefore $H^*(\Omega\mathbb{S}^n)$ is a divided polynomial algebra $\Gamma_{\mathbb{Z}}[x]$ with $|x| = n-1$.
When n is even, the situation is a little bit more complicated. Since $|x_1|$ is odd, we have $x_1^2 = -x_1^2$, that is $x_1^2 = 0$. Then

$$d_n(x_1x_2) = yx_2 - x_1yx_1 = yx_2 + x_1^2 y = yx_2 = d_n(x_3)$$

that is $x_1x_2 = x_3$ and hence $x_1x_3 = x_1^2x_2 = 0$. Assume by induction on $i > 1$ that $d_n(x_1x_{2i-2}) = x_{2i-1}$ so that $x_1x_{2i-1} = x_1^2x_{2i-2} = 0$, then

$$d_n(x_1x_{2i}) = d_n(x_1)x_{2i} - x_1 d_n(x_{2i}) = yx_{2i} - x_1x_{2i-1} = yx_{2i} = d_n(x_{2i+1})$$

hence $x_1x_{2i} = x_{2i+1}$. Furthermore, we compute for $i > 1$, $d_n(x_2^i) = x_1yx_2^{i-1} + x_2 d_n(x_2^{i-1})$ and inductively $d_n(x_2^i) = ix_1yx_2^{i-1}$. Thus assuming, by induction on $i \geq 1$ that $x_2^{i-1} = (i-1)!x_{2i-2}$, we get

$$d_n(x_2^i) = ix_1yx_2^{i-1} = ix_1y(i-1)!x_{2i-1} = i!x_1x_{2i-2} = i!x_{2i-1}y = i!d_n(x_{2i})\,.$$

Hence $x_2^i = i!x_{2i}$. In conclusion, the relations $x_1^2 = 0$, $x_1x_{2i} = x_{2i+1}$ and $x_2^i = i!x_{2i}$ say that when n is even, $H^*(\Omega\mathbb{S}^n)$ is isomorphic to the tensor product of an exterior algebra with a divided polynomial algebra $\Lambda_{\mathbb{Z}}[x_1] \otimes \Gamma_{\mathbb{Z}}[x_2]$ with $|x_1| = n-1$ and $|x_2| = 2n-2$.

We now restrict our attention to mod p cohomology for p a prime. We quickly introduce the notion of transgression, which will allow us, by means of the Kudo transgression theorem, to compute p^{th} powers of elements $x \in H^{2i}(F;\mathbb{F}_p)$ where F is still the fibre of a fibration $F \longrightarrow X \longrightarrow B$.

Definition 3.25.
Let $\{E_r^{*,*}, d_r\}$ be a spectral sequence of cohomology type and let $u \in E_2^{s,t}$. We say that u is *transgressive* if $d_2(u) = d_3(u) = \ldots = d_t(u) = 0$ and $d_{t+1}(u) \neq 0$ in $E_{t+1}^{s+t+1,0}$. Moreover, we say that u *transgresses* to an element $v \in E_2^{s+t+1,0}$ if v survives to represent $d_{t+1}(u)$.

Kudo Transgression Theorem 3.26.
Let $\{E_r^{*,*}, d_r\}$ be a Serre spectral sequence. Let $x \in E_2^{0,2k} \cong H^{2k}(F;\mathbb{F}_p)$ be transgressive and transgresses to the element represented by y in $E_2^{2k+1,0} \cong H^{2k+1}(B;\mathbb{F}_p)$, then $\mathcal{P}^k x = x^p$ is also transgressive with $d_{2pk+1}(x^p) = \mathcal{P}^k y$.

Proof. Although details are omitted, most of the ideas of the proof are given in [McC01]. A complete proof requires a careful description of the action of the Steenrod algebra at the cochain level. □

CHAPTER 4

The p-local splitting of $\Sigma\mathbb{C}P^\infty$

1. A Foreword on the Infinite Complex Projective Space

The nicest way to see the infinite projective space is certainly to see it through its CW-complex decomposition as $\mathbb{C}P^\infty = e^0 \cup e^2 \cup e^4 \cup \cdots$ with a single cell in each even dimension. We can then use the fact that for a CW-complex having no two of its cells in adjacent dimensions, the n-th homology group is free abelian with basis in one-to-one correspondence with the n-cells (see [Hat02, Chapter 2]). It yields

$$H_n(\mathbb{C}P^\infty) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 2, 4, \dots; \\ 0 & \text{if } n \text{ odd} \end{cases}$$

and consequently

$$\widetilde{H}_n(\mathbb{C}P^\infty) = \begin{cases} \mathbb{Z} & \text{if } n = 2, 4, 6, \dots; \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore the suspension isomorphism $\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(\Sigma X)$ yields

$$\widetilde{H}_n(\Sigma\mathbb{C}P^\infty) = \begin{cases} \mathbb{Z} & \text{if } n = 3, 5, 7, \dots; \\ 0 & \text{otherwise.} \end{cases}$$

We have shown, by means of a Serre spectral sequence, that the cohomology ring $H^*(\mathbb{C}P^\infty; \mathbb{Z})$ is a polynomial ring $\mathbb{Z}[x]$ with x a two-dimensional generator.

Another interesting fact about $\mathbb{C}P^\infty$ is that it happens to be an Eilenberg-McLane space $K(\mathbb{Z}, 2)$. This comes nicely from the fibration $\mathbb{S}^1 \longrightarrow \mathbb{S}^\infty \longrightarrow \mathbb{C}P^\infty$. For its associated long exact sequence in homotopy

$$\cdots \xrightarrow{\cong} \pi_n(\mathbb{S}^1) \longrightarrow \underbrace{\pi_n(\mathbb{S}^\infty)}_{0} \longrightarrow \pi_n(\mathbb{C}P^\infty) \xrightarrow{\cong} \pi_{n-1}(\mathbb{S}^1) \longrightarrow \underbrace{\pi_{n-1}(\mathbb{S}^\infty)}_{0} \longrightarrow \cdots$$

$$\cdots \xrightarrow{\cong} \pi_1(\mathbb{S}^1) \longrightarrow \underbrace{\pi_1(\mathbb{S}^\infty)}_{0} \longrightarrow \pi_1(\mathbb{C}P^\infty) \xrightarrow{\cong} \pi_0(\mathbb{S}^1) \longrightarrow \underbrace{\pi_0(\mathbb{S}^\infty)}_{0} \longrightarrow \underbrace{\pi_0(\mathbb{C}P^\infty)}_{0}$$

yields for all $i \geq 1$

$$\pi_i(\mathbb{C}P^\infty) \cong \pi_{i-1}(\mathbb{S}^1) = \begin{cases} \mathbb{Z} & i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

2. The Actual Splitting

Let p be a fixed odd prime. To begin with we described the p-local splitting of the infinite projective space obtained by C.A. McGibbon in [McG81]. We shall then investigate possible co-H-structures on each piece of the splitting.

THEOREM 4.1.
There is a homotopy equivalence: $\Sigma\mathbb{C}P^{\infty}_{(p)} \simeq \bigvee_{j=1}^{p-1} K_j$ *such that for each* $j \in \mathbb{N}_{p-1}$ *the integral homology of* K_j *is given by*

$$\widetilde{H}_q(K_j, \mathbb{Z}) = \begin{cases} \mathbb{Z}_{(p)} & \text{if } q = 2n+1 \text{ and } n(\geq 1) \equiv j \pmod{p-1} \\ 0 & \text{otherwise.} \end{cases}$$

To help bear in mind this result, we note that pictorially the integral homology of $\Sigma\mathbb{C}P^{\infty}_{(p)}$ is divided according to the following pattern:

	K_1	K_2	K_3	$\cdots$	K_{p-2}	K_{p-1}
H_1						
H_3	$\mathbb{Z}_{(p)}$					
H_5		$\mathbb{Z}_{(p)}$				
H_7			$\mathbb{Z}_{(p)}$			
$\vdots$				$\ddots$		
$H_{2(p-2)+1}$					$\mathbb{Z}_{(p)}$	
$H_{2(p-1)+1}$						$\mathbb{Z}_{(p)}$
H_{2p+1}	$\mathbb{Z}_{(p)}$					
H_{2p+3}		$\mathbb{Z}_{(p)}$				
$\vdots$			$\ddots$			

Here blanks obviously mean that the corresponding homology group is trivial.

REMARK 4.2.
As an immediate consequence of theorem 4.1, the universal coefficient theorem

yields that the cohomology of each K_j with coefficients in $\mathbb{F}_p$ is

$$\widetilde{H}^q(K_j, \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_p & \text{if } q = 2n+1 \text{ and } n(\geq 1) \equiv j \pmod{p-1}; \\ 0 & \text{otherwise.} \end{cases}$$

since $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} \mathbb{F}_p \cong \mathbb{F}_p$. It shall also be useful to notice that, for the same reason, cohomology with $\mathbb{F}_p$ coefficients does not distinguish between the space $\Sigma\mathbb{C}P^\infty$ and its p-localization:

$$H^*(\Sigma\mathbb{C}P^\infty_{(p)}; \mathbb{F}_p) \cong H^*(\Sigma\mathbb{C}P^\infty; \mathbb{F}_p)\,.$$

PROOF OF THEOREM 4.1. ([McG80, proposition 2.2].)
The spaces K_j are constructed as mapping telescopes. From the natural bijection $T : [\mathbb{C}P^\infty, K(\mathbb{Z}, 2) = \mathbb{C}P^\infty] \longrightarrow H^2(\mathbb{C}P^\infty; \mathbb{Z}) \cong \mathbb{Z}$, we see that for each integer $k \in \mathbb{Z}$ we can choose a continuous map $\beta_k : \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty$ with degree[1] k in dimension 2. Let λ be an integer whose equivalence class modulo p is a generator of the group of units $\mathbb{F}_p^*$. Now let θ and ν denote the coproduct and the homotopy inverse induced by the suspension co-H-structure on $\Sigma\mathbb{C}P^\infty$. For continuous self-maps of $\Sigma\mathbb{C}P^\infty$, as described in chapter 1, θ gives raise to a sum operation $f + g = \nabla(f \vee g)\theta$ and ν allows to consider opposites $f\nu$ for this operation $+$. Therefore, for each integer $k \in \mathbb{Z}$, we can define a map $f_k : \Sigma\mathbb{C}P^\infty_{(p)} \longrightarrow \Sigma\mathbb{C}P^\infty_{(p)}$ as the p-localization of the map $\Sigma\beta_\lambda - \lambda^k\Sigma\beta_1$.

Let us compute its degree in dimension $2n+1$, $n \geq 1$. In dimension 2 the map β_1 has degree 1 and the map β_λ has degree λ , thus recalling that $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[x]$ with $|x| = 2$, we see that in all even dimensions $2i$, $i \geq 1$, β_1 has degree 1 and β_λ has degree λ^i. The suspension isomorphism raises dimensions by 1, thus in dimensions $2i+1$, $i \geq 1$, $\Sigma\beta_1$ has degree 1 and $\Sigma\beta_\lambda$ has degree λ^i. In dimensions $2n+1$ the cohomology and homology groups of $\Sigma\mathbb{C}P^\infty$ are isomorphic to $\mathbb{Z}$ and are dual to each other, so that if a self-map of $\Sigma\mathbb{C}P^\infty$ induces multiplication by an integer m on $H^{2n+1}(\Sigma\mathbb{C}P^\infty)$, it induces multiplication by m on $H_{2n+1}(\Sigma\mathbb{C}P^\infty)$ as well, thus the homological and the cohomological degree are equal. In addition, since the induced map in homology by the p-localization $f_{(p)}$ of a continuous map f can be seen as $f_* \otimes \mathrm{Id}_{\mathbb{Z}_{(p)}}$, the degree remains unchanged. In consequence, using the result of remark 1.16 (2) we finally get that the degree of the map $f_k : \Sigma\mathbb{C}P^\infty_{(p)} \longrightarrow \Sigma\mathbb{C}P^\infty_{(p)}$ is $\lambda^n - \lambda^k$ in dimension $2n+1$.

Let $F(j,t) := \{f_k \,|\, 1 \leq k \leq t(p-1) \text{ and } k \not\equiv j \pmod{p-1}\}$. We set K_j to be the infinite mapping telescope of the sequence of maps $\{F(j,1), F(j,2), F(j,3), \ldots\}$, say $\mathcal{T}_j$. We first check that its homology is as stated in 4.1.

First, in dimensions $2n+1$ with $n(\geq 1) \not\equiv j \pmod{p-1}$ the degrees of all the maps in the sequence $\mathcal{T}_j$ are invertible modulo p. For in these dimensions the degrees of the maps f_k in $\mathcal{T}_j$ are $\lambda^n - \lambda^k = \lambda^k(\lambda^{n-k} - 1)$ where k is never j modulo $(p-1)$. Hence $n-k$ is never a multiple of $p-1$ so that $\lambda^{n-k} \not\equiv 1 \pmod{p}$. Moreover, by definition, λ is not a multiple of p, thus neither is λ^k and we finally get that $\lambda^n - \lambda^k$ is not divisible by p and hence is invertible in $\mathbb{F}_p$. Therefore these maps

[1] Here the terms *degree m in dimension n* mean that the induced homomorphism is multiplication by m on the n^th cohomology group.

induce isomorphisms in integral homology and the homology of the mapping telescope, which is isomorphic to the corresponding algebraic colimit (see theorem 2.9), is isomorphic to the homology of the first space $\Sigma\mathbb{C}P^\infty_{(p)}$ in the sequence. That is to say

$$H_{2n+1}(K_j) \cong H_{2n+1}(\Sigma\mathbb{C}P^\infty_{(p)}) \cong \mathbb{Z}_{(p)}$$

for all $n(\geq 1) \not\equiv j \pmod{p-1}$.

In the other dimensions q, an infinite number of f_k have degree divisible by p. Thus every class in $\widetilde{H}_q(\Sigma\mathbb{C}P^\infty_{(p)})$ is mapped to zero by an infinite subsequence of maps in $\mathcal{T}_j$ so that the colimit becomes trivial and $H_q(K_j) = 0$. Let us now check that the required homotopy equivalence holds. Let ι_j be the canonical inclusion of $\Sigma\mathbb{C}P^\infty_{(p)}$ in the left-hand extremity of the mapping telescope K_j. Observe that it induces an epimorphism in homology:

- If q is odd with $q = 2n+1$ and $n \equiv j \pmod{p-1}$ then by construction $(\iota_j)_* : H_q(\Sigma\mathbb{C}P^\infty_{(p)}) \cong \mathbb{Z}_{(p)} \longrightarrow H_q(K_j) \cong \mathbb{Z}_{(p)}$ is an isomorphism.
- If q is odd with $q = 2n+1$ and $n \not\equiv j \pmod{p-1}$ then $H_q(K_j) = 0$ so that $(\iota_j)_* : H_q(\Sigma\mathbb{C}P^\infty_{(p)}) \cong \mathbb{Z}_{(p)} \longrightarrow 0$ is surjective.
- If q is even, then $H_q(K_j) = H_q(\Sigma\mathbb{C}P^\infty_{(p)}) = 0$ and $(\iota_j)_*$ is the map $0 \longrightarrow 0$ which is clearly surjective.

We now wish to use Whitehead's theorem to see that the composition

$$\Sigma\mathbb{C}P^\infty_{(p)} \xrightarrow{\theta_{(p)}} \bigvee_{i=1}^{p-1} \Sigma\mathbb{C}P^\infty_{(p)} \xrightarrow{\vee \iota_j} \bigvee_{i=1}^{p-1} K_i$$

where $\theta_{(p)}$ denotes the iterated coproduct, is a homotopy equivalence. Therefore we have to show that it is an integral homology equivalence. In even dimensions this is clear since homology is trivial. In odd dimensions $q = 2n+1$ with $n \equiv j \pmod{p-1}$ for some $j \in \mathbb{N}_{p-1}$, consider the diagram

$$\begin{array}{ccccccc} H_q(X) & \xrightarrow{\theta_{(p)*}} & H_q(\bigvee_{i=1}^{p-1} X) & \underset{\cong}{\xleftarrow{(j_{1*},\ldots,j_{p-1*})}} & \bigoplus_{i=1}^{p-1} H_q(X) & \xrightarrow{(\iota_{1*},\ldots,\iota_{p-1*})} & \bigoplus_{i=1}^{p-1} H_q(K_i) \cong H_q(K_j) \\ & & & \searrow^{(\iota_1 \vee \ldots \vee \iota_{p-1})_*} & & & \cong \downarrow (j_{1*},\ldots,j_{p-1*}) \\ & & & & & & H_q(\bigvee_{i=1}^{p-1} K_i) \end{array}$$

where X stands for $\Sigma\mathbb{C}P^\infty_{(p)}$. By remark 1.16 (2), the triangle commutes. By part (1) of the same remark, the composition $(j_{1*},\ldots,j_{p-1*})^{-1}\theta_{(p)*}$ is the $(p-1)$-fold diagonal map on $H_q(X)$. Moreover only one of the maps $\iota_{1*},\ldots,\iota_{p-1*}$ is non-trivial, so that the top arrow going from $H_q(X)$ to $H_q(K_j)$ is an isomorphism, which in turn forces $(\iota_1 \vee \ldots \vee \iota_{p-1})_*\theta_{(p)*} = ((\iota_1 \vee \ldots \vee \iota_{p-1})\theta_{(p)})_*$ to be an isomorphism and completes the proof. □

Furthermore, we can define a *linking relation* on a cohomology ring with $\mathbb{F}_p$ coefficients as follows. Let $u, v \in \widetilde{H}^*(Y;\mathbb{F}_p)$ be two generators. Then we say that u and v are *linked* by the Steenrod algebra $\mathcal{A}_p$ if there is a sequence of non-trivial classes $u = x_0, x_1, \ldots, x_n = v$ such that for each $i = 0,\ldots,n-1$, up to a unit of $\mathbb{F}_p$ either $\mathcal{P}^r x_i = x_{i+1}$ or $x_i = \mathcal{P}^r x_{i+1}$ for some reduced power $\mathcal{P}^r$. This clearly defines an equivalence relation.

The next result, which is in fact part (b) of theorem 4.1 given in [McG81], asserts that the splitting of $\Sigma\mathbb{C}P^{\infty}_{(p)} \simeq K_1 \vee \cdots \vee K_{p-1}$ is maximal, in the sense that none of the K_j's could be split in a wedge of two or more terms all with non-trivial cohomology.

Proposition 4.3.
For all $j \in \mathbb{N}_{p-1}$, any two generators $u, v \in \widetilde{H}^*(K_j; \mathbb{F}_p)$ are linked by the Steenrod algebra $\mathcal{A}_p$.

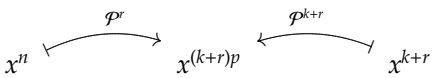
Proof. By definition, Steenrod reduced powers link classes whose degrees are congruent modulo $p-1$, so that the graded isomorphism $\widetilde{H}^*(\mathbb{C}P^{\infty}; \mathbb{F}_p) \cong \bigoplus_{j=1}^{p-1} \widetilde{H}^*(K_j; \mathbb{F}_p)$, provided by the splitting of theorem 4.1, is an isomorphism of modules over the Steenrod Algebra $\mathcal{A}_p$. This amounts to saying that if two nontrivial classes are linked by a Steenrod reduced power in $H^*(\mathbb{C}P^{\infty}; \mathbb{F}_p)$, they correspond to classes in the same summand $\widetilde{H}^*(K_j; \mathbb{F}_p)$. Furthermore, the reduced powers $\mathcal{P}^i$ commute with the suspension isomorphisms $\Sigma_q : H^q(\mathbb{C}P^{\infty}; \mathbb{F}_p) \longrightarrow H^q(\Sigma\mathbb{C}P^{\infty}; \mathbb{F}_p)$. These observations imply that the linking relation partitions $\widetilde{H}^*(\mathbb{C}P^{\infty}; \mathbb{F}_p)$ into at least $p-1$ equivalence classes.
Thus to see that any two generators in $\widetilde{H}^*(K_j; \mathbb{F}_p)$ are linked, we need to show that there are exactly $p-1$ equivalence classes. But since $H^*(\mathbb{C}P^{\infty}; \mathbb{F}_p) \cong \mathbb{F}_p[x]$ with $|x| = 2$, all the generators in $\widetilde{H}^*(K_j; \mathbb{F}_p)$ are, up to a unit of $\mathbb{F}_p$, a power of x and in consequence it suffices to prove that the powers of the generator x lie in at most $p-1$ classes.
It is clear that $x, x^2, \ldots x^{p-1}$ do lie in at most $p-1$ classes, so assume by induction on n that the classes $x, x^2, \ldots x^{n-1}$ with $n \geq p$ lie in at most $p-1$ classes. Write n modulo p, $n = kp + r$ with $k \geq 1$ and $0 \leq r < p$. By remark 3.19 we have

$$\mathcal{P}^r(x^n) = \binom{n}{r} x^{n+r(p-1)} = \binom{n}{r} x^{(k+p)r}$$

where $\binom{n}{r} = \frac{(n-r+1)\cdots n}{r!} = \frac{(kp+1)\cdots(kp+r)}{r!}$ which equals 1 when $r = 0$ and is visibly not divisible by p when $0 < r < p$ so that $\mathcal{P}^r(x^n) = \mu x^{(k+p)r}$ where μ is a unit in $\mathbb{F}_p$. Moreover

$$\mathcal{P}^{k+r}(x^{k+r}) = \binom{k+r}{k+r} x^{k+r+(k+r)(p-1)} = x^{(k+p)r}\,.$$

Thus $\mathcal{P}^r(x^n) = \mu\mathcal{P}^{k+r}(x^{k+r})$, that is x^n and x^{k+r} are linked by the Steenrod algebra according to the following pattern:



Since $k + r < n$, the generator u^n is linked to one of the $(p-1)$ classes described above, which proves the induction and completes the proof. □

3. Co-H-structure on the K_j's

Recall that we assume the prime p to be odd, so that $\Sigma\mathbb{C}P^\infty_{(p)}$ splits into a wedge of at least two spaces. Every space K_j, $j \in \mathbb{N}_{p-1}$ has a co-H-space structure inherited from the suspension co-H-stucture on $\Sigma\mathbb{C}P^\infty$, say θ. Let $\iota_j : K_j :\longrightarrow \bigvee_{i=1}^{p-1} K_j \simeq \Sigma\mathbb{C}P^\infty_{(p)}$ be the canonical inclusion of K_j in $\Sigma\mathbb{C}P^\infty_{(p)}$ and $q_j : \Sigma\mathbb{C}P^\infty_{(p)} \longrightarrow K_j$ be the retraction on the j^{th} factor. Then we have the following coproduct on K_j

$$K_j \xhookrightarrow{\iota_j} \Sigma\mathbb{C}P^\infty_{(p)} \xrightarrow{\theta_{(p)}} \Sigma\mathbb{C}P^\infty_{(p)} \vee \Sigma\mathbb{C}P^\infty_{(p)} \xrightarrow{q_j \vee q_j} K_j \vee K_j\,.$$

The space K_{p-1} is special in the sense that it can be shown that it has the homotopy type of a suspension. This follows from an article by D. Sullivan [Sul74], in which he shows that $B\mathbb{S}^{2n-1}_{(p)}$ is a loop structure on $\mathbb{S}^{2n-1}_{(p)}$ and that there is a map $\pi : \mathbb{C}P^\infty_{(p)} \longrightarrow B\mathbb{S}^{2p-3}_{(p)}$ which induces an isomorphism in cohomology in degrees divisible by $2(p-1)$ and which is zero otherwise. This result is summed up in [McG81], where McGibbon also shows that for p an odd prime and n dividing $p-1$, then $\Sigma B\mathbb{S}^{2n-1}_{(p)}$ is a retract of $\Sigma\mathbb{C}P^\infty_{(p)}$, which he expresses in terms of the splitting as

$$\Sigma B\mathbb{S}^{2n-1}_{(p)} \simeq \bigvee_{i=1}^{(p-1)/n} K_{ni}\,.$$

Thus taking $n = p-1$ yields the result. Our purpose is to study the possible co-H-structures on the other spaces $K_1, \ldots, K_{p-2}$, which are in fact not particularly nice. A first hint of the type of co-H-structures we cannot expect is given by the following theorem.

THEOREM 4.4.
The spaces K_j, $j \in \mathbb{N}_{p-2}$, do not have the homotopy type of a suspension.

PROOF. Let $j \in \mathbb{N}_{p-2}$. Ab absurdo, assume that there is a topological space X_j such that $\Sigma X_j \simeq K_j$. Therefore the cohomology modulo p of X_j, obtained through the suspension isomorphism, is according to remark 4.2:

$$\widetilde{H}^q(X_j;\mathbb{F}_p) \cong \widetilde{H}^{q+1}(K_j;\mathbb{F}_p) = \begin{cases} \mathbb{F}_p & \text{if } q = 2n \text{ and } n(\geq 1) \equiv j \pmod{p-1} \\ 0 & \text{otherwise.} \end{cases}$$

Using, twice, the fact that reduced powers commute with the suspension isomorphism Σ on cohomology, we get the following commutative diagram:

$$\begin{array}{ccccccc}
\mathbb{F}_p < x^j > \cong H^{2j}(\mathbb{C}P^\infty;\mathbb{F}_p) & \xrightarrow[\cong]{\Sigma} & H^{2j+1}(\Sigma\mathbb{C}P^\infty;\mathbb{F}_p) & \cong & H^{2j+1}(K_j;\mathbb{F}_p) & \xleftarrow[\cong]{\Sigma} & H^{2j}(X_j;\mathbb{F}_p) \\
\downarrow \mathcal{P}^j & \circlearrowleft & \downarrow \mathcal{P}^j & \circlearrowleft & & & \downarrow \mathcal{P}^j \\
\mathbb{F}_p < x^{jp} > \cong H^{2jp}(\mathbb{C}P^\infty;\mathbb{F}_p) & \xrightarrow[\Sigma]{\cong} & H^{2jp+1}(\Sigma\mathbb{C}P^\infty;\mathbb{F}_p) & \cong & H^{2jp+1}(K_j;\mathbb{F}_p) & \xleftarrow[\Sigma]{\cong} & H^{2j}(X_j;\mathbb{F}_p)
\end{array}$$

where we assume $H^*(\mathbb{C}P^\infty;\mathbb{F}_p) \cong \mathbb{F}_p[x]$ with $|x| = 2$. Let k_j be a generator of $H^{2j+1}(K_j;\mathbb{F}_p)$. Up to a unit in $\mathbb{F}_p$ we can assume that k_j is the image of x^j under the isomorphism Σ. In addition $\mathcal{P}^j(x^j) = x^{jp} \neq 0$. Thus letting k_j go round the left-hand square, we get first that $P^j(k_j) \neq 0$. On the other hand, letting k_j go round the right-hand square we get that $\mathcal{P}^j(\Sigma^{-1}(k_j)) = (\Sigma^{-1}(k_j))^p = 0$. For the square of any element in $H^{2j}(X_j;\mathbb{F}_p)$ is zero as $H^{4j}(X_j;\mathbb{F}_p)$ is trivial, thus so are the p$^{\text{th}}$ powers. But secondly $\mathcal{P}^j\Sigma^{-1}(k_j) = \Sigma^{-1}\mathcal{P}^j(k_j) \neq 0$ since Σ is bijective. Hence we obtain a contradiction!
Because we shall use this argument again, for the sake of clarity, we summarize it by drawing the journey of k_j through the diagram above:

$$\begin{array}{ccccccc}
x^j & \xleftarrow{\cong} & k_j & \xmapsto{\cong} & \Sigma^{-1}(k_j) \\
\downarrow{\scriptstyle \mathcal{P}^j} & & \downarrow{\scriptstyle \mathcal{P}^j} & & \downarrow{\scriptstyle \mathcal{P}^j} \\
x^{pj} \neq 0 & \xmapsto{\cong} & \mathcal{P}^j(k_j) \neq 0 & \xmapsto{\cong} & \underbrace{\Sigma^{-1}\mathcal{P}^j(k_j)}_{\neq 0} = \mathcal{P}^j\Sigma^{-1}(k_j) = (\Sigma^{-1}(k_j))^p = 0 \ \lightning .
\end{array}$$

□

Remark 4.5.
At this point, it is interesting to point out why the argument used in the proof above doesn't apply to K_{p-1}. As we easily see the reason happens to be purely computational. Assuming that $K_{p-1} \simeq \Sigma X_{p-1}$, it follows that the non-trivial cohomology groups of X_{p-1} are the groups $H^{2i(p-1)}(X_{p-1},\mathbb{F}_p)$ with $i \geq 1$. Hence taking any non trivial element $x \in H^*(X_{p-1},\mathbb{F}_p)$ and elevating it to the power of m ($m \in \mathbb{N}$) does not send it to a nontrivial group: $|x| = 2i(p-1)$ implies $|x^m| = 2im(p-1)$. Thus it cannot be concluded that the p^{th} powers, perforce, are trivial because a smaller one necessarily is.

In consequence, the K_j's, $j \in \mathbb{N}_{p-2}$ fail the first test for the title of "nice" co-H-space. Though, thus far, this does not prevent them from bearing a co-H-group structure. In fact, as we now show, for $j \in \mathbb{N}_{p-2}$, the K_j's do not even possess one coassociative coproduct, let alone co-H-group structures.

To achieve this, the idea is to imitate an argument given by Israel Berstein in [Ber64], where he shows that the space $X := \mathbb{S}^3 \cup_f e^{2p+1}$, with p is an odd prime and f belonging to a class α of order p in $\pi_{2p}(\mathbb{S}^3)$, does not admit a coassociative coproduct.

A major argument of this proof uses the Bott-Samelson theorem. For essentially two reasons, I don't wish to write an explanation section on this theorem. First, because I have already studied this theorem (in a somewhat more particular case) in a previous semester project (see [DKKL06]). Secondly, and mainly, because Berstein has written a very detailed article, [Ber65], on the subject. Nonetheless, for the sake of clarity, we give a quick summary of the main results which he needs

for his proof concerning $\mathbb{S}^3 \cup_f e^{2p+1}$ and which we shall need as well to copy his argument.

3.1. A word on the Bott-Samelson Theorem.

In [Ber65, Corollary 3.3], Berstein restates, as follows, the Bott-Samelson theorem, for the case that the space considered is a co-H-space whose base point has a contractible open neighbourhood and that homology is considered with coefficients in a field $\mathbb{F}$.

THEOREME 4.6 (Bott-Samelson).
Let (X, θ) be a co-H-space satisfying the aforementioned hypothesis. Then $H_*(\Omega X; \mathbb{F})$ is a free Hopf algebra over $\mathbb{F}$. Furthermore, if θ is coassociative, then $H_*(\Omega X; \mathbb{F})$ is freely generated, as an algebra, by the module $N := \mathrm{Ker}[(\Omega\theta)_* - \rho_*]$ where

$$\rho = \mu(\Omega j_1 \times \Omega j_2)\Delta : \Omega X \longrightarrow \Omega(X \vee X)\,,$$

μ is the multiplication of loops in $\Omega(X \vee X)$, Δ the diagonal map on ΩX and j_1, j_2 the inclusions in $X \vee X$.

In the original proof of the Bott-Samelson theorem, [BS53, Theorem III.1.A], it is shown that $H_*(\Omega X; \mathbb{F})$ is isomorphic to a tensor algebra $T(M)$ where M is a submodule of $H_*(\Omega X; \mathbb{F})$ mapped isomorphically onto $H_*(X; \mathbb{F})$ by the homology suspension, which lowers dimensions by one. Berstein shows that we can choose $M = N$, which leads him to the next lemma he uses in [Ber64] to prove that the space $\mathbb{S}^3 \cup_f e^{2p+1}$ does not admit a coassociative coproduct.

LEMMA 4.7 ([Ber65, Lemma 4.2).
Let (X, θ) be a coassociative co-H-space. Then $H_*(\Omega X; \mathbb{F})$ is a free Hopf algebra over $\mathbb{F}$, admitting a free system of generators $\{a_i\}$ satisfying the formula

$$(\Omega\theta)_*(a_i) = a_i + \overline{a_i} + \sum \lambda_{lk} a_l \overline{a_k}$$

where $H_*(\Omega(X \vee X); \mathbb{F})$ is identified with the free product of algebras $H_*(\Omega X; \mathbb{F}) * H_*(\Omega X; \mathbb{F})$, $\lambda_{lk} \in \mathbb{F}$, $0 < |a_j|, |a_k|, |a_l\overline{a_k}| = |a_i|$ and a and $\overline{a}$ denote the same element considered in the two copies of $H_*(\Omega X; \mathbb{F})$. Moreover, the free system of generators $\{a_i\}$ can be chosen to be in one-to-one correspondence with a system of generators of the graded $\mathbb{F}$-module $H_*(X; \mathbb{F})$.

Let us finally note that Berstein also shows that an element $n \in N$ is primitive precisely when $(\Omega\theta)_*(n) = n + \overline{n}$, so that in the preceding lemma, $H_*(\Omega X; \mathbb{F})$ is primitively generated when the system of generators $\{a_i\}$ satisfies for all i,

$$(\Omega\theta)_*(a_i) = a_i + \overline{a_i}\,.$$

3.2. About coassociative co-H-structures.

The result concerning the non-existence of coassociative co-H-structures on the spaces $K_1, \dots, K_{p-2}$ is stated in the following theorem. As mentioned, the proof is suggested by the article [Ber64]. The argument presented above makes use of the Serre spectral sequence and was written by A. Baker, K. Hess Bellwald and Birgit Richter in their *Informal notes on co-H space structures on retracts of* $\mathbb{C}P^\infty$,[BHR06].

THEOREM 4.8.
Let $j \in \mathbb{N}_{p-2}$, then the space K_j does not possess any coassociative coproduct.

PROOF. The proof is made by contradiction. Assume that K_j does admit a coassociative coproduct θ_j. It follows from the splitting theorem 4.1 and the universal coefficient theorem that the homology with coefficients in $\mathbb{F}_p$ of K_j is

$$H_q(K_j, \mathbb{F}_p) = \begin{cases} \mathbb{F}_p & \text{if } q = 2n+1 \text{ and } n(\geq 1) \equiv j \pmod{p-1} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, applying the Bott-Samelson theorem, we get that $H_*(\Omega K_j; \mathbb{F}_p)$ is a free Hopf algebra over $\mathbb{F}_p$ with a generator $a_i \in H_{2j+2i(p-1)}(\Omega K_j; \mathbb{F}_p)$ for all integers $i \geq 0$. Furthermore, according to Berstein's lemma 4.7, for all $i \in \mathbb{N}$, the generators satisfy the formula

$$(\Omega\theta_j)_*(a_i) = a_i + \overline{a_i} + \sum \lambda_{lk} a_l \overline{la_k} \tag{1}$$

where $\lambda_{lk} \in \mathbb{F}_p$, $0 < |a_l|, |a_k|$ and $|a_l\overline{a_k}| = |a_i|$ for all l, k. In addition, since

$$|a_l a_k| = |a_l| + |a_k| = (2j + 2l(p-1)) + (2j + 2k(p-1)) = 2j + 2(l+k)(p-1) + 2j \tag{2}$$

and $j \neq p-1$ there cannot be any cross term in formula (1) and the generators a_i, $i \geq 1$ all satisfy the formula

$$(\Omega\theta_j)_*(a_i) = a_i + \overline{a_i}\,.$$

Consequently $H_*(\Omega K_j; \mathbb{F}_p)$ is a primitively generated Hopf algebra, whose dual Hopf algebra is $H^*(\Omega K_j; \mathbb{F}_p)$, which according to corollary 3.9 has only trivial p^{th} powers.

Thus in order to copy Berstein's argument and get a contradiction, we need to find an element in $H^*(\Omega K_j; \mathbb{F}_p)$ whose p^{th} power is nontrivial. In [Ber64] this was achieved through the study of the geometry of the space and its impact on the first Steenrod reduced power $\mathcal{P}^1$. In our case, the geometry of the K_j's is far too complicated, thus we need to find a somewhat more algebraic argument.

Since the K_j's are simply connected, there is a Serre spectral sequence associated to the pathspace fibration $\Omega K_j \longrightarrow PK_j \longrightarrow K_j$ and that satisfies the properties of theorem 3.21 . The E_∞-page has the form

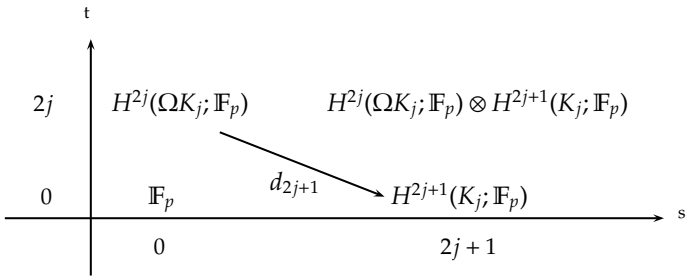
$$E_\infty^{s,t} \cong H^{s+t}(*; \mathbb{F}_p)$$

with only a $\mathbb{F}_p$ in the (0,0)-position. The E_2-page is

$$E_2^{s,t} \cong H^s(K_j; H^t(\Omega K_j; \mathbb{F}_p)) \cong H^s(K_j; \mathbb{F}_p) \otimes_{\mathbb{F}_p} H^t(\Omega K_j; \mathbb{F}_p)$$

with $H^*(K_j; \mathbb{F}_p)$ on the horizontal axis, $H^*(\Omega K_j; \mathbb{F}_p)$ on the vertical axis, nontrivial columns only in degree s equal to $2j+1$ modulo $2(p-1)$ and nontrivial lines in

degree t equal to $2j$ modulo $2(p-1)$. The lower left-hand corner of the E_2-page has the following form:



By the same arguments as the ones used in example 3.24 to compute the cohomology ring of $\mathbb{S}^n$, we obtain that the pages $E_3,\dots,E_{2j+1}$ remain equal to the E_2-page whereas the entries $E_{2j+2}^{0,2j}$ and $E_{2j+2}^{2j+1,0}$ of the E_{2j+2}-page are already equal to the corresponding ones appearing in the E_∞-page, which implies that the differential $d_{2j+1}: H^{2j}(\Omega K_j;\mathbb{F}_p) \longrightarrow H^{2j+1}(K_j;\mathbb{F}_p)$ is an isomorphism.

In view of these observations there must be a class $\omega \in H^{2j}(\Omega K_j;\mathbb{F}_p)$ transgressing to the generator $k_j \in H^{2j+1}(K_j;\mathbb{F}_p)$ introduced in the proof of theorem 4.4. Then by the Kudo transgression theorem (3.26), the class $\mathcal{P}^j(\omega) \in H^{2jp}(\Omega K_j;\mathbb{F}_p)$ also transgresses to the class $\mathcal{P}^j(k_j)$ in $H^{2jp+1}(\Omega K_j;\mathbb{F}_p)$, which is nontrivial according to the proof of 4.4. Hence the desired contradiction to the fact that p^{th} powers are trivial in $H^*(\Omega K_j;\mathbb{F}_p)$:

$$\omega^p = \mathcal{P}^j(\omega) \neq 0 \ \lightning$$

□

Remark 4.9.
Notice that once again the proof couldn't be applied to K_{p-1}. For if $j = p-1$, formula (2) does not prevent the existence of nonzero cross terms in formula (1). Therefore, it cannot be concluded that $H_*(\Omega K_{p-1};\mathbb{F}_p)$ is primitively generated and corollary 3.9 can't be used to say that the p^{th} powers are trivial in $H^*(\Omega K_{p-1};\mathbb{F}_p)$. Hence it can't lead to the contradiction we have obtained for the other K_j's.
As a matter of fact, knowing that K_{p-1} has the homotopy type of a suspension assures that there must be at least one nontrivial p^{th} power in $H^*(\Omega K_{p-1};\mathbb{F}_p)$ and thus, by corollary 3.9, that $H_*(\Omega K_{p-1};\mathbb{F}_p)$ is not primitively generated.

Finally, we suggest to simplify the proof of theorem 4.8, by making it free of spectral sequences and suppressing the argument linked to the Kudo transgression theorem, which we have left unproven in this piece of work. The first part of the proof remains unchanged, but we describe another way to obtain a nontrivial p^{th} power in $H^*(\Omega K_j;\mathbb{F}_p)$.
Since Steenrod reduced powers commute with the suspension isomorphism and

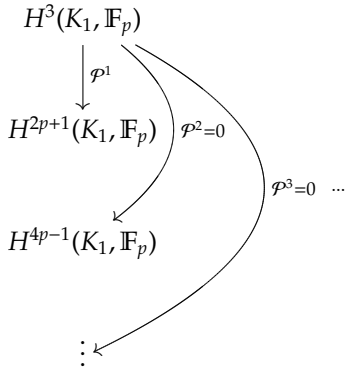
cohomology suspension, we have the commutative diagram

$$\begin{array}{ccccccc}
\mathbb{F}_p < x^j > \cong H^{2j}(\mathbb{C}P^\infty;\mathbb{F}_p) & \xrightarrow[\cong]{\Sigma} & H^{2j+1}(\Sigma\mathbb{C}P^\infty;\mathbb{F}_p) & \cong & H^{2j+1}(K_j;\mathbb{F}_p) & \xhookrightarrow{\sigma^*} & H^{2j}(\Omega K_j;\mathbb{F}_p) \\
\downarrow \mathcal{P}^j & \circlearrowleft & \downarrow \mathcal{P}^j & \circlearrowleft & & & \downarrow \mathcal{P}^j \\
\mathbb{F}_p < x^{jp} > \cong H^{2jp}(\mathbb{C}P^\infty;\mathbb{F}_p) & \xrightarrow[\Sigma]{\cong} & H^{2jp+1}(\Sigma\mathbb{C}P^\infty;\mathbb{F}_p) & \cong & H^{2jp+1}(K_j;\mathbb{F}_p) & \xhookrightarrow[\sigma^*]{} & H^{2j}(\Omega K_j;\mathbb{F}_p)\,.
\end{array}$$

Therefore choosing a generator $k_j \in H^{2j+1}(K_j;\mathbb{F}_p)$, letting it go round the diagram and using the fact that cohomology suspension is injective yields the desired contradiction. By the same arguments as the ones used in the proof of theorem 4.4, k_j goes round the diagram as follows:

$$\begin{array}{ccccccc}
x^j & \xleftarrow{\cong} & k_j & \longmapsto & \sigma^*(k_j) & & \\
\downarrow \mathcal{P}^j & & \downarrow \mathcal{P}^j & & \downarrow \mathcal{P}^j & & \\
x^{pj} \neq 0 & \xmapsto{\cong} & \mathcal{P}^j(k_j) \neq 0 & \longmapsto & \underbrace{\sigma^*\mathcal{P}^j(k_j)}_{\neq 0} = \mathcal{P}^j\sigma^*(k_j) = (\sigma^*(k_j))^p = 0 & \lightning &
\end{array}$$

Moreover, note that for the space K_1 the linking relation described in proposition 4.3 shows at once that $\mathcal{P}^1(k_1) \neq 0$. For by axiom (4) of Steenrod reduced powers, we see that on $H^3(K_1,\mathbb{F}_p)$ only $\mathcal{P}^1$ can be non-zero.



In consequence, the only way to link a generator u in $H^3(K_1,\mathbb{F}_p) \cong \mathbb{F}_p$ to a generator v in $H^{3+2(p-1)}(K_1,\mathbb{F}_p) \cong \mathbb{F}_p$ is to use $\mathcal{P}^1$, which means that $\mathcal{P}^1$ has to be a group isomorphism.

Conclusion and Further Developments

Now that we have figured out common properties that co-H-structures on the spaces K_j, $j = 1, \dots p-2$ do not have, to go further in the study of these spaces as well as of K_{p-1} , one could actually wonder how "many" different co-H-structures do exist on these spaces. It could be interesting to know whether the coproduct described at the beginning of section 4.3 and inherited from the one given by the suspension stucture on $\Sigma\mathbb{C}P^\infty$ is the only one existing on K_j, up to some equivalence, or if we could define other ones with other properties. But this would imply to use or define a good equivalence definition for coproducts.

Besides, the proof we have adapted from the one given by Berstein in [Ber64] might in fact provide a general method to determine whether a space X can possess a coassociative coproduct or not. Although, as we have seen, it is crucial that the homology of the space X is concentrated in "good" degrees, so that the Hopf algebra $H_*(\Omega X; \mathbb{F}_p)$ is primitively generated, otherwise it can't be deduced that the p^{th} powers are trivial in the dual Hopf algebra. Then it is also necessary to be able to work out the behaviour of some well-chosen Steenrod reduced power on the cohomology ring $H^*(X; \mathbb{F}_p)$ in order to obtain a contradiction with the previous observation on the p^{th} powers in $H^*(\Omega X; \mathbb{F}_p)$. As well in Berstein's proof as in the one we have given, it is only necessary to know that the reduced power $\mathcal{P}^i$, for some good i is nontrivial. In Berstein's proof this was deduced from the geometry of the space and in the case of the spaces K_j's this was deduced from algebraic properties. In addition we have also seen two different methods to obtain a contradiction to the initial assumption that the space possesses a coassociative coproduct, the first one using the fact that cohomology suspension homomorphism is an injection for spaces of LS-category smaller or equal to one and the second one using a Serre spectral sequence and the Kudo transgression theorem. Therefore, their are a few variations at one's disposal to adapt these proofs again to other spaces.

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