Examples of Involutions on Algebraic K-Theory of Bimonoidal Categories

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Introduction

Algebraic K-theory of bimonoidal categories arises as a structure of interest in "Two-vector bundles and forms of elliptic cohomology" [3] by Nils Baas, Bjørn Ian Dundas and John Rognes and its follow-up papers "Ring completion of rig categories" [1] and "Stable bundles over rig categories"[2] by these three authors joint with Birgit Richter. (These papers formerly were the one paper "Two-vector bundles define a form of elliptic cohomology".) Specifically they investigate the category of two-vector-bundles. By analogy to the case of principal bundles over a topological space they study two-vector-bundles by examining a represented functor $[X, K(\mathcal{V})]$, where $K(\mathcal{V})$ is the algebraic K-theory spectrum of the bimonoidal category of finite-dimensional complex vector spaces $\mathcal{V}$. This furthermore embeds into the context of interpolating between the complexity of singular homology, which only captures few phenomena, and the complexity of complex bordism, which detects all levels of periodic phenomena at once. The cohomology theory defined by $K(\mathcal{V})$ is in a precise manner one level more complex than topological K-theory is.

The topic of involutions is introduced by the fact that it would be negligent to ignore that the category of complex vector spaces is equipped with an involution naturally induced by conjugation in complex numbers.

This diploma thesis studies the K-theory and Hochschild homology of rings with involution. In detail the diploma thesis arose from the motivation to examine non-trivial involutions on K-theory of bimonoidal categories by studying non-trivial involutions on rings. To that end after introducing the basic concepts I define the algebraic K-theory of strict bimonoidal categories following Birgit Richter's "An involution on the K-theory of bimonoidal categories with anti-involution"[21]. Furthermore if an object has an associated anti-involution, this gives an associated involution on its K-theory, which we define according to [21] as well.

As an example I explicitly present K-theory and involutions of group rings and more specifically Laurent polynomials. These examples illustrate the limitations that involutions can be studied on K-groups directly with the same difficulty that K-groups can be computed. There is a need for further tools.

The last chapter is the heart of this diploma thesis. I investigate the Dennis trace map

$$\mathrm{Dtr}\colon K_n(R) \to HH_n(R)$$

and the map associated to an anti-involution on Hochschild homology. The main result 5.4.3 of this diploma thesis is that the trace map Dtr does commute with the induced involutions. Thus the trace map provides an additional tool to study involutions on K-theory and can help to prove non-triviality. The result should be compared to the statement by Bjørn Ian Dundas in the introduction of [8] that his functorial definition of a trace map in particular implies that it respects involutions on K-theory and topological Hochschild homology. This diploma thesis serves as an algebraic analogue of a fact known on the topological level, although it is not directly implied by that.

Finally I discuss the example of the integers with an adjoined prime root of unity $\mathbb{Z}[\zeta_p]$ and discuss the non-triviality in that case in contrast to Laurent polynomials, which goes to show the usefulness of the Dennis trace map as well as its defects.

This text is organised as follows: The first two chapters are dedicated to study algebraic K-theory in the standard context of rings and specifically study rings with involution by providing an induced involution on the K-groups. The third chapter essentially is a summary of Birgit Richter's "An involution on the K-theory of bimonoidal categories with anti-involution"[21]. Furthermore the relations between a strict bimonoidal category and its associated ring embed this paper into the non-triviality statements of the following chapters. I provide examples of rings with families of involutions in Chapter 4. All of these examples have non-trivial involutions on K_1. Finally, chapter 5 introduces the Dennis trace map as an example of a useful tool which can be extended to the context of rings with involution and provide examples to evaluate its usefulness in the ring context.

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Contents

1 Introduction to K-Theory of Rings

In order to investigate K-theory of rings with involution I first want to give a solid foundation of K-theory of general rings. My approach will diverge drastically from the historical development in an effort to homogenise the appearance of K-theory throughout this diploma thesis.

But in order to do the actual history justice, let me give some remarks on the history of (algebraic) K-theory. The K in the designation K-theory is reminiscent of Grothendieck's approach to the Riemann-Roch Theorem in which he studied coherent sheaves over an algebraic variety and looked at classes - which is "Klassen" in German - of those sheaves modulo exact sequences in 1957. This approach has been published by Borel and Serre in "Le théorème de Riemann-Roch" [6] (1958). This first group is what is nowadays called K_0.

About seven years later, Bass provided a definition of the group he called $K_1(A)$ for A a ring and provided an exact sequence

$$K_1(A, \mathfrak{q}) \to K_1(A) \to K_1(A/\mathfrak{q}) \to K_0(A, \mathfrak{q}) \to K_0(A) \to K_0(\mathfrak{q})$$

in the paper "K-Theory and Stable Algebras" [4] (1964). Since this exact sequence was built upon to define higher K-theory, the name K_1 for this group is still the common one.

The next group K_2 was defined by Milnor in 1968 according to Bass' "K_2 and symbols" [5] and again had an exact sequence connecting it to K_1, along with a pairing

$$K_1(A) \otimes K_1(A) \to K_2(A)$$

which is bilinear and antisymmetric, thus looked like a segment of a graded ring. Since the exact sequence, this pairing and further relations look far too natural to be a coincidence, one was looking for a general sequence of K-groups $K_n(R)$, which could be associated to a ring and which would yield a more structural explanation of the known results.

This was given by Milnor in "Algebraic K-Theory and Quadratic Forms" [19], but Milnor himself described his definition as "purely ad hoc" (in [19] as well). Furthermore Milnor's extension is restricted to be K-theory of fields, otherwise it would not agree with the first three known K-groups.

The next attempt at defining K-theory for each natural number was given by Quillen in 1973 in his paper "Higher Algebraic K-Theory: I" [20]. Quillen defines a topological space, proves that its homotopy groups agree with the known definitions and derives some structural results, which translate algebraic relations into topological relations between these newly defined spaces. These are the K-groups that are studied in this diploma thesis.

1.1 K-Theory of Rings

I will generally assume rings to be unital but not necessarily commutative rings.

As said before, I will deviate from the historical viewpoint and just define the K-theory space in order to give a more linear concise summary for K-theory of rings. In preparation for that there are some necessary prerequisites in group homology.

1.1.1 Classifying Space of a Group

The first investigation focuses on a space $|BG|$, which arises in the context of classifying principal G-bundles for a group G. It yields the result that isomorphism classes of G-bundles over a space X are classified by homotopy classes of maps from X to $|BG|$. It is relevant in the context of this thesis however because of its homotopy groups (cf. Lemma 1.1.2), so I will not elaborate on bundles any further. For simplicial methods consult Loday's Appendix B in "Cyclic Homology"[15], the "basic definitions" in Chapter I of Goerss-Jardine "Simplicial homotopy theory" [12] and May's "Simplicial Objects in Algebraic Topology"[17].

Definition 1.1.1. For G a group define a simplicial set as follows:

- The set of n-simplices is given as

$$BG_n := G^n = G \times \ldots \times G,$$

  i.e. n-simplices are n-tuples of elements of G,

- The face maps $d_i \colon BG_n \to BG_{n-1}$ are given by the equations

$$d_i(g_1, \ldots, g_n) := \begin{cases} (g_2, \ldots, g_n) & \text{for } i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & \text{for } i = 1, \ldots, n-1 \\ (g_1, \ldots, g_{n-1}) & \text{for } i = n. \end{cases}$$

- The degeneracies $s_i \colon BG_n \to BG_{n+1}$ are given by the equations

$$s_i(g_1, \ldots, g_n) := (g_1, \ldots, g_{i-1}, 1, g_i, \ldots, g_n) \quad \text{for } i = 0, \ldots, n$$

which are putting a 1 next to each component including the left- and rightmost position.

This constitutes a simplicial set (cf. May's "Concise Course in Algebraic Topology" [18, Chapter 16, Section 5]).

Proposition 1.1.2. [18, Section 16.5] (1) Let G be a topological group, then each homotopy group of the realisation of BG is identified as follows

$$\pi_n(|BG|) \cong \pi_{n-1}(G) \ \ (n \geq 1).$$

(2) If in particular G is discrete, this implies

$$\pi_n(|BG|) = \begin{cases} G & \text{for } \ n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

□

Remark 1.1.3. In particular this proposition answers the question whether each discrete group can be realised as a fundamental group affirmatively.

1.1.2 Bar Construction on the Group Ring

There is a simplicial abelian group closely related to the simplicial set discussed before.

Definition 1.1.4. Let G be a discrete group and $\mathbb{Z}[G]$ its group ring with integer coefficients. Define the following simplicial abelian group:

- Its group of n-simplices consists the free abelian group on n-tuples of G

$$B_n(G) := \mathbb{Z}[G]^{\otimes n} \cong \mathbb{Z}[G^n]$$

- Its face and degeneracy maps are given by the linear extension of the respective maps given before in definition 1.1.1.

The chain complex associated to this simplicial abelian group is called the bar complex associated to the group G.

Remark 1.1.5. The common notations BG for both objects are quite unfavourable, but on the other hand from the simplicial perspective legitimate. I will denote the simplicial set with $BG_\bullet$ and the simplicial abelian group with $B_*(G)$. The context in which they are used should provide sufficient grounds for just one interpretation to make sense.

The bar complex defines group homology.

Definition 1.1.6. Let G be a group, then the group homology of G is defined to be the homology of the bar complex G, i.e.

$$H_n(G) := H_n(B_*(G)).$$

The following result shows how well the two bar constructions interact:

Theorem 1.1.7. [24, Theorem 6.10.5] For G a group, $BG_\bullet$ the associated simplicial set and $B_*(G)$ its associated bar complex, the singular homology with $\mathbb{Z}$-coefficients of $|BG|$ and group homology of G, which is homology of $B_*(G)$, are isomorphic, i.e.

$$H_n(|BG|, \mathbb{Z}) \cong H_n(G) = H_n(B_*(G)).$$

□

Remark 1.1.8. I will use this isomorphism in quite an inexplicit manner, since chains in $B_*(G)$ and the cellular chain complex of $|BG|$ are isomorphic in a natural way. A very elegant proof via singular chains can be found in Weibel's "An Introduction to Homological Algebra" [24] Theorem 6.10.5.

There is an interpretation of the first homology group as follows:

Proposition 1.1.9. [24, Theorem 6.1.11] For each group G, the first homology group of G is naturally identified with the abelianised group

$$H_1(G) = G_{ab}.$$

□

1.1.3 Plus-Construction on Classifying Spaces

Since by lemma 1.1.2 the classifying space of a discrete group only provides one non-trivial homotopy group, the space has to be modified in order to get $K_n(R) = \pi_n(KR)$ coinciding with the old definitions of K_1 and K_2. Furthermore it ought to represent arithmetic information about the ring R. Therefore it would be quite counterintuitive to expect the K-groups in all higher degrees to be trivial. Since the first three groups measure the linear algebra over a ring, it seems natural to approach the definition of K-theory by looking at the classifying space of general linear groups over a ring.

In order to organise the linear groups of a ring into one structure, define the index category $\mathcal{N}$ with objects $[n] := \{1, \ldots, n\}$ and morphisms $f\colon [n] \to [n+k]$ given by $f(i) = i$ for each $i \in [n]$.

Definition 1.1.10. For R a unital ring and $F\colon \mathcal{N} \to Grp$ the functor given on objects by

$$F([n]) := GL_n(R)$$

and on morphisms by

$$(F(f\colon [n] \to [n{+}1])(A))_{i,j} = \begin{cases} A_{f^{-1}(i),f^{-1}(j)} & i,j \in f([n]) \\ 0 & (i \notin f([n]) \vee j \notin f([n])) \wedge i \neq j \\ 1 & i = j \wedge i \notin f([n]), \end{cases}$$

define the stabilised linear group of R as the colimit over F

$$GL(R) := \mathrm{colim}_{\mathcal{N}} F.$$

In this case it is evidently even a filtered colimit $GL(R) := \varinjlim GL_n(R)$.

Remark 1.1.11. In less formal terms the stabilised linear group allows to identify each invertible matrix as a top left finite submatrix of an infinite matrix, which otherwise has unit entries on the diagonal and zeroes everywhere else.

Defining the K-theory space of a ring involves studying the subgroup of elementary matrices in $GL(R)$. This was initially motivated by Bass via K_1 in [4]. Recall the following definition:

Definition 1.1.12. Denote by $e_{i,j}(\lambda)$ the matrix with the following components

$$(e_{i,j}(\lambda))_{k,l} := \delta_{k,l} + \lambda\delta_{i,k}\delta_{j,l} \quad i \neq j$$

That is $e_{i,j}(\lambda)$ is defined to be the identity matrix with exactly one off-diagonal component $\lambda \in R$.

The group of $n \times n$-elementary matrices $E_n(R)$ is defined as the subgroup of $GL_n(R)$ generated by $e_{i,j}(\lambda)$ for each $i \neq j$ and each $\lambda \in R$.

Since this is compatible with the stabilisation given in definition 1.1.10, the stabilised group of elementary matrices can be defined in the same fashion as in 1.1.10 by

$$E(R) := \varinjlim E_n(R).$$

It is a subgroup of $GL(R)$ in a natural manner.

The next lemma is essential in constructing the K-theory space $K(R)$.

Lemma 1.1.13. [23, Proposition 1.5] The stabilised elementary matrices generate a normal subgroup of the stabilised general linear group for any ring

$$E(R) \lhd GL(R),$$

which is perfect, i.e.

$$E(R) = [E(R), E(R)].$$

Furthermore the commutator subgroup $[GL(R), GL(R)]$ of the stabilised general linear group is equal to the stabilised subgroup generated by elementary matrices, i.e.

$$E(R) = [GL(R), GL(R)].$$

□

The following result gives the classical construction of Quillen to define higher K-theory:

Theorem 1.1.14. [23, Theorem 2.1] Let X be a CW-complex and $N \lhd \pi_1(X)$ a perfect normal subgroup of the fundamental group of X. Then there is a space X^+ with a map $i\colon X \to X^+$, which are called the plus-construction on X with respect to N. These satisfy the following properties:

1. The map $i\colon X \to X^+$ induces the canonical projection on fundamental groups

$$i_*\colon \pi_1(X) \to \pi_1(X^+) = \pi_1(X)/N.$$

2. Each map $f\colon X \to Z$, which is trivial on the given subgroup N, that is

$$\pi_1(f) \circ (j\colon N \to \pi_1(X)) = 0,$$

   extends to a map on the plus-construction $\bar{f}\colon X^+ \to Z$. The extension is unique up to homotopy in making the diagram

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "f"'] & & X^+ \arrow[dl, "\bar{f}"] \\ & Z & \end{tikzcd}$$

   commute up to homotopy.

3. For each system of local coefficients $\mathcal{L}\colon \Pi_1(X^+) \to Ab$ on X^+ the induced map of the inclusion

$$i_*\colon H_n(X, i^*\mathcal{L}) \to H_n(X^+, \mathcal{L})$$

   induces an isomorphism on singular homology with local coefficients for each $n \geq 0$.

□

Remark 1.1.15. Let me emphasise some facts that the proof of this result yields. It is essential that the actual construction can be given by attaching 2-cells which precisely take care of the subgroup N and additional 3-cells to correct the defect on homology the 2-cells might have caused. In that manner it is legitimate to think of $i\colon X \to X^+$ as an inclusion.

I will not go into detail about local coefficients, but state that 1.1.14.(3) implies $H_n(X, G) \cong H_n(X^+, G)$ for each abelian coefficient group G understood as constant coefficients as well, specifically for $G = \mathbb{Z}$.

Property (2) in particular implies that any two plus-constructions to a fixed perfect normal subgroup are homotopy-equivalent by the usual argument.

The results 1.1.13 and 1.1.14 combine to Quillen's definition of higher K-theory.

Definition 1.1.16. For R a ring define the K-theory space of R to be the plus-construction on the classifying space of $GL(R)$ with respect to the elementary matrices $E(R)$, i.e.

$$K(R) := |BGL(R)|^+$$

and for $n \geq 1$ define the K groups of R by

$$K_n(R) := \pi_n(K(R)) \quad (n \geq 1).$$

Remark 1.1.17. Be aware that I completely avoid $K_0(R)$ here and in all the diploma thesis, because it is exceptional in most cases.

As a defining property of the plus construction there are immediate reinterpretations of K_1:

Proposition 1.1.18. [23, Theorem 2.1 and Proposition 1.5] There are the following natural identifications

$$K_1(R) = GL(R)/E(R) = GL(R)/[GL(R), GL(R)] = GL(R)_{ab} = H_1(GL(R)).$$

Proof. The first equality is the definition of $K(R)$, the second identification is a consequence of lemma 1.1.13, the third is the usual identification giving a natural model for the abelianised group of any group and the last identification is the classical one already cited in proposition 1.1.9. □

I will only refer to [23] again to note that there are interpretations for K_2 and K_3 as well, which are not used in this diploma thesis.

The identifications given above in particular yield a useful tool for commutative rings.

Proposition 1.1.19. [22, Theorem 2.2.1] For R a commutative ring, the determinant maps for each finite degree $\det_n\colon GL_n(R) \to R^\times$ stabilise to give a map

$$\det\colon GL(R) \to R^\times,$$

which is a group homomorphism with

$$\det(e_{i,j}(\lambda)) = 1_R$$

for each $i \neq j$ and $\lambda \in R$. In particular it satisfies $E(R) \subset \ker(det\colon GL(R) \to R^\times)$ and hence factors as follows

$$\begin{tikzcd}
GL(R) \arrow[rr, "\det"] \arrow[dr] & & R^\times \\
& GL(R)/E(R) = K_1(R). \arrow[ur] &
\end{tikzcd}$$

□

It is thus legitimate to write $\det\colon K_1(R) \to R^\times$ as well and to call it the determinant map as well. This is useful because there is an obvious inclusion $j\colon R^\times = GL_1(R) \to GL(R) \to GL(R)/E(R)$, which yields $\det \circ j = id_{R^\times}$ and hence gives a natural splitting exact sequence

$$0 \to SK_1(R) \to K_1(R) \to R^\times \to 0,$$

which in particular implies

$$K_1(R) \cong SK_1(R) \oplus R^\times$$

for $SK_1(R) := SL(R)/E(R)$, where $SL(R)$ is the usual special linear group stabilised as $GL(R)$ and $E(R)$ before. For commutative rings R the problem of computing $K_1(R)$ thus reduces to computing units — which may be a very hard problem (cf. Chapter 4) — and computing the reduction of matrices of determinant one by elementary matrices — which is hard in general as well, but can be feasible.

Remark 1.1.20. Let me emphasise that there is a natural map $K_i(R) \to H_i(GL(R))$ given as follows:

The K-theory of a ring is defined as $K_i(R) := \pi_i(|BGL(R)|^+)(i \geq 1)$ and the plus-construction does not change homology

$$H_i(|BGL(R)|^+, \mathbb{Z}) \cong H_i(|BGL(R)|, \mathbb{Z}) = H_i(GL(R)).$$

Hence the Hurewicz-homomorphism gives a natural map

$$K_i(R) = \pi_i(|BGL(R)|^+) \to H_i(GL(R))$$

and specifically in degree 1 this can be understood as the identity on $GL(R)/E(R)$, since there are canonical identifications given in proposition 1.1.18

$$H_1(GL(R)) = GL(R)_{ab} = GL(R)/[GL(R), GL(R)] = GL(R)/E(R) = K_1(R).$$

2 K-Theory of Rings with (Anti-)Involution

The central notion for this diploma thesis is the following definition.

Definition 2.0.21. Let R be a unital (not necessarily commutative) ring. A map $\tau\colon R \to R$, which is additive and satisfies the conditions:

- $\tau(ab) = \tau(b)\tau(a)$,
- $\tau(1) = 1$,
- $\tau^2 = id_R$

is called an (anti-)involution on R. Of course τ can as well be understood as a unital ring homomorphism $\tau\colon R \to R^{op}$.

Naturally the anti-involution on R should induce a map on the K-groups of R. In what follows I am mainly following the lines of Burghelea and Fiederowicz [7], although I can drastically simplify their approach for an algebraic, discrete ring instead of a simplicial ring. To unify the vocabulary, I will mostly speak of involutions, which for a ring will mean anti-involution without exception and for groups just a self-inverse homomorphism.

2.1 The Induced Involution on the K-Theory Space

In order to induce an involution on the K-theory of R it is useful to induce a map on the general linear group of R first. This involves the following maps:

Lemma 2.1.1. Transposition is a morphism of matrix rings

$$T\colon M_r(R) \to M_r(R^{op})^{op},$$

and hence also induces a morphism of the linear groups $T\colon GL_r(R) \to GL_r(R^{op})^{op}$.

Inverting group elements is a morphism

$$\iota\colon G \to G^{op}.$$

Each anti-involution of R induces a morphism

$$M_r(\tau)\colon M_r(R) \to M_r(R^{op})$$

and in particular analogous to the transposition thus induces a homomorphism of linear groups $GL_r(\tau)\colon GL_r(R) \to GL_r(R^{op})$.

Proof. In the case of the transposition, denote by $\circ$ the multiplication of R^{op} and $M_r(R^{op})^{op}$ and find

$$\begin{aligned} T(A \cdot B)_{i,j} &= (A \cdot B)_{j,i} \\ &= \sum_{k=1}^{r} A_{j,k} B_{k,i} \\ &= \sum_{k=1}^{r} TA_{k,j} TB_{i,k} \\ &= \sum_{k=1}^{r} TB_{i,k} \circ TA_{k,j} \\ &= (TB \cdot TA)_{i,j} = (TA \circ TB)_{i,j}, \end{aligned}$$

which proves that transposition is a ring homomorphism

$$T\colon M_r(R) \to M_r(R^{op})^{op}.$$

In case of the inverse map the relation $(ab)^{-1} = b^{-1}a^{-1}$ is a standard fact.

Per definition an anti-involution opposes the ring structure componentwise and assigning to each ring its $r \times r$-matrices (for fixed $r \in \mathbb{N}$) is an endofunctor of unital rings $M_r(\_)\colon Rng_1 \to Rng_1$, thus the claim follows. $\square$

Lemma 2.1.2. Each of the three maps of lemma 2.1.1 commute, that is $\iota \circ T = T \circ \iota$, $T \circ GL_r(\tau) = GL_r(\tau) \circ T$ and $GL_r(\tau) \circ \iota = \iota \circ GL_r(\tau)$.

Proof. The equality $\iota \circ T = T \circ \iota$ is equivalent to the claim $(TA)^{-1} = T(A^{-1})$, which is equivalent to the statement $T(A^{-1})TA = 1_r$, but this simplifies as follows, since $T(1_r) = 1_r$

$$T(A^{-1})TA = T(AA^{-1}) = T(1_r) = 1_r.$$

Hence follows $T(A^{-1}) = (TA)^{-1}$.

The fact that transposition and componentwise involution commute follows by the calculation

$$\begin{aligned} (GL_r(\tau) \circ T(A))_{i,j} &= \tau(TA_{i,j}) \\ &= \tau(A_{j,i}) \\ &= (GL_r(\tau)(A))_{j,i} \\ &= T(GL_r(\tau)(A))_{i,j}. \end{aligned}$$

The inverse map and componentwise involution commute, since $\tau(1) = 1$, which implies $1_r = GL_r(\tau)(AA^{-1}) = GL_r(\tau)(A)GL_r(\tau)(A^{-1})$ and hence gives the equation $GL_r(\tau)(A^{-1}) = (GL_r(\tau)(A))^{-1}$.

□

Definition 2.1.3. For R a ring with anti-involution τ, the endomorphism of the linear group induced by composition of inverting, transposition and componentwise involution

$$\tau_*\colon GL_r(R) \to GL_r(R)$$
$$\tau_* := T \circ \iota \circ GL_r(\tau)$$

is defined to be the homomorphism induced by the involution τ.

Remark 2.1.4. By the preceding lemma τ_* is again its own inverse, since each of the factors is self-inverse and they commute. Furthermore note that τ_* is stable with respect to the inclusions $j_r\colon GL_r(R) \to GL_{r+1}(R)$, since each of the factors evidently is. This shows that there is an induced homomorphism on $GL(R)$ as well.

It is essential to see that τ_* also preserves elementary matrices:

Lemma 2.1.5. The induced involution on $GL(R)$ is a map $\tau_*\colon GL(R) \to GL(R)$, which preserves elementary matrices, i.e. $\tau_*(E(R)) \subset E(R)$.

Proof. The inverse of an elementary matrix is another elementary matrix by the equation $e_{ij}(\lambda)e_{ij}(-\lambda) = 1_n$. Quite obviously the transposition gives $e_{ji}(\lambda)^t = e_{ij}(\lambda)$. Each involution fixes the unit $\tau(1) = 1$, which implies $GL_r(\tau)(e_{ij}(\lambda)) = e_{ij}(\tau(\lambda))$. So each factor preserves elementary matrices and hence the induced involution τ_* preserves elementary matrices as well. □

Corollary 2.1.6. For R a (not necessarily commutative) ring with anti-involution τ, there is an induced involution on the K-theory of R.

Proof. As noted there is an induced map $\tau_*\colon BGL(R) \to BGL(R)$, hence there is a map of geometric realisations as well $|\tau_*|\colon |BGL(R)| \to |BGL(R)|$. Composed with the inclusion $|BGL(R)| \to |BGL(R)|^+$, this induces the canonical projection on fundamental groups

$$GL(R) \to GL(R)/E(R).$$

The universal property of the plus construction (with respect to $E(R)$) describes a map

$$f\colon |BGL(R)| \to Z$$

to a space Z with $\ker(f_*\colon \pi_1(|BGL(R)|) \to \pi_1(Z)) \subset E(R)$ as a product of the inclusion into the plus-construction $|BGL(R)| \to |BGL(R)|^+$ and a map $\bar{f}$ as follows

$$\begin{tikzcd} {|BGL(R)|} \arrow[r, "f"] \arrow[d] & Z \\ {|BGL(R)|^+.} \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

Furthermore $\bar{f}$ is determined uniquely up to homotopy (cf. Theorem 1.1.14). In particular, since the induced homomorphism $\tau_*\colon GL(R) \to GL(R)$ preserves elementary matrices, there is a map on the plus-construction $\tau_*^+\colon |BGL(R)|^+ \to |BGL(R)|^+$ such that the following diagram commutes up to homotopy

$$\begin{tikzcd} {|BGL(R)|} \arrow[r, "\tau_*"] \arrow[d] & {|BGL(R)|} \arrow[d] \\ {|BGL(R)|^+} \arrow[r, "{\tau^+}_*"] & {|BGL(R)|^+} \end{tikzcd}$$

and thus gives a map $\tau_*\colon K_i(R) \to K_i(R)$ for $i \geq 1$. □

By the same argument there is an induced involution on group homology of $GL(R)$ as well, which will later be useful in the investigation of trace maps.

Corollary 2.1.7. For R a (not necessarily commutative) ring with anti-involution τ, there is an induced involution on group (co-)homology of $GL(R)$.

Proof. The map $\tau_*\colon BGL(R) \to BGL(R)$ passes on to geometric realisation. In singular homology (with arbitrary coefficients) this yields the induced involution (cf. Theorem 1.1.7). □

It is quite obvious that even in this basic stage there are a lot of identifications, which ought to be compatible with each induced involution. So the next statement is just asserting that everything is coherently defined.

Theorem 2.1.8 (Involution on $H_*(GL(R))$)**.** The evident involution on the bar complex (cf. Definition 1.1.4) given by applying the induced involution componentwise

$$\tau_*(g_1, \ldots, g_n) := (\tau_*(g_1), \ldots, \tau_*(g_n))$$

yields the same involution on $H_*(GL(R), \mathbb{Z})$ as singular homology with $\mathbb{Z}$-coefficients on $|BGL(R)|$ does.

Proof. It is legitimate to think of the bar complex $B_*(\mathbb{Z}[GL(R)])$ as the chain complex of abelian groups associated to the simplicial set $BGL(R)$. The induced homomorphism is applied componentwise as $B\tau_*(g_1, \ldots, g_n) := (\tau_*(g_1), \ldots, \tau_*(g_n))$, so on the geometric realisation this is

$|B\tau_*|([(g_1, \ldots, g_n), (t_0, \ldots, t_n)]) = [(\tau_*(g_1), \ldots, \tau_*(g_n)), (t_0, \ldots, t_n)]$.

In particular $|B\tau_*|$ is not just cellular, it maps one cell precisely to one other cell in an orientation-preserving manner. Thus on cellular chains, it maps one basis element to another by the same formula as on $BGL(R)$ before. □

The plus construction preserves homology and thus provides an alternative to induce an involution on the homology of $|BGL(R)|^+$. By the following result this induces the same involution.

Proposition 2.1.9. For $\tau_*\colon |BGL(R)| \to |BGL(R)|$ the induced map on the classifying space of the general linear group and τ_*^+ the induced map on the plus-construction $\tau_*^+\colon |BGL(R)|^+ \to |BGL(R)|^+$ the isomorphism given by the inclusion into the plus-construction $i_*\colon H_*(|BGL(R)|) \to H_*(|BGL(R)|^+)$ transforms one involution into the other.

Proof. The homology-isomorphism is a consequence of the fact that the 1-cells corresponding to elements in $E(R)$ are boundaries of additional 2-cells. By adding another set of 3-cells, the effect of these 2-cells on homology is removed. This shows that the effect of τ_* on homology is the same on non-trivial cycles, since they came from $|BGL(R)|$ anyway. □

For later reference the results summarise to the following statement:

Theorem 2.1.10 (Coherence of the Involutions)**.** The induced involution on K-theory 2.1.6 and group homology 2.1.7 commutes with the Hurewicz homomorphism and the homology isomorphism of the plus-construction.

Proof. For the Hurewicz homomorphism $h\colon K_i(R) \to H_i(GL(R))$ the following commutative diagram is commutative, since the Hurewicz homomorphism is natural with respect to continuous maps

$$\begin{tikzcd} K_i(R) = \pi_i(|BGL(R)|^+) \arrow[r, "\pi_i(\tau_*)"] \arrow[d, "h"] & K_i(R) \arrow[d, "h"] \\ H_i(|BGL(R)|^+, \mathbb{Z}) \arrow[r, "H_i(\tau_*)"] & H_i(|BGL(R)|^+, \mathbb{Z}). \end{tikzcd}$$

Furthermore the identification chain

$$H_i(|BGL(R)|^+, \mathbb{Z}) \cong H_i(|BGL(R)|, \mathbb{Z}) \cong H_i(BGL(R), \mathbb{Z}) = H_i(GL(R))$$

commutes with induced involution as well by theorem 2.1.8 . □

2.2 Involution and Determinant

Since in the following I mainly concentrate on K_1 for explicit calculations, I will need the following results on how the determinant behaves with respect to the induced involution on $GL(R)$.

Lemma 2.2.1. For R a commutative ring with τ an involution, $R^\times$ its group of units and

$$\det\colon GL(R) \to R^\times$$

the determinant map, there are the following equalities:

(1) $\det \circ GL_r(\tau) = \tau \circ \det$,

(2) $\det \circ T = \det$,

(3) $\det \circ \iota_{GL_r(R)} = \iota_{R^\times} \circ \det$.

Proof. (1) For $A \in GL_r(R)$ calculate

$$\begin{aligned}
\det(GL_r(\tau)(A)) &= \sum_{\sigma \in \Sigma_r} sgn(\sigma)(GL_r(\tau)(A))_{1,\sigma(1)} \cdot \ldots \cdot (GL_r(\tau)(A))_{r,\sigma(r)} \\
&= \sum_{\sigma \in \Sigma_r} sgn(\sigma)\tau(A_{1,\sigma(1)}) \cdot \ldots \cdot \tau(A_{r,\sigma(r)}) \\
&= \tau \left(\sum_{\sigma \in \Sigma_r} sgn(\sigma)A_{1,\sigma(1)} \cdot \ldots \cdot A_{r,\sigma(r)} \right) \\
&= \tau(\det(A)).
\end{aligned}$$

In particular note that this of course needed commutativity.

(2) The fact that transposition does not change the value of the determinant is a well-known fact of linear algebra, which is still true for commutative rings.

(3) The determinant transfers inverting matrices into inverting ring units, since the determinant is multiplicative $\det(AB) = \det(A)\det(B)$ for arbitrary commutative unital rings. This implies

$$1 = \det(1_r) = \det(A^{-1}A) = \det(A^{-1})\det(A)$$

and hence $\det(A^{-1}) = \det(A)^{-1}$. □

The induced involution on K_1 is thus transformed in the following manner:

Corollary 2.2.2. The determinant map and the induced involution commute as follows

$$\det \circ \tau_* = i \circ \tau \circ \det .$$

□

Therefore on the group of units, included as a subgroup in K_1, induced involutions look precisely as expected:

Proposition 2.2.3. For $j\colon R^\times \to K_1(R)$ the inclusion of units with $\det \circ j = id_{R^\times}$, the involution restricts as

$$\tau_*|_{j(R^\times)} = \iota_{R^\times} \circ \tau.$$

□

3 Involutions on Bimonoidal Categories

So far the involution defined in Richter [21] does not directly apply to the preceding chapters. This section summarises the results and definitions of [21] such that later chapters of this diploma thesis provide non-trivial examples of involutions on K-theory, which are induced by involutions of simplicial rings according to Burghelea and Fiedorowicz [7] as well as involutions on bimonoidal categories as defined by Richter [21].

Let $\mathcal{R}$ be a small category in this section. Following [21, Introduction of Chapter 2] and [10, Definition 3.3] I define a (strict) bimonoidal category as follows:

Definition 3.0.4. A strict bimonoidal category $\mathcal{R}$ is a category with two functors $\oplus, \otimes \colon \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ and two distinguished objects $0, 1$ together with a natural transformation $c_{\oplus}^{A,B} \colon A \oplus B \to B \oplus A$ and a natural isomorphism $d_l \colon A \otimes B \oplus A \otimes B' \to A \otimes (B \oplus B')$, which are subject to the following conditions:

- addition and multiplication $\oplus$ and $\otimes$ are strictly associative

$$A \oplus (B \oplus C) = (A \oplus B) \oplus C \quad A \otimes (B \otimes C) = (A \otimes B) \otimes C,$$

- the zero and the unit element are strictly neutral

$$A \oplus 0 = 0 \oplus A = A \quad 1 \otimes A = A \otimes 1 = A,$$

- the zero element strictly multiplies to zero and there is one strict distributivity

$$A \otimes 0 = 0 \otimes A = 0 \quad A \otimes B \oplus A' \otimes B = (A \oplus A') \otimes B,$$

- and the additive twist is a self-inverse map

$$c_{\oplus}^{B,A} \circ c_{\oplus}^{A,B} = id_{A \oplus B}.$$

Furthermore the natural transformations have to satisfy a (long) list of coherence conditions, which are spelled out in Laplaza [14] on pages 31-35 .

The following example illustrates why this is a suitable category analogue for rings.

Example 3.0.5. To each ring R associate a discrete category $\mathcal{R}_R$ with the following data:

- $Ob\mathcal{R}_R = R$,
- $r \oplus s := r + s$,
- $r \otimes s := rs$.

This is a strict bimonoidal category, where each of the natural transformations is the identity.

Of course there are also more interesting examples:

Example 3.0.6. [1, Example 2.3] For each natural number n denote by $[n]$ the following set $[n] := \{1, \dots, n\}$. Consider the following category E with

- $ObE = \{[n] | n \in \mathbb{N}\}$
- $E([n], [m]) = \{f : \{1, \dots, n\} \to \{1, \dots, m\} \mid f \in Set([n], [m])\}$,

which is a skeleton of the category of finite sets. Set $[n] \oplus [m] := [n + m]$, which extends to morphisms in the following fashion (for $f : [n] \to [n']$ and $g : [m] \to [m']$)

$$(f \oplus g)(i) := \begin{cases} f(i) & \text{for } 1 \le i \le n \\ n' + g(n - i) & \text{for } n + 1 \le i \le n + m. \end{cases}$$

Define $c_\oplus : [n + m] \to [m + n]$ by the formula

$$\begin{aligned} (c_\oplus)_{n,m} : [n + m] &\to [m + n] \\ i &\mapsto \begin{cases} m + i & \text{for } 1 \le i \le n \\ i - n & \text{for } n + 1 \le i \le n + m, \end{cases} \end{aligned}$$

which is obviously natural in n and m. Furthermore observe that $\oplus$ is strictly associative and $[0] := \emptyset$ is a strict unit with respect to $\oplus$.

Consider the multiplication $[n] \otimes [m] := [nm]$. This is extended to morphisms by choosing a natural bijection $[n] \times [m] \cong [nm]$ and applying morphisms componentwise, which I will not display in detail. The multiplication has a twist $c_\otimes$ as well, is strictly associative and has $[1]$ as a strict unit.

Restricted to isomorphisms this gives

$$E([n], [m]) = \begin{cases} \Sigma_n & \text{for } m = n \\ \emptyset & \text{otherwise} \end{cases}$$

Example 3.0.7. Depending on the context the objects $[n]$ can also be understood as dimensions or ranks of modules to give another category. The particular example $\mathcal{V}$ of Baas, Dundas and Rognes [3] is given by

$$\mathcal{V}([n],[m]) := \begin{cases} U(n) & \text{for } n = m \\ \emptyset & \text{otherwise,} \end{cases}$$

which is a skeleton of the category finite dimensional complex (unitary) vector spaces with just isomorphisms, which preserve the scalar product.

There is a category of matrices for strict bimonoidal categories:

Definition 3.0.8. The category of $n \times n$-matrices over $\mathcal{R}$, denoted as $M_n(\mathcal{R})$, is the following

- Objects are matrices of objects in $\mathcal{R}$

$$ObM_n(\mathcal{R}) := \{(A_{i,j})_{i,j=1,\dots,n} | A_{i,j} \in Ob\mathcal{R}\},$$

- Morphisms are matrices of morphisms between the respective components

$$Mor(A,B) := \{(\varphi_{i,j})_{i,j=1,\dots,n} | \varphi_{i,j} \in \mathcal{R}(A_{i,j}, B_{i,j})\}.$$

Checking the following lemma from [21] is tedious, but straightforward:

Lemma 3.0.9. [21, Lemma 2.2] For a bimonoidal category $(\mathcal{R}, \oplus, 0, c_\oplus, \otimes, 1_\mathcal{R})$ the category of $n \times n$-matrices is a monoidal category with the usual matrix multiplication

$$(A \cdot B)_{i,j} := \bigoplus_{k=1}^{n} A_{i,k} \otimes B_{k,j}.$$

Its unit is the unit matrix 1_n with $1_\mathcal{R}$ on the diagonal and zeroes everywhere else. Its associator α can be given by the distributivity morphisms in $\mathcal{R}$, whereas the left and right unitor morphisms $\lambda\colon 1_n \cdot A \to A$ and $\rho\colon A \cdot 1_n \to A$ are identities. □

Remark 3.0.10. The fact that α can be expressed by distributivity morphisms on the components, in particular implies that in the case, where both distributivity transformations are identities, the monoidal category of matrices $M_n(\mathcal{R})$ will be strict w.r.t. $\otimes$ as well.

Let $\mathcal{R}$ be a small bimonoidal category. Then its set of path components $\pi_0\mathcal{R} = \pi_0(|B\mathcal{R}|)$ is a ring except for additive inverses. (Also called ring without negatives, rig.) Since the sum $\oplus$ induces the structure of an abelian monoid, the usual group completion with respect to $\oplus$, i.e. $Gr(\pi_0(\mathcal{R})) = (-\pi_0\mathcal{R})\pi_0\mathcal{R}$, yields a ring. Set $R := (-\pi_0\mathcal{R})\pi_0 R$ and call R the ring associated to the bimonoidal category $\mathcal{R}$.

Denoting by $GL_n(\mathcal{R})$ the matrices which have inverse matrices, would yield far too few matrices to be interesting for a general bimonoidal category. For example, for $\mathbb{N}$ regarded as a bimonoidal category this would only yield permutation matrices. In order to get some more variety, weaken the invertibility as follows:

Definition 3.0.11. The monoid of weakly invertible $n \times n$-matrices over the rig $\pi_0(\mathcal{R})$, denoted by $GL_n(\pi_0(\mathcal{R}))$ is defined to be matrices in $M_n(\pi_0(\mathcal{R}))$, which are invertible if included into matrices over the ring associated to $\mathcal{R}$, i.e. in $M_n(R) = M_n(Gr(\pi_0\mathcal{R}))$.

With this define the category of weakly invertible matrices:

Definition 3.0.12. The category of weakly invertible $n \times n$-matrices over $\mathcal{R}$ is defined as the full subcategory of $M_n(\mathcal{R})$ with matrices A such that the projection to π_0-classes $[A]$ is in the weakly invertible matrices over $\pi_0(\mathcal{R})$, i.e. $GL_n(\pi_0(\mathcal{R}))$. Denote this category by $GL_n(\mathcal{R})$.

Less formal, since $\pi_0(\mathcal{R})$ is usually just a rig and not a ring, this additional step just collects all matrices which are invertible up to connecting chains of morphisms.

Lemma 3.0.13. [21, Remark between Definition 2.4 and Definition 2.5] Matrix multiplication respects weak invertibility, that is $\cdot\colon M_n(\mathcal{R}) \times M_n(\mathcal{R}) \to M_n(\mathcal{R})$ restricts to $GL_n(\mathcal{R})$.

Proof. First note that, if $\mathcal{R}$ is small, then each of the defined matrix categories is small as well, hence intersections are defined. The preceding sequence of definitions then gives the equivalences

$$\begin{aligned} A \in GL_n(\mathcal{R}) &\Leftrightarrow A \in M_n(\mathcal{R}) \wedge [A] \in GL_n(\pi_0(\mathcal{R})) \\ &\Leftrightarrow A \in M_n(\mathcal{R}) \wedge [A] \in M_n(\pi_0(\mathcal{R})) \wedge [A] \in GL_n((-\pi_0(\mathcal{R})\pi_0(\mathcal{R}))) \\ &\Leftrightarrow A \in M_n(\mathcal{R}) \wedge [A] \in GL_n((-\pi_0(\mathcal{R})\pi_0(\mathcal{R}))) \end{aligned}$$

and both of these conditions are compatible with matrix multiplication. $\square$

Remark 3.0.14. The stabilisation of $GL_n(R)$ for a ring R (cf. Definition 1.1.10) generalises to this case by mimicking the stabilisation of linear groups as follows. Define on objects

$$J_n\colon Ob(GL_n(\mathcal{R})) \to Ob(GL_{n+1}(\mathcal{R}))$$
$$J_n(A) := \begin{pmatrix} A & 0_n \\ 0_n^t & 1_{\mathcal{R}} \end{pmatrix}$$

(where 0_n^t denotes the horizontal zero n-tuple) and extend this on morphisms in the obvious fashion via

$$J_n(f_{i,j}\colon A \to B)_{k,l} := \begin{cases} f_{k,l}\colon A_{k,l} \to B_{k,l} & \text{for } 0 \leq k,l \leq n \\ id_{1_{\mathcal{R}}} & \text{for } k = l = n+1 \\ id_{0_{\mathcal{R}}} & \text{otherwise} \end{cases}$$

This defines a functor and hence defines in the usual fashion $GL(\mathcal{R})$ as the colimit in small categories $GL(\mathcal{R}) := \mathrm{colim}_{\mathcal{N}} GL_n(\mathcal{R})$, which still is a monoidal category.

3.1 Bar Construction for Monoidal Categories

From [3] I take the following bar construction for monoidal categories (no strictness assumed), so in this context for $GL(\mathcal{R})$.

Definition 3.1.1. [21, Definition 2.5] Let $(\mathcal{C}, \cdot, 1_{\mathcal{C}}, \alpha, \lambda, \rho)$ be a monoidal category (for α the associativity transformation, λ, ρ the unit transformations). Let $B_q(\mathcal{C})$ be the following category:

- Its objects are given by triangular matrices of objects in $\mathcal{C}$, such that for each $0 \leq i < j \leq q$ there is an object $A^{i,j} \in \mathcal{C}$, i.e.

$$\begin{pmatrix} A^{0,1} & A^{0,2} & \dots & A^{0,q} \\ & A^{1,2} & \dots & A^{1,q} \\ & & \ddots & \vdots \\ & & & A^{q-1,q} \end{pmatrix}.$$

- Furthermore for each $0 \leq i < j < k \leq q$ there is a (chosen) isomorphism

$$\varphi^{i,j,k}\colon A^{i,j} \cdot A^{j,k} \to A^{i,k}$$

subject to the coherence

$$\begin{tikzcd} (A^{i,j}A^{j,k})A^{k,l} \arrow[rr, "\alpha"] \arrow[d, "\varphi^{i,j,k}\cdot id"] & & A^{i,j}(A^{j,k}A^{k,l}) \arrow[d, "id\cdot\varphi^{j,k,l}"] \\ A^{i,k}A^{k,l} \arrow[r, "\varphi^{i,k,l}"] & A^{i,l} & A^{i,j}A^{j,l}. \arrow[l, "\varphi^{i,j,l}"'] \end{tikzcd}$$

- A morphism f in $B_q(\mathcal{C})$ is a set of morphisms $f^{i,j}\colon A^{i,j} \to B^{i,j}$ for each pair (i,j) satisfying $0 \leq i < j \leq q$, such that for all $0 \leq i < j < k \leq q$ and $\psi_{i,j,k}$ the isomorphisms of B the maps $f^{i,j}$ satisfy the following coherence

$$f^{i,k}\varphi^{i,j,k} = \psi^{i,j,k}(f^{i,j} \cdot f^{j,k}).$$

Example 3.1.2. The isomorphisms $\varphi^{i,j,k}$ are a necessary part of the data because of the additional choices which are involved by a non-strict associativity. The 1-simplices are the objects of $\mathcal{C}$, so no additional data. In $B_2\mathcal{C}$ there are typical simplices

$$\begin{pmatrix} a & ab \\ & b \end{pmatrix},$$

which are one specific instance of a chosen representative given the diagonal entries a, b. But for objects in $B_3\mathcal{C}$ and even fixed diagonal objects a, b, c and binary products ab, bc, this triangle can only be built up as follows

$$\begin{pmatrix} a & ab & ? \\ & b & bc \\ & & c \end{pmatrix},$$

which is the first occasion, where in the top right there is a choice of a representative for the triple product involved. The first coming to mind might be $a(bc)$ and $(ab)c$ and both are isomorphic via α, but there might be even more and the $\varphi^{i,j,k}$ fix the chosen isomorphisms on the way.

I will not prove the next lemma, but I want to exhibit the statement very clearly.

Lemma 3.1.3. The categories $B_q(\mathcal{C})$ form a simplicial category with the following face and degeneracy functors

$$\begin{aligned} d_i \colon B_q(\mathcal{C}) &\to B_{q-1}(\mathcal{C}) \;\; i = 0, \dots, q \\ (d_i(A_{k,l}))_{m,n} &= A_{\delta_i(m),\delta_i(n)} \\ s_i \colon B_q(\mathcal{C}) &\to B_{q+1}(\mathcal{C}) \;\; i = 0, \dots, q \\ (s_i(A_{k,l}))_{m,n} &= A_{\sigma_i(m),\sigma_i(n)}, \end{aligned}$$

where $\delta_i \colon [q-1] \to [q]$ is the monotonic map that skips i and $\sigma_i \colon [q+1] \to [q]$ is the monotonic map that hits i (and only i) twice. Since degeneracies s_i might be "hitting twice", I use the convention $A_{i,i} = 1_{\mathcal{C}}$ for each A and i.

The isomorphisms for $d_i(A)$ are just the isomorphisms $\varphi^{\delta_i(j),\delta_i(k),\delta_i(l)}$, since d_i does not change anything about strictness of $0 \leq j < k < l \leq q$. By the convention to take $A^{i,i} = 1_{\mathcal{C}}$ either $\lambda \colon 1 \cdot c \to c$ or $\rho \colon c \cdot 1 \to c$ are natural choices for isomorphisms φ, according to the position of the unit in the degenerate simplex. This is coherent by the coherence of the left and right unit transformation given by $\mathcal{C}$ being a monoidal category.

The extension of d_i and s_i to morphisms is just restricting a given $(f^{i,j})$ or inserting $f^{i,i} = id_{1_{\mathcal{C}}}$ at equal indices. $\square$

For strict monoidal categories however one can formally mimic the bar construction given in definition 1.1.1 and get the following result:

Theorem 3.1.4. [3, Prop 3.9] The bar construction $B\mathcal{C}$ is equivalent to the strict bar construction $[n] \mapsto \mathcal{C}_s^n$ for any strictly monoidal rigidification $\mathcal{C}_s$ of $\mathcal{C}$. □

Remark 3.1.5. One could either directly use the bar construction given for monoidal categories or strictify the monoidal category first, i.e. replace the associativity, left and right unit transformation by identities in an organised fashion, and then apply the usual bar construction (definition 1.1.1). This theorem implies that both approaches yield equivalent results.

Remark 3.1.6. In order to avoid confusion let me point out explicitly that there is a functor $U\colon \mathrm{CAT} \to Sets$ from the (1-)category of small categories to sets, which just sends each category to its set of objects and just forgets morphisms. This then extends to a forgetful functor $U\colon s\mathrm{CAT} \to sSets$ from simplicial categories to simplicial sets, which defines the geometrical realisation of $B_\bullet\mathcal{C}$ by setting $|B\mathcal{C}| := |UB\mathcal{C}|$.

3.2 K-Theory of a Strict Bimonoidal Category

By the preceding section there already is a classifying space associated to the category of weakly invertible matrices over a bimonoidal category, but so far there is no obvious extension to the meaning of using a Plus-construction $|BGL(\mathcal{R})|^+$. Fortunately the additional simplices do not affect the first homotopy group. The following lemma from Baas, Dundas and Rognes [3] simplifies the proof of lemma 3.2.2.

Lemma 3.2.1. [3, Proposition 5.3] Let B be a rig with additive Grothendieck group completion $A := Gr(B)$. Then the rig-homomorphism $B \to A$ induces a weak equivalence of classifying spaces

$$|BGL(B)| \to |BGL(A)|,$$

where $GL(B)$ denotes weakly invertible matrices over a rig. □

The fundamental group of $BGL(\mathcal{R})$ can hence be described explicitly by its associated ring R:

Lemma 3.2.2. Let $\mathcal{R}$ be a bimonoidal category and $R := (-\pi_0(\mathcal{R}))\pi_0(\mathcal{R})$ its associated ring.

(1) The natural projection

$$GL_n(\mathcal{R}) \to GL_n(R)$$

from weakly invertible $\mathcal{R}$-matrices to the general linear group over R induces a natural simplicial map of the (underlying set of the) bar construction of the monoidal category $GL_n(\mathcal{R})$ to the ordinary bar construction of the group $GL_n(R)$

$$p\colon B_\bullet(GL_n(\mathcal{R})) \longrightarrow B_\bullet(GL_n(R))$$

given by projecting to diagonal entries

$$p\left(\begin{pmatrix} A^{0,1} & A^{0,2} & \dots & A^{0,q} \\ & A^{1,2} & \dots & A^{1,q} \\ & & \ddots & \vdots \\ & & & A^{q-1,q} \end{pmatrix}, (\varphi^{i,j,k})\right) := ([A^{0,1}], [A^{1,2}], \dots, [A^{q-1,q}])$$

(2) Stabilisation of $GL_n(\mathcal{R})$ and $GL_n(R)$ with regard to n and geometrical realisation yield an isomorphism of fundamental groups

$$p_*\colon \pi_1(|BGL(\mathcal{R})|) \to \pi_1(|BGL(R)|) = GL(R).$$

In particular there is a natural inclusion of the elementary matrices over R as a perfect normal subgroup of $\pi_1|BGL(\mathcal{R})|$, i.e. $E(R) \hookrightarrow \pi_1(|BGL(\mathcal{R})|)$.

Proof. (1) Since it is not an ordinary induced map between bar constructions of the same kind, I explicitly present the compatibility with face maps in the case $l = 1, \dots, q-1$. Note that for matrices in $B_q(GL_n(\mathcal{R}))$ the associated isomorphisms give the relation $[A^{i,j}][A^{j,k}] = [A^{i,k}]$ in $\pi_0(\mathcal{R})$ for indices $i < j < k$, which implies

$$\begin{aligned} p(((d_l(A))^{i,j})_{0\le i<j\le q-1}) &= p(((A)^{d_l(i),d_l(j)})_{0\le i<j\le q-1}) \\ &= ([A^{0,1}], \dots, [A^{l-1,l+1}], \dots, [A^{q-1,q}]) \\ &= ([A^{0,1}], \dots, [A^{l-1,l}][A^{l,l+1}], \dots, [A^{q-1,q}]) \\ &= d_l([A^{0,1}], \dots, [A^{q-1,q}]) \\ &= (d_l \circ p)(A). \end{aligned}$$

The other cases are also readily derived from the existence of those morphisms.

(2) By the preceding lemma inspect

$$p\colon |B_\bullet(GL(\mathcal{R}))| \longrightarrow |B_\bullet(GL(\pi_0(\mathcal{R})))|$$

Without loss of generality assume that a given loop $\bar{\gamma}\colon \mathbb{S}^1 \to |BGL(\pi_0(\mathcal{R}))|$ is a cellular map, such that $\bar{\gamma}$ passes through finitely many 1-cells, since $\mathbb{S}^1$ is compact. More precisely, denote the unique 0-cell of $|BGL(\pi_0(\mathcal{R}))|$ by $*$, then the image of the loop except for $*$, i.e. $\bar{\gamma}(\mathbb{S}^1) \setminus *$, can be decomposed into a sequence of finitely many 1-cells. In particular for $\theta \in \mathbb{S}^1$ define $[A^{\bar{\gamma}(\theta)}]$ as either the unique cell of $\bar{\gamma}(\theta)$ or as $*$. It is clear that between two occurrences of the base-point $[A^{\bar{\gamma}(\theta)}]$ is constant. Choose a sequence of representatives $(A^{\gamma(\theta)})$ with $A^{\gamma(\theta)} = * \in BGL(\mathcal{R})$, if $\bar{\gamma}(\theta) = * \in |BGL(\pi_0(\mathcal{R}))|$ and $p(A^{\gamma(\theta)}) = [A^{\gamma(\theta)}]$, which are chosen to be locally constant as well. This gives a map

$$\gamma\colon \mathbb{S}^1 \to |BGL(\mathcal{R})|$$

by setting $\gamma(\theta) := [A^{\gamma(\theta)}, (\bar{\gamma})_2(\theta)]$, where $(\bar{\gamma})_2$ denotes the simplex coordinate in the image of Δ^1 modulo identification. This is a well-defined and hence a continuous map $\gamma\colon \mathbb{S}^1 \to |BGL(\mathcal{R})|$, which satisfies $p \circ \gamma = \bar{\gamma}$ per construction. Thus the induced map $p_*\colon \pi_1(|BGL(\mathcal{R})|) \to \pi_1(|BGL(\pi_0(R))|)$ is surjective.

By lifting homotopies in the same fashion with the useful choice

$$p \begin{pmatrix} A_1 & A_1 A_2 \\ & A_2 \end{pmatrix} = ([A_1], [A_2]),$$

the induced map of the projection is injective on π_1 as well. Therefore on fundamental groups there is an isomorphism

$$p_*\colon \pi_1(|BGL(\mathcal{R})|) \to \pi_1(|BGL(R)|) = GL(R).$$

□

Another strategy to prove the preceding lemma is to apply the result 3.1.4 and try to sufficiently understand the strictified monoidal category $GL(\mathcal{R})_s$. Since I only need this result in degree 1, lemma 3.2.2 allows to associate a K-theory space to a bimonoidal category:

Definition 3.2.3. Let $\mathcal{R}$ be a bimonoidal category and $R = (-\pi_0(\mathcal{R}))\pi_0(\mathcal{R})$ its associated ring. By the preceding lemmas there is a natural inclusion $E(R) \to \pi_1|BGL(\mathcal{R})|$ and hence the plus-construction can be used with respect to the perfect normal subgroup $E(R)$ to define

$$K(\mathcal{R}) = |BGL(\mathcal{R})|^+$$

and thus the K-groups by

$$K_n(\mathcal{R}) := \pi_n(K(\mathcal{R})) \quad (n \geq 1).$$

Remark 3.2.4. Again recall that I do not define K_0 here, since $K(\mathcal{R})$ is evidently connected.

3.3 Bimonoidal Categories with Involution

I will not go through all the details of the construction of an induced involution for a category with involution, since it is very explicitly given in Birgit Richter's paper [21, Chapter 4]. But I will state the relevant definitions to give the comparison theorem from [21, Proposition 4.12].

Definition 3.3.1. An anti-involution in a strict bimonoidal category $\mathcal{R}$ consists of a functor $\zeta\colon \mathcal{R} \to \mathcal{R}$ with $\zeta \circ \zeta = id$ together with natural isomorphisms

$$\mu\colon \zeta(A \otimes B) \to \zeta(B) \otimes \zeta(A)$$

for all $A, B \in \mathcal{R}$. These have to satisfy the following conditions The functor ζ is strictly symmetric monoidal with respect to $(\mathcal{R}, \oplus, O_{\mathcal{R}}, c_{\oplus})$, fixes the unit by $\zeta(1_{\mathcal{R}}) = 1_{\mathcal{R}}$ and $\mu_{1_{\mathcal{R}},A} = id_A = \mu_{A,1_{\mathcal{R}}}$ and satisfies coherences spelled out in [21, Definition 3.1]. Denote a bimonoidal category with involution by $(\mathcal{R}, \zeta, \mu)$.

Remark 3.3.2. Recall that the definition 3.0.4 of strict bimonoidal categories requires one of the distributivity morphisms to be strict while the other distributivity is not. This hinders a generalisation of the interpretation of anti-homomorphisms as morphisms into opposite structures. Opposing the multiplication to a category $\mathcal{R}^{op_{\otimes}}$ does not give a strict bimonoidal category in the sense of definition 3.0.4. Relaxing each distributivity to be just isomorphisms helps in this specific spot, but I will not elaborate on this.

I define the induced involution with a certain ignorance to all the technicalities involved in [21].

Definition 3.3.3. For $\mathcal{R}$ a bimonoidal category with involution $\zeta\colon \mathcal{R} \to \mathcal{R}$ let

$$\zeta_*\colon |BGL(\mathcal{R})| \to |BGL(\mathcal{R})|$$

be given by

$$\begin{aligned}\zeta_*&\left[\begin{pmatrix} A^{0,1} & \dots & A^{0,q} \\ & \ddots & \vdots \\ & & A^{q-1,q}\end{pmatrix}, (t_0, \dots, t_q)\right] \\ &:= \left[\begin{pmatrix} (A^{q-1,q})^t & \dots & \zeta(A^{0,q})^t \\ & \ddots & \vdots \\ & & \zeta(A^{0,1})^t\end{pmatrix}, (t_q, \dots, t_0)\right]\end{aligned}$$

It is quite clear that the natural inclusion of elementary matrices over R is respected by this involution, so there is an induced map on K-spaces

$$\zeta_*\colon K(\mathcal{R}) = |BGL(\mathcal{R})|^+ \to |BGL(\mathcal{R})|^+.$$

Call this the involution on K-theory induced by $\zeta\colon \mathcal{R} \to \mathcal{R}$.

Remark 3.3.4. This means the involution is induced by transposing each matrix (beware that transposition is a functor now), applying the involution to each matrix componentwise and reversing the triangle (of matrices) along the secondary diagonal. Of course the involved isomorphisms are to be changed accordingly as well.

In order to compare the involution on the linear group of a ring with involution with the one on a bimonoidal category with involution, one obviously needs to replace the inverse map. This is provided by the following lemma from [7] (in the proof of Proposition 4.5), which compares the canonical homeomorphism $|BG| \to |B(G^{op})|$ for a group G given by the inverse map to another one, which generalises to monoids easily.

Lemma 3.3.5. [7, Proof of Proposition 4.5] There is a homotopy between the homeomorphism

$$\kappa\colon |BG| \to |B(G^{op})|$$

given by $\kappa([(x_1, \dots, x_n), (t_0, \dots, t_n)]) := [(x_n, \dots, x_1), (t_n, \dots, t_0)]$ and the canonical homeomorphism $|B\iota|\colon |BG| \to |BG^{op}|$ given by the induced map of the inverse map.

Proof. (Sketch) The homotopy $H\colon |BG| \times I \to |BG^{op}|$ is given by

$$\begin{aligned} &H([(g_1, \dots, g_n), (t_0, \dots, t_n)], s) \\ &\quad := [(g_n, \dots, g_1, {g_1}^{-1}, \dots, {g_n}^{-1}), s(t_n, \dots, t_0, 0_n) + (1-s)(0_n, t_0, \dots, t_n)] \end{aligned}$$

where 0_n denotes a zero tuple with n positions. It is tedious to check but true that this is well-defined and a homotopy between the given maps. □

The following results compare the induced involutions on K-theory of strict bimonoidal categories and the K-theory of rings. Of course the comparison should be induced by the natural projection $p\colon GL_n(\mathcal{R}) \to GL_n(R)$. The induced involution defined in 3.3.3 extends the earlier definition for rings given in 2.1.6, if the projection induces a map on K-groups respecting the involution.

Theorem 3.3.6. [21, Corollary 4.11] For $(\mathcal{R}, \zeta, \mu)$ a bimonoidal category with involution let $R = (-\pi_0(\mathcal{R}))\pi_0(\mathcal{R})$ be its associated ring. Then R is a ring with anti-involution and the induced map of projecting to components in π_0

$$p\colon |BGL(\mathcal{R})| \to |BGL(R)|$$

induces a map of K-groups

$$K_i(\mathcal{R}) \to K_i(R)$$

respecting the two given induced involutions. More precisely, the diagram

$$\begin{array}{ccc} |BGL(\mathcal{R})| & \xrightarrow{p} & |BGL(R)| \\ \downarrow{\scriptstyle \zeta_*} & & \downarrow{\scriptstyle \zeta_*} \\ |BGL(\mathcal{R})| & \xrightarrow{p} & |BGL(R)| \end{array}$$

commutes.

Proof. Study the following relations

$$
\begin{aligned}
&(p\zeta_*)\left[\begin{pmatrix} A^{0,1} & \dots & A^{0,q} \\ & \ddots & \vdots \\ & & A^{q-1,q} \end{pmatrix}, (t_0, \dots, t_q)\right] \\
=&p\left[\begin{pmatrix} \zeta(A^{q-1,q})^t & \dots & \zeta(A^{0,q})^t \\ & \ddots & \vdots \\ & & \zeta(A^{0,1})^t \end{pmatrix}, (t_q, \dots, t_0)\right] \\
=&[([\zeta(A^{q-1,q})^t], \dots, [\zeta(A^{0,1})^t]), (t_q, \dots, t_0)]
\end{aligned}
$$

But R is an ordinary ring, so the fact $\kappa \simeq |B\iota|$ by 3.3.5 gives the following equality on homotopy groups

$$
\begin{aligned}
p_* \circ \zeta_* &= \kappa_* \circ |BT|_* \circ |B\zeta|_* \\
&= |B\iota| \circ |BT|_* \circ |B\zeta|_* \\
&= |B(\iota \circ T \circ \zeta)|_*,
\end{aligned}
$$

which is the induced involution defined for a ring with involution. □

In particular, the associated bimonoidal category to a ring with involution yields the same K-theory.

Corollary 3.3.7. [21, Proposition 4.12] Let R be any ring with involution and $\mathcal{R}_R$ its associated bimonoidal category (cf. Example 3.0.5), then there is an isomorphism of K-groups compatible with the induced involutions. The isomorphism is induced by the projection $p\colon Ob(\mathcal{R}_R) \to R$ and gives

$$K_*(\mathcal{R}_R) \cong K_*(R).$$

Proof. Inspect the bar construction of matrices over $\mathcal{R}_R$ first. For the bar construction over the discrete category $\mathcal{R}_R$ associated to the ring R the existence of a morphism $\varphi^{i,j,k}\colon A^{i,j}A^{j,k} \to A^{i,k}$ is equivalent to the statement $A^{i,j}A^{j,k} = A^{i,k}$. So the isomorphisms are redundant information, thus each simplex has the following form

$$
\begin{pmatrix} A^{0,1} & A^{0,1}A^{1,2} & \dots & A^{0,1}A^{1,2}\dots A^{q-1,q} \\ & A^{1,2} & \dots & A^{1,2}\dots A^{q-1,q} \\ & & \ddots & \vdots \\ & & & A^{q-1,q} \end{pmatrix}.
$$

This implies that each q-simplex is uniquely determined by its diagonal. Furthermore the equality $R = \pi_0(\mathcal{R}_R) = Gr(\pi_0(\mathcal{R}_R))$ implies that restricting the matrix components via p is just a plain identity.

This extends to an isomorphism of simplicial sets $Ob(B_\bullet(GL(\mathcal{R}_R))) \cong B_\bullet(GL(R))$ and theorem 3.3.6 implies that restricting the matrix entries induces a morphism of K-groups, which is compatible with the induced involutions, thus the claim follows. □

Theorem 3.3.6 of course also implies that one can detect non-trivial involutions on bimonoidal categories $\mathcal{R}$, which are not discrete. First reduce the category $\mathcal{R}$ to the rig $\pi_0(\mathcal{R})$, then group complete to $R = (-\pi_0(\mathcal{R}))\pi_0(\mathcal{R})$. This is a ring and hence might give a starting point for tools developed in algebraic K-theory of rings (as for example the trace map (cf. 5.2.4)).

Remark 3.3.8. An explicit example of that may be found in the proof of Proposition 8.1 in [21], where Richter proves that the involution on algebraic K-theory of complex topological K-theory ku is non-trivial. In "Two-Vector Bundles and Forms of Elliptic Cohomology" [3] the authors construct a bimonoidal category $\mathcal{V}$, which satisfies $K(ku) \simeq K(\mathcal{V})$ by the results of "Ring Completion of Rig Categories"[1] and "Stable Bundles over Rig Categories"[2]. This interpretation allows to give a map $K(ku) \to K^f(\mathbb{Z})$ (where $K^f(\mathbb{Z})$ is just a minor modification of $K(\mathbb{Z})$) and this map is induced by a map of bimonoidal categories $\mathcal{V} \to \mathcal{R}_{\mathbb{Z}}$.

The K-theory of $\mathbb{Z}$ is, although or maybe because it is still not completely known, intensively studied. In particular there is a result of Farrell and Hsiang (Lemma 2.4 in [11]) that the involution on $K_*(\mathbb{Z})$ is non-trivial, where the involution is induced by the identity. This result now implies that the involution on $K(ku)$ is non-trivial.

Indeed the actual Proposition 8.1 in [21] also states the non-triviality of the involution on $K(ko)$ for ko the spectrum of real topological K-theory and the non-triviality of the involution on Waldhausen's A-Theory of the double classifying space of an abelian group $A(BBG)$. The proof proceeds by an analogous strategy to identify $K(ko)$ and $A(BBG)$ as homotopy-equivalent to K-theory spaces of bipermutative categories, which also have a non-trivial map to $\mathcal{R}_{\mathbb{Z}}$.

Of course a detailed investigation of all those structures would require knowledge of spectra, various identifications of K-theory and other techniques, which are beyond the scope of this diploma thesis.

4 Non-trivial Involutions

In this section I want to construct some examples of rings with non-trivial involutions and study their induced involution on K_1. This will provide some non-trivial examples of induced involutions on K-theory. In particular it will provide examples for induced involution on K-theory of simplicial rings as defined in the paper of Burghelea and Fiedorowicz [7] and for involutions associated to a bipermutative category with involution as defined by Richter [21].

Obviously group rings (with commutative coefficients) provide a large class of examples, but in order to have a good understanding of how the units contribute to K_1, I will restrict to the commutative case.

4.1 Involution on Group Rings

For K_1 of a commutative ring R it is quite natural to investigate the determinant (cf. Proposition 1.1.19)

$$det\colon K_1(R) \to R^\times$$

and this is split by the inclusion $R^\times = GL_1(R) \to GL(R)$. This implies that for a group ring $R[G]$ with commutative coefficients R on an abelian group G, there is the following natural inclusion of groups

$$R^\times \times G \to (R[G])^\times$$

where it is quite common to call $R^\times \times G$ the trivial units in a group ring. But be warned that it is still an open problem to determine, under which restrictions all units of a group ring are just the trivial ones. In fact the unit conjecture according to a survey article by Lück and Reich [16] very modestly reads as follows:

"Let R be an integral domain and G be a torsion free group. Then every unit in $R[G]$ is trivial, i.e. of the form rg for some unit $r \in R^\times$ and $g \in G$."

Nonetheless group rings are a useful class of examples of rings with involutions by the fact that they carry a natural and in a manner universal involution.

Proposition 4.1.1. For R a commutative ring and G any group there is the following bijection

$$Hom(R[G], R[G]) \to Hom(R[G], (R[G])^{op})$$
$$\varphi \mapsto \iota \circ \varphi$$

where ι denotes the inverse map on G.

Proof. There is mainly one point that establishes this bijection and that is the identification

$$(R[G])^{op} = R[G^{op}]$$

which in this simplicity only works with commutative coefficients, otherwise one might need an involution on R itself. So the inverse map gives a map $R[G] \to R[G^{op}] \to (R[G])^{op}$, which can be understood as a canonical involution and hence provides the bijection as claimed. $\square$

So in this manner for $R[G]$ a commutative ring involutions on $R[G]$ are given by self-inverse homomorphisms on $R[G]$. Thus investigate involutions induced by compositions of self-inverse homomorphisms with the inverse map in the special case of commutative group rings.

Proposition 4.1.2. Let R be a commutative ring and G an abelian group. For a self-inverse homomorphism $\varphi\colon G \to G$ and the inverse map $\iota\colon G \to G^{op}$ the induced involution $\varphi \circ \iota$ on $R[G]$ yields a non-trivial involution on $K_1(R[G])$.

Proof. Proposition 2.2.1 gives that the determinant transforms an involution by the formula $\det \circ \tau_* = \iota \circ \tau \circ \det$ and by 4.1 the units are a natural subgroup of K_1. This implies (for $r \in R^\times$ and $g \in G$)

$$\begin{aligned}(\det \circ \tau_*)(rg) &= (\iota_{R^\times} \circ \tau)(rg) \\ &= (\iota_{R^\times} \circ \varphi \circ \iota_G)(rg) \quad = \iota_{R^\times}(r\varphi(g^{-1})) = r^{-1}\varphi(g).\end{aligned}$$

Therefore the induced involution is non-trivial on the subgroup of trivial units of K_1. $\square$

In particular note that the units in the coefficient ring are always inverted, independent of the chosen homomorphism.

4.2 Involutions on Laurent Polynomials

The formula before was quite explicit, but restricting gives an even more explicit class of examples.

4.2.1 Units in Laurent Polynomials

Proposition 4.2.1. For R a commutative ring there is an inclusion

$$\varphi\colon R^\times \oplus (\mathbb{Z}, +) \to R[t, t^{-1}]^\times$$

given by $\varphi(r, k) := rt^k$. ☐

But if R is commutative, then $R[t, t^{-1}]$ is commutative as well and so induction gives the following result:

Corollary 4.2.2. For R a commutative ring there is a natural inclusion

$$\varphi\colon R^\times \oplus (\mathbb{Z}^k, +) = R^\times \oplus \langle e_1, \ldots, e_k\rangle_{\mathbb{Z}} \to (R[t_1{}^\pm, \ldots, t_k{}^\pm])^\times$$

with $\varphi(r, m_1 e_1, \ldots, m_k e_k) := rt_1^{m_1} \ldots t_k^{m_k}$. ☐

So for the first K-group of such a ring the units include into the first summand of the well-known splitting

$$K_1(R[t_1^\pm, \ldots, t_k^\pm]) \cong (R[t_1^\pm, \ldots, t_k^\pm])^\times \oplus SK_1(R[t_1^\pm, \ldots, t_k^\pm]) \supseteq R^\times \oplus (\mathbb{Z}^k, +),$$

where SK_1 is the usual quotient

$$SL(R[t_1^\pm, \ldots, t_k^\pm])/E(R[t_1^\pm, \ldots, t_k^\pm]).$$

The preceding inclusions of trivial units even extend to equalities, if the coefficient ring R is an integral domain. To that end study the following results.

Lemma 4.2.3. For $R[t]$ the polynomial ring with coefficients in an integral domain R the indeterminate t is a prime element.

Proof. Suppose t is a divisor of a product of polynomials pq. This is equivalent to the assumption that pq has no constant term, which evidently means that at least one of the factors has no constant term as well and is therefore divisible by t. ☐

This result can be extended to study divisors of powers of t:

Lemma 4.2.4. Powers of the indeterminate in a polynomial ring $R[t]$ with coefficients in an integral domain R have no further divisors than powers of lower degree.

Proof. This is a proof by induction. For $t^0 = 1$ it is evident that only polynomials of degree zero, i.e. constant polynomials, can divide t^0. But if they divide 1 they are units and hence trivial divisors. Let $t^n = pq$, then by the preceding proposition t divides either p or q. Without loss of generality assume $t|p$, then there is a polynomial $\bar{p}$ satisfying the equality $\bar{p}t = p$, which implies $t^{n-1} = \bar{p}q$, which by the induction hypothesis yields $\bar{p} = rt^i$ and $q = r^{-1}t^{n-1-i}$. As a consequence the factors of t^n are $p = \bar{p}t = rt^{i+1}$ and $q = r^{-1}t^{n-1-i}$, hence follows the claim. ☐

These results extend furthermore to give all units in Laurent polynomials for coefficients in an integral domain:

Theorem 4.2.5. The units in a Laurent polynomial ring with coefficients in an integral domain are trivial, i.e. if $p \in R[t^{\pm 1}]$ is a unit, then p is of the form $p = rt^n$ with $r \in R^\times$ and $n \in \mathbb{Z}$.

Proof. Let $p, q \in R[t^{\pm 1}]$ satisfy the equality $pq = 1$. Write both in the following form

$$p = t^{-k}\bar{p} \quad \text{and} \quad q = t^{-l}\bar{q} \quad \text{for} \quad k, l \in \mathbb{N}.$$

Then this implies

$$1 = pq = t^{-(k+l)}\bar{p}\bar{q},$$

which is equivalent to the equality

$$t^{k+l} = \bar{p}\bar{q}.$$

But since this holds in the ordinary polynomials, the preceding lemma gives

$$\bar{p} = rt^i \quad \text{and} \quad q = rt^{k+l-i}$$

and therefore the initially given Laurent polynomials are of the form

$$p = t^{-k}\bar{p} = rt^{i-k} \quad \text{and} \quad q = t^{-l}\bar{q} = rt^{k-i},$$

which proves the claim. □

With the induction $R[t_1^\pm, \dots, t_k^\pm] = R[t_1^\pm, \dots, t_{k-1}^\pm][t_k^\pm]$ this gives the following corollary:

Proposition 4.2.6. For R an integral domain the units in Laurent polynomials of finitely many variables $R[t_1^\pm, \dots, t_k^\pm]$ are just the trivial ones

$$(R[t_1^\pm, \dots, t_k^\pm])^\times \cong R^\times \oplus \mathbb{Z}^k.$$

□

4.2.2 Involutions on $R[t_1^\pm, \dots, t_k^\pm]$

For simplicity I will restrict to the case of involutions, which are degree-preserving in the following manner:

Definition 4.2.7. For $p = \sum_{i \in \mathbb{Z}} a_i t^i \in R[t^\pm]$ define the degree $\deg p$ to be the following

$$\deg p := \max\{|i| \mid a_i \neq 0\}.$$

This can be extended to Laurent polynomials in finitely many variables. For $I = (i_1, \dots, i_k) \in \mathbb{Z}^k$ set $t^I := t_1^{i_1} \cdot \ldots \cdot t_k^{i_k}$.

Definition 4.2.8. For $p = \sum_{I \in \mathbb{Z}^k} a_I t^I \in R[t_1^{\pm}, \dots, t_k^{\pm}]$ define the degree $\deg p$ to be

$$\deg p := \max \left\{ \sum_{j=1}^{k} |i_j| \;\middle|\; a_I \neq 0 \right\}.$$

It is quite natural to expect involutions to preserve this degree and to be the identity on coefficients. In this case the involutions can be completely described as follows:

Proposition 4.2.9. For R an integral domain a degree-preserving involution

$$\varphi\colon R[t_1^{\pm}, \dots, t_k^{\pm}] \to R[t_1^{\pm}, \dots, t_k^{\pm}]$$

with $\varphi|_R = id_R$ determines and is determined by the following data:

- a permutation $f\colon \{1, \dots, n\} \to \{1, \dots, n\}$,
- a choice of signs $s\colon \{1, \dots, n\} \to \{-1, +1\}$,
- a choice of ring units $r_\bullet\colon \{1, \dots, n\} \to R^\times$,

which are subject to the following conditions:

- The permutation is its own inverse $f^2(i) = i \;\; \forall i \in \{1, \dots, n\}$.
- Transposed elements carry the same sign $s(i) = s(f(i)) \;\; \forall i \in \{1, \dots, n\}$.
- Transposed elements carry inverse units $r_{f(i)} = r_i^{-1} \;\; \forall i \in \{1, \dots, n\}$.

Proof. Let φ be an involution, which is the identity on coefficients and degree-preserving, then on generators φ is of the form

$$\varphi(t_i) = r_i t_{f(i)}^{s(i)}.$$

The condition $\varphi^2 = 1$ then reads as follows

$$t_i = \varphi^2(t_i) = \varphi(r_i t_f^{s(i)}(i)) = r_i r_{f(i)} t_{f^2(i)}^{s(f(i))f(i)}.$$

This gives the following restrictions

$$\begin{aligned} f^2(i) &= i, \\ s(i)s(f(i)) = 1 \Leftrightarrow \quad s(i) &= s(f(i)), \\ r_i r_{f(i)} = 1 \Leftrightarrow \quad r_i &= r_{f(i)}^{-1}. \end{aligned}$$

Furthermore it is clear that maps $f, s, r_\bullet$ satisfying these conditions yield degree-preserving involutions by the formula $\varphi(t_i) = r_i t_{f(i)}^{s(i)}$. $\square$

If the group of units of R is finite, then this gives the following corollary:

Corollary 4.2.10. For R an integral domain with finitely many units, i.e. $|R^\times| < \infty$, there are only finitely many involutions on $R[t_1^\pm, \dots, t_k^\pm]$, which are degree-preserving and trivial on coefficients.

Proof. According to the preceding proposition each involution, which is trivial on coefficients and degree-preserving, is uniquely determined by three maps in Σ_n, $Set(\{1, \dots, n\}, \{-1, +1\})$ and $Set(\{1, \dots, n\}, R^\times)$. The first two sets are finite and the last is finite by the assumption that there are just finitely many units. Therefore there are only finitely many involutions of the given type. □

Example 4.2.11. One sees that the group of arbitrary involutions on Laurent polynomials with at least two variables cannot be finite by the following observation. For R a commutative ring consider the maps $\varphi\colon R[t_1^\pm, t_2^\pm]$ given by

$$\varphi(t_1) = t_1 t_2^k \text{ and } \varphi(t_2) = t_2^{-1}.$$

This map is an involution for each $k \in \mathbb{Z}$, since $\varphi^2(t_1) = \varphi(t_1 t_2^k) = t_1 t_2^k t_2^{-k} = t_1$. Therefore the group of involutions cannot be finite and since the group of involutions is a $\mathbb{Z}/2\mathbb{Z}$-vector space it cannot even be finitely generated.

Of course the main interest is again, if these involutions yield non-trivial involutions on K-theory. Indeed they all do:

Theorem 4.2.12. Let R be an integral domain (with at least one non-trivial unit) and φ a degree-preserving involution $\varphi\colon R[t_1^\pm, \dots, t_k^\pm] \to R[t_1^\pm, \dots, t_k^\pm]$ with $\varphi|_R = id_R$. Then φ induces a non-trivial involution on the units of $R[t_1^\pm, \dots, t_k^\pm]$ and therefore on $K_1(R[t_1^\pm, \dots, t_k^\pm])$.

Proof. Since by theorem 4.2.5 the units are completely the determined as $R^\times \oplus \mathbb{Z}^k$ and by proposition 4.2.9 an involution can be described by a self-inverse permutation $f\colon \{1, \dots, n\} \to \{1, ..., n\}$ a choice of signs $s\colon \{1, \dots, n\} \to \{-1, +1\}$ and a choice of units $r_\bullet\colon \{1, \dots, n\} \to R^\times$, calculate the induced involution on the units $R^\times \oplus \mathbb{Z}^k$.

By proposition 2.2.3 an involution induces the following map on units:

- for $u \in R^\times$ this is

$$\varphi_*(u) = (\iota \circ \varphi)(u) = \iota(u) = u^{-1},$$

- for t_i a generator of a $\mathbb{Z}$-factor

$$\varphi_*(t_i) = (\iota \circ \varphi)(t_i) = \iota(r_i t_{f(i)}^{s(i)}) = (r_i)^{-1} t_{f(i)}^{s(i)} = r_{f(i)} t_{f(i)}^{s(i)}.$$

In particular the induced involution is non-trivial for every involution φ, since it is non-trivial on $R^\times$. $\square$

The induced involution of course remains non-trivial in the case of just commutative coefficients, which are not an integral domain. But it is neither clear, whether there are additional units in that case and hence, whether there are additional involutions in that case, since $\varphi(t_i) \in (R[t_1^\pm, \dots, t_k^\pm])^\times$ is not that useful to restrict the image of t_i. Nonetheless the non-triviality statement remains true in the following form:

Proposition 4.2.13. For R a commutative ring (with at least one non-trivial unit) and $\varphi\colon R[t_1^\pm, \dots, t_k^\pm] \to R[t_1^\pm, \dots, t_k^\pm]$ an involution given by maps $f\colon \{1, \dots, n\} \to \{1, \dots, n\}$, $s\colon \{1, \dots, n\} \to \{-1, +1\}$ and $r_\bullet\colon \{1, \dots, n\} \to R^\times$ as described in theorem 4.2.9, the induced involution is non-trivial on trivial units and hence non-trivial on K_1. $\square$

Thus each commutative ring yields a family of rings, which have non-trivial involutions on their K-theory.

Of course it would be nicer, if I could present (non-discrete) categories, which projected to this class of examples, but there seems to be no obvious candidate for a category associated to a group in the same manner as a group-ring.

5 Hochschild Homology of Rings

One of the tools to detect non-trivial classes in K-theory is the Dennis trace map from K-theory to Hochschild homology. In the following chapter I will check that this map commutes with the induced involutions on the groups in question, so that the Dennis trace map is a detection tool for rings with involution as well.

So at least by the formal similarity between Hochschild homology and Topological Hochschild homology visible in Chapter IX, Definition 2.1 of “Rings, Modules and Algebras in stable homotopy theory” [9] this algebraic analogue should not come as a surprise.

5.1 Definition of Hochschild Homology

I will consider Hochschild homology restricted to the case of unital $\mathbb{Z}$-algebras only, more commonly known as rings. In this I am mainly following the chapters 1 and 8.4 of [15].

For R a unital ring, let $C_n(R) := R^{\otimes n+1}$ and let

$$d_i^n \colon C_n(R) \to C_{n-1}(R) \quad \text{for} \quad i = 0, \dots, n$$

be defined as

$$d_i^n(r_0 \otimes r_1 \otimes \dots \otimes r_n) = \begin{cases} r_0 \otimes \dots \otimes r_i r_{i+1} \otimes \dots \otimes r_n & i = 0, \dots, n-1 \\ r_n r_0 \otimes \dots \otimes r_{n-1} & i = n \end{cases}$$

and furthermore

$$s_i^n \colon C_n(R) \to C_{n+1}(R) \quad \text{for} \quad i = 0, \dots, n$$

is

$$s_i^n(r_0 \otimes r_1 \otimes \dots \otimes r_n) = r_0 \otimes \dots \otimes r_i \otimes 1 \otimes r_{i+1} \otimes \dots \otimes r_n.$$

Note that no s_i places the unit on the left of r_0.

This constitutes a simplicial module $C_*(R)$ and hence allows to associate a chain-complex with boundary map $d^n := \sum\limits_{i=0}^{n} (-1)^i d_i^n$ and find a subcomplex of degenerate elements via:

$$D_n(R) := \sum_{i=0}^{n-1} \mathrm{im} s_i^{n-1}$$

Then the following is a standard fact about simplicial modules:

Proposition 5.1.1 (Normalising modules [cf. Loday Prop. 1.6.5])**.** The canonical projection

$$C_* \to C_*/D_*$$

is a quasi-isomorphism, i.e. it induces an isomorphism of homology groups.

Definition 5.1.2. For R a unital ring the Hochschild-homology of R is defined as the homology of the chain complex associated to $C_*(R)$ given above:

$$HH_n(R) := H_n(C_*(R)) \quad n \in \mathbb{N}_0.$$

Remark 5.1.3. The preceding proposition thus allows the calculation of Hochschild-homology via $C_*(R)/D_*(R)$ as well.

Since I focused a lot on group rings so far it is natural to investigate Hochschild homology of group rings as well. For group rings with integer coefficients there is a very satisfactory identification of Hochschild homology.

Theorem 5.1.4. ([15], Proposition 7.4.2) Let G be a group, then the Hochschild homology of the group ring $\mathbb{Z}[G]$ is naturally isomorphic to the group homology of G (cf. Theorem 1.1.7)

$$HH_*(\mathbb{Z}[G]) \cong H_*(G).$$

□

Example 5.1.5. This result and proposition 1.1.9 in particular give a natural identification of the first Hochschild homology group, which is of particular interest for the following examples

$$HH_1(\mathbb{Z}[G]) \cong H_1(G) = G_{ab}.$$

Group rings might seem like an artificial class of examples, but they determine the first Hochschild homology of $\mathbb{Z}[\zeta_p]$ for ζ_p a root of unity to a prime number $p \geq 3$. For convenience I first identify this ring as a quotient of $\mathbb{Z}[X]$.

Proposition 5.1.6. For $p \in \mathbb{N}$ a prime number, $\zeta_p \in \mathbb{C}$ a p-th root of unity and

$$\varphi_p(X) = \sum_{i=0}^{p-1} X^i \in \mathbb{Z}[X] \subset \mathbb{Q}[X]$$

the cyclotomic polynomial of degree p, there is an isomorphism

$$\mathbb{Z}[X]/(\varphi_p) \cong \mathbb{Z}[\zeta_p]$$

induced by the evaluation homomorphism $X \mapsto \zeta_p$.

Proof. This fact is essentially what one would expect from the case of field extensions, because there is an isomorphism $\mathbb{Q}[X]/(\varphi_p) \cong \mathbb{Q}[\zeta_p]$ given by identification of φ_p as the minimal polynomial of ζ_p. Nonetheless there is a technical point to show that the principal ideal generated in $\mathbb{Q}[X]$ by φ_p restricts to the principal ideal generated by φ_p in $\mathbb{Z}[X]$. Formally this is

$$(\varphi_p)_{\mathbb{Q}[X]} \cap \mathbb{Z}[X] \subset (\varphi_p)_{\mathbb{Z}[X]},$$

i.e. one cannot generate more polynomials with integer coefficients, if one allows polynomials with rational coefficients. This is a consequence of the fact that the coefficients of the cyclotomic polynomial for a prime degree are all 1, which is a unit in $\mathbb{Z}$. $\square$

Remark 5.1.7. There is the elementary identification

$$\mathbb{Z}[\mathbb{Z}/p\mathbb{Z}] \cong \mathbb{Z}[X]/(X^p - 1)$$

and hence a projection

$$\pi\colon \mathbb{Z}[\mathbb{Z}/p\mathbb{Z}] \cong \mathbb{Z}[X]/(X^p - 1) \longrightarrow \mathbb{Z}[X]/(\varphi_p) \cong \mathbb{Z}[\zeta_p],$$

because of the factorisation $X^p - 1 = \varphi_p(X)(X-1)$.
In the next proof I will use both descriptions of $\mathbb{Z}[\zeta_p]$ as either a subring of $\mathbb{C}$ or as a quotient of $\mathbb{Z}[X]$ without explicit mention of the isomorphism.

I use this projection to determine the Hochschild homology of $\mathbb{Z}[\zeta_p]$ in degree 1.

Theorem 5.1.8. The first Hochschild homology group of the integers with an adjoint p-th root of unity ζ_p for an odd prime number p is given by

$$HH_1(\mathbb{Z}[\zeta_p]) \cong \mathbb{Z}/p\mathbb{Z}$$

with the isomorphism given on generators by $[\zeta_p^i \otimes \zeta_p^j] \mapsto [j]$.

Proof. First check that $HH_1(\mathbb{Z}[\zeta_p])$ is finite. To that end study the chain map induced by π in degrees 0 and 1

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{Z}[\mathbb{Z}/p\mathbb{Z}] \otimes \mathbb{Z}[\mathbb{Z}/p\mathbb{Z}] \arrow[r, "d^1"] \arrow[d, "\pi_*"] & \mathbb{Z}[\mathbb{Z}/p\mathbb{Z}] \arrow[r] \arrow[d, "\pi_*=\pi"] & 0 \\
\cdots \arrow[r] & \mathbb{Z}[\zeta_p] \otimes \mathbb{Z}[\zeta_p] \arrow[r, "d^1"] & \mathbb{Z}[\zeta_p] \arrow[r] & 0.
\end{tikzcd}$$

Evidently π_* is still surjective in each degree, in particular in degree 1. But since both of the involved rings are commutative, the first boundary map is trivial $d^1 = 0$; so π_* is surjective on cycles as well and thus also on homology. Hence the first Hochschild homology group $HH_1(\mathbb{Z}[\zeta_p])$ is a quotient of $\mathbb{Z}/p\mathbb{Z}$ and thus either trivial or $\mathbb{Z}/p\mathbb{Z}$ itself.

Consider the following map

$$\begin{aligned}\Phi\colon \mathbb{Z}[X]/(\varphi_p) \otimes \mathbb{Z}[X]/(\varphi_p) &\to \mathbb{Z}/p\mathbb{Z}\\ [X^i] \otimes [X^j] &\mapsto [j]\end{aligned}$$

It is well-defined with respect to the quotient by φ_p in either component by the following calculations. In the first component this is

$$\Phi\left(\left[\sum_{i=0}^{p-1} X^i\right] \otimes \left[X^j\right]\right) = \sum_{i=0}^{p-1}[j] = [p \cdot j] = 0.$$

In the second component the relation introduced by φ_p gives

$$\Phi\left(\left[X^j\right] \otimes \left[\sum_{i=0}^{p-1} X^i\right]\right) = \left[\sum_{i=0}^{p-1} i\right] = \left[\frac{p(p-1)}{2}\right] = 0,$$

because $(p-1)$ is even, which implies that $\frac{p(p-1)}{2}$ is divisible by p.

It is obvious that this map is surjective and furthermore on boundaries this yields

$$\begin{aligned}\Phi(d^2(X^i \otimes X^j \otimes X^k)) &= \Phi(X^{i+j} \otimes X^k - X^i \otimes X^{j+k} + X^{i+k} \otimes X^j)\\ &= [k] - [j+k] + [j] = 0.\end{aligned}$$

So this map is still well-defined and surjective as a map from $HH_1(\mathbb{Z}[\zeta_p])$ to $\mathbb{Z}/p\mathbb{Z}$ and thus by the preceding calculation it is an isomorphism. □

5.2 Construction of the Trace Map

A tool to detect non-trivial classes in K-groups is the so called Dennis trace map from K-theory to the just defined Hochschild homology of R. It was defined by Keith Dennis, but that paper was never published, hence the first reference is Kiyoshi Igusa [13] in 1984.

Remark 5.2.1. Recall the usual trace map of matrices

$$tr\colon M_r(R) \to R$$

with $tr(A) := \sum\limits_{i=1}^{r} A_{ii}$, which is obviously compatible with the stabilisation of square $r \times r$-matrices given by

$$\begin{aligned}i\colon M_r(R) &\to M_{r+1}(R)\\ A &\mapsto \begin{pmatrix} A & 0_r \\ 0_r^t & 0 \end{pmatrix},\end{aligned}$$

which just adds bordering zeroes.

There is generalisation of the trace map to tensor products of matrix rings (of equal size) as follows:

Definition 5.2.2. The generalised trace map

$$tr\colon M_r(R)^{\otimes n+1} \to R^{\otimes n+1}$$

is given by

$$tr(A^0 \otimes A^1 \otimes \ldots \otimes A^n) := \sum_{i \in J} A^0_{i_0,i_1} \otimes A^1_{i_1,i_2} \otimes \ldots \otimes A^n_{i_n,i_0},$$

where $i = (i_0, \ldots, i_n)$ is from the set $J = \{1, \ldots, r\}^{n+1}$.

Definition 5.2.3. The fusion map

$$\mathrm{fus}\colon \mathbb{Z}[GL_r(R)] \to M_r(R)$$

is given by the extension of the identity on $GL_r(R)$ as follows

$$\mathrm{fus}\left(\sum_i k_i A_i\right) = \sum_i k_i A_i.$$

Remark 5.2.4. This might look quite tautological, but the point is to replace the formal sums in the group ring of $GL_r(R)$ by actual sums in $M_r(R)$. Be aware that $GL_r(R)$ has to stabilise by adding bordering zeroes and a 1 on the diagonal

$$\begin{aligned} GL_r(R) &\to GL_{r+1}(R) \\ A &\mapsto \begin{pmatrix} A & 0_r \\ 0_r^t & 1 \end{pmatrix}. \end{aligned}$$

So the fusion map evidently does not stabilise (cf. Loday [15, 8.4.1], The Fusion Map), but I will take care of that problem later.

Study the following sequence of maps (for B_* the bar complex (cf. Definition 1.1.4) and C_* the Hochschild complex)

$$\begin{tikzcd} B_n(\mathbb{Z}[GL_r(R)]) \arrow[r, "\mathrm{inc}"] & C_n(\mathbb{Z}[GL_r(R)]) \arrow[r, "C_n(\mathrm{fus})"] & C_n(M_r(R)) \arrow[r, "C_n(\mathrm{tr})"] & C_n(R) \arrow[d, "p_n"] \\ & & & C_n(R)/D_n(R) \end{tikzcd}$$

where in sequence there are the following maps

- $\mathrm{inc}\colon Z[(GL_r(R))^n] \to Z[GL_r(R)]^{\otimes n+1}$ is the map given by

$$\mathrm{inc}(g_1, \ldots, g_n) = (g_1 \cdot \ldots \cdot g_n)^{-1} \otimes g_1 \otimes \ldots \otimes g_n,$$

- $\mathrm{fus}^{\otimes n+1}\colon Z[GL_r(R)]^{\otimes n+1} \to M_r(R)^{\otimes n+1}$ is the $(n+1)$-fold tensor product of the fusion map,
- $tr\colon M_r(R)^{\otimes n+1} \to R^{\otimes n+1}$ is the generalised trace map defined before,
- $p_n\colon C_n(R) \to C_n(R)/D_n(R)$ is the canonical projection seen before.

Lemma 5.2.5. Each of these is a simplicial map, hence a chain map on the induced chain complexes.

Proof. The fusion map evidently is a simplicial morphism, since it is just an extension of the identity on generators, and the identity is a simplicial map. The canonical projection is simplicial as well, since the quotient is taken by a simplicial subcomplex.

For the inclusion of the bar complex into the Hochschild complex I check the interesting case $i = n$

$$\begin{aligned}(\mathrm{inc} \circ d_n(g_1 \otimes \ldots \otimes g_n)) &= \mathrm{inc}(g_1 \otimes \ldots \otimes g_{n-1}) \\ &= (g_1 \ldots g_{n-1})^{-1} \otimes g_1 \otimes \ldots \otimes g_{n-1} \\ &= g_n(g_1 \ldots g_n)^{-1} \otimes g_1 \otimes \ldots \otimes g_{n-1} \\ &= d_n((g_1 \ldots g_n)^{-1} \otimes g_1 \otimes \ldots \otimes g_n) \\ &= d_n(\mathrm{inc}(g_1 \otimes \ldots \otimes g_n)),\end{aligned}$$

which shows quite well, how the factor $(g_1 \ldots g_n)^{-1}$ contributes to the inclusion, while the other relations are straightforward calculations.

For the trace map from the Hochschild complex on $r \times r$-matrices over R to the Hochschild complex over R itself, check the exceptional case $i = n$ again

$$\begin{aligned}(d_n \circ tr)(A^0 \otimes \ldots \otimes A^n) &= d_n\left(\sum_{i \in J} A^0_{i_0,i_1} \otimes A^1_{i_1,i_2} \otimes \ldots \otimes A^n_{i_n,i_0}\right) \\ &= \sum_{i \in J} A^n_{i_n,i_0} A^0_{i_0,i_1} \otimes A^1_{i_1,i_2} \otimes \ldots \otimes A^{n-1}_{i_{n-1},i_n} \\ &= \sum_{i_0=1}^{r} \sum_{i' \in \{1,\ldots,r\}^n} A^n_{i'_n,i_0} A^0_{i_0,i'_1} \otimes A^1_{i'_1,i'_2} \otimes \ldots \otimes A^{n-1}_{i'_{n-1},i'_n} \\ &= \sum_{i' \in \{1,\ldots,r\}^n} (A^n A^0)_{i'_n,i'_1} \otimes A^1_{i'_1,i'_2} \otimes \ldots \otimes A^{n-1}_{i'_{n-1},i'_n} \\ &= tr(A^n A^0 \otimes A^1 \otimes \ldots \otimes A^{n-1}) \\ &= (tr \circ d_n)(A^0 \otimes A^1 \otimes \ldots \otimes A^{n-1} \otimes A^n).\end{aligned}$$

The same argument at each index gives the other relations as well. Thus follows the claim that all the maps given above are maps of simplicial abelian groups. $\square$

So far I have only defined all this on matrices of fixed degrees. Since the fusion map on its own does not stabilise, it is quite remarkable that the trace map does.

Theorem 5.2.6 (Dennis Trace Map)**.** [15, Proposition 8.4.3] The sequence of simplicial modules and maps defines a natural map (in unital rings R)

$$\overline{Dtr}\colon H_n(GL_r(R)) \to HH_n(R)$$

for all $n, r \in \mathbb{N}$, which is compatible with the inclusions $i_r\colon GL_r(R) \to GL_{r+1}(R)$ and as a consequence gives a natural map

$$\overline{Dtr}\colon H_n(GL(R)) \to HH_n(R)$$

which composed with the Hurewicz homomorphism $K_n(R) \to H_n(GL(R))$ is called the Dennis trace map $Dtr := \overline{Dtr} \circ h$.

Proof. This proof is directly taken from [15, Proposition 8.4.3].

It is evident that all the factors are natural maps in unital rings and via application of the functor $H_n\colon \mathrm{sAb} \to \mathrm{Ab}$, there is an induced map $\overline{Dtr}$. So the only point is to show that $\overline{Dtr}$ is stable. It is clear that $\mathrm{inc}\colon Z[GL_r(R)]^{\otimes n} \to Z[GL_r(R)]^{\otimes n+1}$ is natural and stable, so focus on the remainder

$$Z[GL_r(R)]^{\otimes n+1} \to M_r(R)^{\otimes n+1} \to R^{\otimes n+1} = C_n(R) \to C_n(R)/D_n(R)$$

and calculate

$$(p \circ tr \circ \mathrm{fus})\left(\begin{pmatrix} A^0 & \\ & 1 \end{pmatrix} \otimes \begin{pmatrix} A^1 & \\ & 1 \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} A^n & \\ & 1 \end{pmatrix}\right)$$

$$= (p \circ tr)\left(\begin{pmatrix} A^0 & \\ & 1 \end{pmatrix} \otimes \begin{pmatrix} A^1 & \\ & 1 \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} A^n & \\ & 1 \end{pmatrix}\right)$$
$$= p\left(\sum_{i \in J} \begin{pmatrix} A^0 & \\ & 1 \end{pmatrix}_{i_0,i_1} \otimes \begin{pmatrix} A^1 & \\ & 1 \end{pmatrix}_{i_1,i_2} \otimes \ldots \otimes \begin{pmatrix} A^n & \\ & 1 \end{pmatrix}_{i_n,i_0}\right)$$

If for any j the index equals $i_j = r+1$, then the only case, in which $\begin{pmatrix} A^j & \\ & 1 \end{pmatrix}_{i_j,i_{j+1}}$ can be non-zero, is when $i_{j+1} = r+1$ as well. Repeat this argument for each j, then the only non-trivial additional index family i is $i = (r+1, r+1, \ldots, r+1)$, each other index family can only contribute non-trivial factors, if $i_j \leq r \;\; \forall j \in \{1, \ldots, r\}$. This recovers the trace map of the $GL_r(R)$ and gives the equality

$$= p\left(tr(A^0 \otimes A^1 \otimes \ldots \otimes A^n) + 1 \otimes 1 \otimes \ldots \otimes 1\right)$$

But $\ker p$ contains each element which has at least on unit entry in any position except for the first. This yields

$$= (p \circ tr)(A^0 \otimes A^1 \otimes \ldots \otimes A^n) = (p \circ tr \circ \text{fus})(A^0 \otimes A^1 \otimes \ldots \otimes A^n).$$

Therefore $\overline{Dtr}$ is stable, if regarded as a map $B_n(\mathbb{Z}[GL_r(R)]) \to C_n(R)/D_n(R)$, so applying homology yields the induced map

$$\overline{Dtr}\colon H_n(GL(R)) \to H_n(C_n(R)/D_n(R)) \overset{5.1.1}{\cong} HH_n(R).$$

□

As a recurring example I specifically study degree 1 of Hochschild homology for $\mathbb{Z}[\zeta_p]$ for ζ_p a p-th root of unity and p prime. So far I presented that $K_1(R[G])$ contains the units of $\mathbb{Z}[\zeta_p]$, which includes the powers of ζ_p. This gives the following result:

Theorem 5.2.7. The Dennis trace map

$$Dtr\colon K_1(\mathbb{Z}[\zeta_p]) \to HH_1(\mathbb{Z}[\zeta_p])$$

is non-trivial.

Proof. Consider ζ_p as a unit, then this gives the equality

$$Dtr(\zeta_p) = tr(fus(\zeta_p^{-1} \otimes \zeta_p)) = \zeta_p^{p-1} \otimes \zeta_p,$$

which under the isomorphism of theorem 5.1.8 yields

$$\Phi(Dtr(\zeta_p)) = \Phi(\zeta_p^{p-1} \otimes \zeta_p) = [1] \in \mathbb{Z}/p\mathbb{Z} \cong HH_1(\mathbb{Z}[\zeta_p]),$$

so Dtr is even surjective, in particular it is not the zero map. □

5.3 Involution on Hochschild Homology

So far it is not clear, whether there is an induced involution on $HH_*(R)$, if there is one on R. Again I am following Loday [15], specifically Section 5.2.

5.3.1 Opposing the Simplicial Structure on the Hochschild Complex

The following complex gives a comparison on how opposing the ring changes the simplicial structure of the Hochschild complex:

Definition 5.3.1. For $\tilde{C}_n(R) := R^{\otimes n+1}$ set

$$\tilde{d}_i := d_{n-i} \quad \text{and} \quad \tilde{s}_i := s_{n-i}.$$

This still is a simplicial module, call it the opposite Hochschild complex.

Lemma 5.3.2. (Adapted from [15, 5.2.1]) There is a natural isomorphism of simplicial modules

$$C_*(R) \to \tilde{C}_*(R^{op})$$

given by

$$w_{HH}(r_0 \otimes r_1 \otimes \ldots \otimes r_n) = r_0 \otimes r_n \otimes \ldots \otimes r_1,$$

and w_{HH} is a map of simplicial abelian groups.

Proof. This is a calculation what w_{HH} does to face and degeneracy maps, where again I denote the opposed multiplication by $\circ$.

The map w_{HH} commutes with face maps by the following calculation

$$\begin{aligned}
w_{HH}d_i(r_0 \otimes r_1 \otimes \ldots \otimes r_n) &= \begin{cases} w_{HH}(r_0 \otimes \ldots \otimes r_i r_{i+1} \otimes \ldots \otimes r_n) & 0 \leq i < n \\ w_{HH}(r_n r_0 \otimes \ldots \otimes r_{n-1}) & i = n \end{cases} \\
&= \begin{cases} r_0 \otimes r_n \otimes \ldots \otimes r_i r_{i+1} \otimes \ldots \otimes r_1 & i = 0, \ldots, n-1 \\ r_n r_0 \otimes r_{n-1} \otimes \ldots \otimes r_1 & i = n \end{cases} \\
&= \begin{cases} r_0 \otimes r_n \otimes \ldots \otimes r_{i+1} \circ r_i \otimes \ldots \otimes r_1 & i = 0, \ldots, n-1 \\ r_0 \circ r_n \otimes r_{n-1} \otimes \ldots \otimes r_1 & i = n \end{cases} \\
&= \begin{cases} d_{n-i}(r_0 \otimes r_n \otimes \ldots \otimes r_1) & i = 0, \ldots, n-1 \\ d_0(r_0 \otimes r_n \otimes \ldots \otimes r_1) & i = n \end{cases} \\
&= \left.\begin{cases} d_{n-i}w_{HH}(r_0 \otimes \ldots \otimes r_n) & i = 0, \ldots, n-1 \\ d_0 w_{HH}(r_0 \otimes \ldots \otimes r_n) & i = n \end{cases}\right\} = \tilde{d}_i w_{HH}(r_0 \otimes \ldots \otimes r_n).
\end{aligned}$$

Furthermore w_{HH} commutes with degeneracies by the following equalities

$$\begin{aligned}
w_{HH}s_i(r_0 \otimes \ldots \otimes r_n) &= w_{HH}(r_0 \otimes \ldots \otimes r_i \otimes 1 \otimes r_{i+1} \otimes \ldots \otimes r_n) \\
&= r_0 \otimes r_n \otimes \ldots \otimes r_{i+1} \otimes 1 \otimes r_i \otimes \ldots \otimes r_1 \\
&= s_{n-i}(r_0 \otimes r_n \otimes \ldots \otimes r_1) = \tilde{s}_i w_{HH}(r_0 \otimes \ldots \otimes r_n).
\end{aligned}$$

The fact that w_{HH} is a natural isomorphism is evident. □

So far there is the following sequence of morphisms

$$C_*(R) \xrightarrow{w_{HH}} \tilde{C}_*(R^{op}) \xrightarrow{\tilde{C}_*(\tau)} \tilde{C}_*(R)$$

This does not give an endomorphism of the Hochschild complex associated to R (or its opposite complex), but the associated chain complexes give a comparison.

Lemma 5.3.3. (cf. [15, 5.2.1]) For M_* a simplicial module and $\tilde{M}_*$ the associated opposite module (defined via $\tilde{d}_i := d_{n-i}$ and $\tilde{s}_i := s_{n-i}$ as before), there is an isomorphism of the associated chain complexes $j\colon M_* \to \tilde{M}_*$ given by $j_n := (-1)^{\frac{n(n+1)}{2}} id$.

Proof. The following equalities hold

$$
\begin{aligned}
(-1)^{\frac{n(n+1)}{2}}\tilde{d} &= (-1)^{\frac{n(n+1)}{2}}\sum_{i=0}^{n}(-1)^i\tilde{d}_i \\
&= (-1)^{\frac{n(n+1)}{2}}\sum_{i=0}^{n}(-1)^i d_{n-i} \\
&= (-1)^{\frac{n(n+1)}{2}}\sum_{i=0}^{n}(-1)^{n-i} d_i \\
&= (-1)^{\frac{n(n+1)}{2}}(-1)^n\sum_{i=0}^{n}(-1)^i d_i \\
&= (-1)^{\frac{n(n+3)}{2}} d = (-1)^{\frac{(n-1)n}{2}} d
\end{aligned}
$$

which give the result that $(-1)^{\frac{n(n+1)}{2}}$ is indeed a chain map. $\square$

Remark 5.3.4. This lemma informally gives that opposing the simplicial structure changes the associated chain complex only up to sign. This implies the following result.

Theorem 5.3.5. (cf. [15, E.5.2.2]) The Hochschild homology groups of any ring R and its opposed ring R^{op} are naturally isomorphic

$$HH_*(R) \cong HH_*(R^{op}).$$

Proof. By lemma 5.3.2 and lemma 5.3.3 the isomorphism is given by composition of the simplicial isomorphism w_{HH} and the chain isomorphism j as follows

$$C_*(R) \xrightarrow{w_{HH}} \tilde{C}_*(R^{op}) \xrightarrow{j} C_*(R^{op}) \;,$$

which gives the result. $\square$

Much more importantly the identification 5.3.3 allows to induce an involution on Hochschild homology for rings with involution.

Corollary 5.3.6. Let R be a ring with involution τ. Then there is an induced map on its Hochschild homology, which is given by the following composition

$$C_*(R) \xrightarrow{w_{HH}} \tilde{C}_*(R^{op}) \xrightarrow{\tilde{C}_*(\tau)} \tilde{C}_*(R) \xrightarrow{j} C_*(R).$$

Call this map the involution induced by τ on Hochschild homology.

Beware that this is not a simplicial statement, but one in chain complexes. Furthermore the choice of sign is not the only possible choice, but in this case dictated by the application to the trace map.

In 5.1.8 I focused on extensions of the integers by a prime root of unity and proved that the Dennis trace map is non-trivial in degree 1. Investigate the two evident involutions on their Hochschild homology in degree 1.

Theorem 5.3.7. For $\mathbb{Z}[\zeta_p] \cong \mathbb{Z}[X]/(\varphi_p)$ an extension of the integers by a p-th root of unity for $p \in \mathbb{N}$ a prime number, there are the following involutions on Hochschild homology:

1. For $\overline{\cdot}$ the map induced by complex conjugation, i.e.

$$\begin{aligned} \overline{\cdot}\colon \mathbb{Z}[X]/(\varphi_p) &\to \mathbb{Z}[X]/(\varphi_p) \\ \left[\sum_{i=0}^{p-1} a_i X^i\right] &\mapsto \left[\sum_{i=0}^{p-1} a_i X^{p-i}\right], \end{aligned}$$

   the induced involution on $HH_1(\mathbb{Z}[X]/(\varphi_p))$ is trivial.

2. The identity map induces a non-trivial involution on the first Hochschild homology group.

Proof. 1. On generators of the form $[[X^i] \otimes [X^j]]$ complex conjugation induces a map as follows

$$(\overline{\cdot})_*([[X^i] \otimes [X^j]]) = (-1)^{\frac{1(1+1)}{2}}([[X^{p-i}] \otimes [X^{p-j}]]),$$

which by the isomorphism of theorem 5.1.8 reduces as follows

$$\pi_*((\overline{\cdot})_*([[X^i] \otimes [X^j]])) = -[p-j] = [j] = \pi_*([[X^i] \otimes [X^j]]).$$

Since π_* is an isomorphism by theorem 5.1.8 this implies that $(\overline{\cdot})_*$ is the identity.
2. For the identity map calculate the following

$$(id)_*([[X^i] \otimes [X^j]]) = -[[X^i] \otimes [X^j]],$$

and $\pi_*(-[[X^i] \otimes [X^j]]) = -[j] = [p-j]$ implies that the identity induces the inverse map on $\mathbb{Z}/p\mathbb{Z}$. □

5.4 The Dennis Trace Map Commutes with Involutions

I first present how the induced involution on Hochschild homology of a ring with involution can be unravelled. In particular recall the identification $(R[G])^{op} = R[G^{op}]$ for a commutative coefficient ring R and any group G (cf. Proposition 4.1.1). Furthermore recall the opposition of structures given by inverting, transposition and pointwise involution (cf. Lemma 2.1.1) and the opposing of simplicial structure as in definition

5.3.1. Note that S_* does not denote singular chains but cellular chains, B_* is the bar complex introduced in chapter 1 and C_* is the Hochschild complex defined in section 1 of this chapter. Finally $\tilde{\cdot}$ denotes the respective opposite simplicial structure for each of these. Inspect the following diagram

$$
\begin{tikzcd}
S_*(|BGL_r(R)|,\mathbb{Z}) \arrow[r, "\sim"] \arrow[d, "S_*|w_{Bar}|"] & B_*(\mathbb{Z}[GL_r(R)]) \arrow[rr, "\mathrm{inc}"] \arrow[d, "w_{Bar}"] \arrow[dr, phantom, "(1)"] & & C_*(\mathbb{Z}[GL_r(R)]) \arrow[r, "\mathrm{fus}"] \arrow[d, "w_{HH}"] & C_*(M_r(R)) \arrow[r, "tr"] \arrow[d, "w_{HH}"] \arrow[dr, phantom, "(2)"] & C_*(R) \arrow[dd, "w_{HH}"] \\
S_*(|\tilde{B}GL_r(R)^{op}|,\mathbb{Z}) \arrow[r, "\sim"] \arrow[d, "S_*|\tilde{B}(T)|"] & \tilde{B}_*(\mathbb{Z}[GL_r(R)^{op}]) \arrow[r, "\mathrm{inc}"] \arrow[d, "\tilde{B}_*(\mathbb{Z}[T])"] & \tilde{C}_*(\mathbb{Z}[GL_r(R)^{op}]) \arrow[r, equal] \arrow[d, "\tilde{C}_*(\mathbb{Z}[T])"] & \tilde{C}_*(\mathbb{Z}[GL_r(R)]^{op}) \arrow[r, "\mathrm{fus}"] & \tilde{C}_*(M_r(R)^{op}) \arrow[d, "\tilde{C}_*(T)"] & \\
S_*(|\tilde{B}GL_r(R^{op})|,\mathbb{Z}) \arrow[r, "\sim"] \arrow[d, "S_*|\tilde{B}(GL_r(\tau))|"] & \tilde{B}_*(\mathbb{Z}[GL_r(R^{op})]) \arrow[r, "\mathrm{inc}"] \arrow[d, "\tilde{B}_*(\mathbb{Z}[GL_r(\tau)])"] & \tilde{C}_*(\mathbb{Z}[GL_r(R^{op})]) \arrow[rr, "\mathrm{fus}"] \arrow[d, "\tilde{C}_*(\mathbb{Z}[GL_r(\tau)])"] & & \tilde{C}_*(M_r(R^{op})) \arrow[r, "tr"] \arrow[d, "\tilde{C}_*(M_r(\tau))"] & \tilde{C}_*(R^{op}) \arrow[d, "\tilde{C}_*(\tau)"] \\
S_*(|\tilde{B}GL_r(R)|,\mathbb{Z}) \arrow[r, "\sim"] \arrow[d, "S_*(\Gamma)"] \arrow[dr, phantom, "(3)"] & \tilde{B}_*(\mathbb{Z}[GL_r(R)]) \arrow[r, "\mathrm{inc}"] \arrow[d, "j"] & \tilde{C}_*(\mathbb{Z}[GL_r(R)]) \arrow[rr, "\mathrm{fus}"] \arrow[d, "j"] & & \tilde{C}_*(M_r(R)) \arrow[r, "tr"] \arrow[d, "j"] & \tilde{C}_*(R) \arrow[d, "j"] \\
S_*(|BGL_r(R)|,\mathbb{Z}) \arrow[r, "\sim"] & B_*(\mathbb{Z}[GL_r(R)]) \arrow[r, "\mathrm{inc}"] & C_*(\mathbb{Z}[GL_r(R)]) \arrow[rr, "\mathrm{fus}"] & & C_*(M_r(R)) \arrow[r, "tr"] & C_*(R)
\end{tikzcd}
$$

Lemma 5.4.1. The diagram above commutes.

Proof. Most squares are instances of naturalities. More precisely the naturality of the isomorphism of theorem 1.1.7 in the left-most column, and naturality of the inclusion map inc, the fusion map fus and the trace map tr . There are only three exceptions, namely:

The complete reversal of coordinates in the bar complex is included into fixing the first coordinate and reversing the remaining coordinates (top middle). Furthermore the generalised trace map does not notice transposition of matrices (top right). Finally the homeomorphism of the classifying space to its opposite is coherent with the isomorphism j of chain complexes (bottom left).

(1) Let $\circ$ denote the multiplication in the opposite group, this yields

$$\begin{aligned}
(\mathrm{inc} \circ w_{Bar})(g_1 \otimes \ldots \otimes g_n) &= \mathrm{inc}(g_n \otimes \ldots \otimes g_1) \\
&= (g_n \ldots g_1)^{-1} \otimes g_n \otimes \ldots \otimes g_1 \\
&= (g_1 \circ \ldots \circ g_n)^{-1} \otimes g_n \otimes \ldots \otimes g_1 \\
&= w_{HH}((g_1 \circ \ldots \circ g_n)^{-1} \otimes g_1 \otimes \ldots \otimes g_n) \\
&= (w_{HH} \circ \mathrm{inc})(g_1 \otimes \ldots \otimes g_n).
\end{aligned}$$

(2) The familiar invariance of the trace map under transposition extends to this context as follows

$$\begin{aligned}
(tr \circ \tilde{C}_*(T) \circ w_{HH})(A^0 \otimes \ldots \otimes A^n) &= (tr \circ \tilde{C}_*(T))(A^0 \otimes A^n \otimes \ldots \otimes A^1) \\
&= tr(T(A^0) \otimes T(A^n) \otimes \ldots \otimes T(A^1)) \\
&= \sum_{i \in J} T(A^0)_{i_0,i_1} \otimes T(A^n)_{i_1,i_2} \otimes \ldots \otimes T(A^1)_{i_n,i_0} \\
&= \sum_{i \in J} {A^0}_{i_1,i_0} \otimes {A^n}_{i_2,i_1} \otimes \ldots \otimes {A^1}_{i_0,i_n} \\
&= w_{HH} \left(\sum_{i \in J} {A^0}_{i_1,i_0} \otimes {A^1}_{i_0,i_n} \otimes \ldots \otimes {A^n}_{i_2,i_1} \right) \\
&= (w_{HH} \circ tr)(A^0 \otimes \ldots \otimes A^n),
\end{aligned}$$

so the transposition just permuted the summands, which does not have any effect on the sum.

(3) In order to check, whether the square (3) commutes, inspect what the map

$$\begin{aligned}
\Gamma \colon BGL_r(R) &\to \tilde{B}GL_r(R) \\
[x, (t_0, \ldots, t_n)] &\mapsto [x, (t_n, \ldots, t_0)]
\end{aligned}$$

does to orientations of cells, since in this context there is a meaningful way of speaking of equal bases in the cellular complexes of $BGL_r(R)$ and $\tilde{B}GL_r(R)$. The induced map yields

$$S_*(\Gamma)(e_x) = (-1)^{\frac{n(n+1)}{2}} e_x$$

for each cell in $BGL_r(R)$, since this amounts to calculating the degree of the map

$$\begin{aligned} \Delta^n/(\partial\Delta^n) &\to \Delta^n/(\partial\Delta^n) \\ [t_0,\dots,t_n] &\mapsto [t_n,\dots,t_0] \end{aligned}$$

and that is precisely $(-1)^{\frac{n(n+1)}{2}}$ as chosen before for j in the following columns. Hence it follows that the whole diagram commutes. $\square$

This process stabilises as well, so the result extends to $GL(R)$.

Remark 5.4.2. Recall from lemma 3.3.5 that reversal of coordinates κ and the inverse map $|B\iota|$ are naturally homotopic as maps from $|BG|$ to $|B(G^{op})|$.

Notice furthermore that κ is equal to the composition of reversing the simplex coordinates Γ and reversal of the coordinates in the bar complex w_{Bar} as follows $\kappa = \Gamma \circ |w_{Bar}| = |w_{Bar}| \circ \Gamma$, which commute, since each map reverses just one type of coordinate. Furthermore Γ commutes with each map which is induced by a group homomorphism, in other words, Γ is a natural transformation $\Gamma\colon |B(\_)| \Rightarrow |\tilde{B}(\_)|$.

These results provide almost every coherence needed in order to show that the Dennis trace map commutes with the induced involutions defined before.

Theorem 5.4.3. The Dennis trace map is a natural transformation from the K-theory of rings with anti-involution to the Hochschild homology of rings with anti-involution. In particular Dtr transfers the involution given on K-theory to the one given on Hochschild homology.

Proof. By the first of the preceding lemmas the diagram above (5.4) reduces to the following square

$$\begin{tikzcd} S_*(|BGL_r(R)|,\mathbb{Z}) \arrow[r, "\overline{Dtr}"] \arrow[d, "J"] & C_*(R) \arrow[d, "\tau_*"] \\ S_*(|BGL_r(R)|,\mathbb{Z}) \arrow[r, "\overline{Dtr}"] & C_*(R) \end{tikzcd}$$

with $J := S_*(\Gamma) \circ S_*(|\tilde{B}(GL_r(\tau))|) \circ S_*(|\tilde{B}(T)|) \circ S_*(|w_{Bar}|)$ the left most sequence of maps and $\tau_*\colon C_*(R) \to C_*(R)$ being the induced involution on the Hochschild complex (cf. Definition 5.3.6). Thus the claim reduces to showing that J indeed induces the same map as the involution on $GL_r(R)$.

Since S_* is a functor, find

$$J = S_*(\Gamma \circ |\tilde{B}(GL_r(\tau))| \circ |\tilde{B}(T)| \circ |w_{Bar}|)$$

but $|\cdot|$ is a functor as well, so

$$J = S_*(\Gamma \circ |\tilde{B}(GL_r(\tau)) \circ \tilde{B}(T) \circ w_{Bar}|)$$

$\tilde{B}$ is a functor, hence

$$J = S_*(\Gamma \circ |\tilde{B}(GL_r(\tau) \circ T) \circ w_{Bar}|).$$

But Γ is a natural transformation between the geometrical realisations of the bar construction and its opposite, which implies

$$J = S_*(|B(GL_r(\tau) \circ T)| \circ |w_{Bar}| \circ \Gamma)$$

and by the earlier remark this is

$$J = S_*(|B(GL_r(\tau) \circ T)| \circ \kappa).$$

Applying homology to the chain complexes in the diagram introduces the liberty to take another representative for that map. Lemma 3.3.5 gives $\kappa \simeq |B\iota|$ and this yields the equations $H_*(J) = H_*(|B(GL_r(\tau) \circ T)| \circ \kappa) = H_*(|B(GL_r(\tau) \circ T)| \circ |B\iota|) = H_*(|B(GL_r(\tau) \circ T \circ \iota)|)$. But the last term is the induced involution on $BGL_r(R)$. Stabilised with respect to r this gives the following diagram

$$\begin{tikzcd}
K_*(R) \arrow[r, "h"] \arrow[d, "\tau_*"] & H_*(GL(R)) \arrow[r, "\overline{Dtr}"] \arrow[d, "\tau_*"] & HH_*(R) \arrow[d, "\tau_*"] \\
K_*(R) \arrow[r, "h"] & H_*(GL(R)) \arrow[r, "\overline{Dtr}"] & HH_*(R).
\end{tikzcd}$$

This proves the claim, because the first square commutes by naturality of the Hurewicz map and the second by the previous calculation. Therefore the Dennis trace map is a natural transformation of functors from unital rings with anti-involution to the category of abelian groups. $\square$

Remark 5.4.4. In other words I have now proved that the Dennis trace map can be used to detect non-trivial involutions on K-theory in arbitrary degrees by calculating induced involutions on Hochschild homology, which is easier in general. Nonetheless one still has to show that the trace map is non-trivial and that the involution is non-trivial on the image of the trace map. Both restrictions can hinder the detection of a non-trivial involution.

5.5 Detecting Involutions with the Trace Map

I have already shown in 5.2.7 for $\mathbb{Z}[\zeta_p]$ that the Dennis trace map is non-trivial. In particular the involution on K-theory given by the identity on $\mathbb{Z}[\zeta_p]$ cannot be trivial, because the involution on the first Hochschild homology group is non-trivial and the Dennis trace map commutes with the induced involutions. This is the case one would hope for, since the non-triviality of the involution in K-theory can be detected by calculating the involution in Hochschild homology.

I want to emphasise that the trace map drastically simplified this detection. If one instead wanted to calculate the induced involution on $K_1(\mathbb{Z}[\zeta_p]) \cong (\mathbb{Z}[\zeta_p])^\times \oplus SK_1(\mathbb{Z}[\zeta_p])$, this involves calculating the units, which is a famous classical result, the Dirichlet Unit Theorem (cf. Rosenberg [22] Theorem 2.3.8), and it needs $SK_1(\mathbb{Z}[\zeta_p]) = 0$, which according to Rosenberg again "is not an easy theorem and there doesn't seem to be an elementary proof" ([22] Remarks after Theorem 2.3.8.

However the Dennis trace map does not always detect non-trivial involutions. In order to investigate the involution on Hochschild homology of Laurent polynomials study the first homology group.

Remark 5.5.1. This part is adapted from Prop 1.1.10 [15]. For a commutative ring R there is the following interpretation of its first Hochschild homology. Everything in degree 1 is a cycle, if R is commutative. The boundaries introduce the following relation

$$R \otimes R \otimes R \to R \otimes R,$$
$$p \otimes q \otimes r \mapsto pq \otimes r - p \otimes qr + rp \otimes q.$$

This inspires the interpretation of $HH_1(R)$ as Kähler-Differentials $\Omega^1(R)$ for a commutative ring (cf. [15] 1.1.9/10), so write $adb := [a \otimes b]$ and $da = 1da$, which implies the relation

$$ad(bc) = abd(c) + acd(b)$$

such that the relation introduced by the boundaries is a Leibniz-rule as in differentials.

In particular this gives the usual result $dr^n = nr^{n-1}dr$ for each $r \in R$ and $n \in \mathbb{N}$, and for units in R even for each $n \in \mathbb{Z}$. Furthermore for $r \in R^\times$ the equality $0 = d1 = d(rr^{-1}) = rdr^{-1} + r^{-1}dr$ gives $rdr^{-1} = -r^{-1}dr$.

By 4.2 the involution induced by $t \mapsto t^{-1}$ on $K_1(R[t^\pm])$ is non-trivial. The following example investigates, whether the Dennis trace map detects that.

Example 5.5.2. By the relation $dr^n = nr^{n-1}dr$ it is clear that $HH_1(R[t^\pm])$ is generated by the elements $(dt, (dr)_{r\in R})$. The first K-groups has the trivial units as a subgroup $K_1(R[t, t^{-1}]) \supset R^\times \oplus \mathbb{Z}$.

For euclidean rings R it is actually even true that $K_1(R[t^\pm]) \cong R^\times \oplus \mathbb{Z}$, because there is a general splitting result for K-theory of Laurent rings, if the category of R-modules is well-behaved (cf. [23] Theorem 5.2).

For a trivial unit rt^k as an element of $K_1(R[t^\pm])$ the preceding example gives the following equations

$$\begin{aligned} Dtr(rt^k) &= tr(\mathrm{fus}(\mathrm{inc}(rt^k))) \\ &= tr(\mathrm{fus}(r^{-1}t^{-k}drt^k)) \\ &= tr(\mathrm{fus}(r^{-1}t^{-k}(rdt^k + t^k dr))) \\ &= tr(\mathrm{fus}(t^{-k}dt^k + r^{-1}dr)) \\ &= tr(\mathrm{fus}(kt^{-k}t^{k-1}dt + r^{-1}dr)) \\ &= tr(\mathrm{fus}(kt^{-1}dt + r^{-1}dr)) = kt^{-1}dt + r^{-1}dr. \end{aligned}$$

Furthermore on elements of this form the involution yields

$$\tau_*(kt^{-1}dt + r^{-1}dr) = -(ktdt^{-1} + r^{-1}dr) = k(-tdt^{-1}) - r^{-1}dr = kt^{-1}dt - r^{-1}dr.$$

So the involution on Hochschild homology of $R[t^{\pm}]$ is non-trivial, if there is a unit in R with $r^{-1}dr \neq 0$. In particular this gives a specific example, which provides an instance of a non-trivial involution on K-theory, which is not detected by the trace map, namely for $R = \mathbb{Z}$. In $\mathbb{Z}$ the only units are $\{\pm 1\}$ which give $1d(-1) = (-1)d1 = 0$. The result mentioned above even shows $K_1(\mathbb{Z}[t^{\pm}]) \cong \mathbb{Z}[t^{\pm}]^{\times}$. Thus there are no further elements in K_1 and so I described the involution on the complete image of the trace map $\mathrm{im}(\mathrm{Dtr})$. But here the involution is trivial, even though it is not trivial on $K_1(\mathbb{Z}[t^{\pm}])$.

I will close these examples with the strongest defect the trace map can have.

Example 5.5.3. By theorem 5.1.4 Hochschild homology of the group ring $\mathbb{Z}[G]$ is isomorphic to group homology

$$HH_*(\mathbb{Z}[G]) = H_*(G).$$

This implies that in principle an involution on K-theory of $\mathbb{Z}[G]$ can be detected by computing group homology, which in most cases is a feasible task. But for $\mathbb{Z}/p\mathbb{Z}$, there is the following well-known result (cf. Weibel [24] Example 6.2.3)

$$H_n(\mathbb{Z}/p\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \textit{for } n = 0 \\ \mathbb{Z}/p\mathbb{Z} & \textit{for } n \textit{ odd} \\ 0 & \textit{for } n \neq 0 \textit{ even}, \end{cases}$$

which in this case yields that the trace map is trivial for each even degree not equal to zero. Thus in even degrees the trace map cannot detect, whether an induced involution on K-theory of the group ring $K_{2n}(\mathbb{Z}[G])$ is non-trivial.

Remark 5.5.4. As a final remark I want to summarise the process of showing non-triviality of induced involutions on K-groups from the point of view of this diploma thesis: One would follow the strategy to consider a strict bimonoidal category, which might be a combinatorial model for K-theory of the object of interest (for example ko or ku as in [21]), then project to its components and group complete additively. This is a ring and hence allows to study the induced involution on the Hochschild homology of this ring. If this involution is non-trivial, one can try to retrace such a class back to the K-theory of the given bimonoidal category, which is an element not fixed by the induced involution on the K-groups of the bimonoidal category.

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