AN INTERPRETATION OF E_n-HOMOLOGY AS FUNCTOR HOMOLOGY

MURIEL LIVERNET AND BIRGIT RICHTER

ABSTRACT. We prove that E_n-homology of non-unital commutative algebras can be described as functor homology when one considers functors from a certain category of planar trees with n levels. For different n these homology theories are connected by natural maps, ranging from Hochschild homology and its higher order versions to Gamma homology.

1. Introduction

By neglect of structure, any commutative and associative algebra can be considered as an associative algebra. More generally, we can view such an algebra as an E_n-algebra, *i.e.*, an algebra over an operad in chain complexes that is weakly equivalent to the chain complex of the little-n-cubes operad of [4] for $1 \leqslant n \leqslant \infty$. Hochschild homology is a classical homology theory for associative algebras and hence it can be applied to commutative algebras as well. Less classically, Gamma homology [15] is a homology theory for E_∞-algebras and Gamma homology of commutative algebras plays an important role in the obstruction theory for E_∞ structures on ring spectra [14, 7, 1] and its structural properties are rather well understood [13].

It is desirable to have a good understanding of the appropriate homology theories in the intermediate range, *i.e.*, for $1 < n < \infty$. A definition of E_n-homology for augmented commutative algebras is due to Benoit Fresse [6] and the main topic of this paper is to prove that these homology theories possess an interpretation in terms of functor homology. We extend the range of E_n-homology to functors from a suitable category Epi_n to modules in such a way that it coincides with Fresse's theory when we consider a functor that belongs to an augmented commutative algebra and show in Theorem 4.1 that E_n-homology can be described as functor homology, so that the homology groups are certain Tor-groups.

As a warm-up we show in section 2 that bar homology of a non-unital algebra can be expressed in terms of functor homology for functors from the category of order-preserving surjections to k-modules. In section 3 we introduce our categories of epimorphisms, Epi_n, and their relationship to planar trees with n-levels. We introduce a definition of E_n-homology for functors from Epi_n to k-modules that coincides with Benoit Fresse's definition of E_n-homology of a non-unital commutative algebra, $\bar{A}$, when we apply our version of E_n-homology to a suitable functor, $\mathcal{L}(\bar{A})$. We describe a spectral sequence that has tensor products of bar homology groups as input and converges to E_2-homology. Section 4 is the technical heart of the paper. Here we prove that E_n-homology has a Tor interpretation. The proof of the acyclicity of a family of suitable projective generators is an inductive argument that uses poset homology.

For varying n, the derived functors that describe E_n-homology are related to each other via a sequence of homology theories

$$H_*^{E_1} \to H_*^{E_2} \to H_*^{E_3} \to \dots$$

In a different context it is well known that the stabilization map from Hochschild homology to Gamma homology can be factored over so called higher order Hochschild homology [9]: for a commutative algebra A there is a sequence of maps connecting Hochschild homology of A, $HH_*(A)$, to Hochschild homology of order n of A and finally to Gamma homology of A, $H\Gamma_{*-1}(A)$. We explain how higher order Hochschild homology is related to E_n-homology for n ranging from 1 to ∞ in 3.1.

Date: January 21, 2010.

2000 *Mathematics Subject Classification.* 13D03, 55P48, 18G15.

Key words and phrases. Functor homology, iterated bar construction, E_n-homology, Hochschild homology, operads.

The first author thanks MIT and the Clay Institute for hosting her and Haynes Miller for conversations on E_n-algebras. The second author thanks the Institut Galilée of Université Paris 13 for an invitation as professeur invité that led to this work. We are grateful to Benoit Fresse for catching a serious sign error.

In the following we fix a commutative ring with unit, k. For a set S we denote by $k[S]$ the free k-module generated by S.

2. Tor interpretation of bar homology

We interpret the bar homology of a functor from the category of finite sets and order-preserving surjections to the category of k-modules as a Tor-functor.

For unital k-algebras, the complex for the Hochschild homology of the algebra can be viewed as the chain complex associated to a simplicial object. In the absense of units, this is no longer possible.

Let $\bar{A}$ be a non-unital k-algebra. The bar-homology of $\bar{A}$, $H_*^{\mathrm{bar}}(\bar{A})$, is defined as the homology of the complex

$$C_*^{\mathrm{bar}}(\bar{A}) : \ldots \to \bar{A}^{\otimes n+1} \xrightarrow{b'} \bar{A}^{\otimes n} \xrightarrow{b'} \ldots \xrightarrow{b'} \bar{A} \otimes \bar{A} \xrightarrow{b'} \bar{A}$$

with $C_n^{\mathrm{bar}}(\bar{A}) = \bar{A}^{\otimes n+1}$ and $b' = \sum_{i=0}^{n-1}(-1)^i d_i$ where d_i applied to $a_0 \otimes \ldots \otimes a_n \in \bar{A}^{\otimes n+1}$ is $a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n$.

The category of non-unital associative k-algebras is equivalent to the category of augmented k-algebras. If one replaces $\bar{A}$ by $A = \bar{A} \oplus k$, then $C_n^{\mathrm{bar}}(\bar{A})$ corresponds to the reduced Hochschild complex of A with coefficients in the trivial module k, shifted by one: $H_*^{\mathrm{bar}}(\bar{A}) = HH_{*+1}(A, k)$, for $* \geqslant 0$.

Definition 2.1. Let Δ^{epi} be the category whose objects are the sets $[n] = \{0, \ldots, n\}$ for $n \geqslant 0$ with the ordering $0 < 1 < \ldots < n$ and whose morphisms are order-preserving surjective functions. We will call covariant functors $F \colon \Delta^{\mathrm{epi}} \to k\text{-mod}$ Δ^{epi}*-modules.*

We have the basic order-preserving surjections $d_i \colon [n] \to [n-1], 0 \leqslant i \leqslant n-1$ that are given by

$$d_i(j) = \begin{cases} j & j \leqslant i, \\ j-1 & j > i. \end{cases}$$

Any order-preserving surjection is a composition of these basic ones.

Definition 2.2. We define the *bar-homology of a* Δ^{epi}*-module* F as the homology of the complex $C_*^{\mathrm{bar}}(F)$ with $C_n^{\mathrm{bar}}(F) = F[n]$ and differential $b' = \sum_{i=0}^{n-1}(-1)^i F(d_i)$.

For a non-unital algebra $\bar{A}$ the functor $\mathcal{L}(\bar{A})$ that assigns $\bar{A}^{\otimes(n+1)}$ to $[n]$ and $\mathcal{L}(d_i)(a_0 \otimes \ldots \otimes a_n) = a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n$ $(0 \leqslant i \leqslant n-1)$ is a Δ^{epi}-module. In that case, $C_*^{\mathrm{bar}}(\mathcal{L}(\bar{A})) = C_*^{\mathrm{bar}}(\bar{A})$.

In the following we use the machinery of functor homology as in [11]. Note that the category of Δ^{epi}-modules has enough projectives: the representable functors $(\Delta^{\mathrm{epi}})^n \colon \Delta^{\mathrm{epi}} \to k\text{-mod}$ with $(\Delta^{\mathrm{epi}})^n[m] = k[\Delta^{\mathrm{epi}}([n], [m])]$ are easily seen to be projective objects and each Δ^{epi}-module receives a surjection from a sum of representables. The analogous statement is true for contravariant functors from Δ^{epi} to the category of k-modules where we can use the functors Δ_n^{epi} with $\Delta_n^{\mathrm{epi}}[m] = k[\Delta^{\mathrm{epi}}([m], [n])]$ as projective objects.

We call the cokernel of the map between contravariant representables

$$(d_0)_* \colon \Delta_1^{\mathrm{epi}} \to \Delta_0^{\mathrm{epi}}$$

b^{epi}. Note that $\Delta_0^{\mathrm{epi}}[n]$ is free of rank one for all $n \geqslant 0$ because there is just one map in Δ^{epi} from $[n]$ to $[0]$ for all n. Furthermore, $\Delta_1^{\mathrm{epi}}[0]$ is the zero module, because $[0]$ cannot surject onto $[1]$. Therefore

$$b^{\mathrm{epi}}[n] \cong \begin{cases} 0 & \text{for } n > 0, \\ k & \text{for } n = 0. \end{cases}$$

Proposition 2.3. *For any* Δ^{epi}*-module* F

$$H_p^{\mathrm{bar}}(F) \cong \mathrm{Tor}_p^{\Delta^{\mathrm{epi}}}(b^{\mathrm{epi}}, F) \text{ for all } p \geqslant 0. \tag{2.1}$$

For the proof recall that a sequence of Δ^{epi}-modules and natural transformations

$$0 \to F' \xrightarrow{\phi} F \xrightarrow{\psi} F'' \to 0 \tag{2.2}$$

is *short exact* if it gives rise to a short exact sequence of k-modules

$$0 \to F'[n] \xrightarrow{\phi[n]} F[n] \xrightarrow{\psi[n]} F''[n] \to 0$$

for every $n \geqslant 0$.

Proof. We have to show that $H_*^{\mathrm{bar}}(-)$ maps short exact sequences of Δ^{epi}-modules to long exact sequences, that $H_*^{\mathrm{bar}}(-)$ vanishes on projectives in positive degrees and that $H_0^{\mathrm{bar}}(F)$ and $b^{\mathrm{epi}} \otimes_{\Delta^{\mathrm{epi}}} F$ agree for all Δ^{epi}-modules F.

A short exact sequence as in (2.2) is sent to a short exact sequence of chain complexes

$$0 \longrightarrow C_*^{\mathrm{bar}}(F') \xrightarrow{C_*^{\mathrm{bar}}(\phi)} C_*^{\mathrm{bar}}(F) \xrightarrow{C_*^{\mathrm{bar}}(\psi)} C_*^{\mathrm{bar}}(F'') \longrightarrow 0$$

and therefore the first claim is true.

In order to show that $H_*^{\mathrm{bar}}(P)$ is trivial in positive degrees for any projective Δ^{epi}-module P it suffices to show that the representables $(\Delta^{\mathrm{epi}})^n$ are acyclic. In order to prove this claim we construct an explicit chain homotopy.

Let $f \in (\Delta^{\mathrm{epi}})^n[m]$ be a generator, *i.e.*, a surjective order-preserving map from $[n]$ to $[m]$. Note that $f(0) = 0$. We can codify such a map by its fibres, *i.e.*, by an $(m+1)$-tuple of pairwise disjoint subsets $(A_0, \dots, A_m)$ with $A_i \subset [n]$, $0 \in A_0$ and $\bigcup_{i=0}^{m-1} A_i = [n]$ such that $x < y$ for $x \in A_i$ and $y \in A_j$ with $i < j$. With this notation $d_i(A_0, \dots, A_n) = (A_0, \dots, A_{i-1}, A_i \cup A_{i+1}, \dots, A_n)$.

We define the chain homotopy $h\colon \Delta^{\mathrm{epi}}([n],[m]) \to \Delta^{\mathrm{epi}}([n],[m+1])$ as

$$h(A_0, \dots, A_m) := \begin{cases} 0 & \text{if } A_0 = \{0\}, \\ (0, A_0', A_1, \dots, A_m) & \text{if } A_0 = \{0\} \cup A_0', A_0' \neq \varnothing. \end{cases} \tag{2.3}$$

If $A_0 = \{0\}$, then

$$(b' \circ h + h \circ b')(\{0\}, \dots, A_m) = 0 + h \circ b'(\{0\}, \dots, A_m) = h(\{0\} \cup A_1, \dots, A_m) = (\{0\}, \dots, A_m).$$

In the other case a direct calculation shows that $(b' \circ h + h \circ b')(A_0, \dots, A_m) = \mathrm{id}(A_0, \dots, A_m)$.

It remains to show that both homology theories coincide in degree zero. By definition $H_0^{\mathrm{bar}}(F)$ is the cokernel of the map

$$F(d_0)\colon F[1] \longrightarrow F[0].$$

A Yoneda-argument [16, 17.7.2(a)] shows that the tensor product $\Delta_n^{\mathrm{epi}} \otimes_{\Delta^{\mathrm{epi}}} F$ is naturally isomorphic to $F[n]$ and hence the above cokernel is the cokernel of the map

$$((d_0)_* \otimes_{\Delta^{\mathrm{epi}}} \mathrm{id})\colon \Delta_1^{\mathrm{epi}} \otimes_{\Delta^{\mathrm{epi}}} F \longrightarrow \Delta_0^{\mathrm{epi}} \otimes_{\Delta^{\mathrm{epi}}} F.$$

As tensor products are right-exact [16, 17.7.2 (d)], the cokernel of the above map is isomorphic to

$$\mathrm{coker}((d_0)_*\colon \Delta_1^{\mathrm{epi}} \to \Delta_0^{\mathrm{epi}}) \otimes_{\Delta^{\mathrm{epi}}} F = b^{\mathrm{epi}} \otimes_{\Delta^{\mathrm{epi}}} F = \mathrm{Tor}_0^{\Delta^{\mathrm{epi}}}(b^{\mathrm{epi}}, F).$$

□

Remark 2.4. The generating morphisms d_i in Δ^{epi} correspond to the face maps in the standard simplicial model of the 1-sphere with the exception of the last face map.

3. Epimorphisms and trees

Planar level trees are used in [2], [6] and [3, 3.15] as a means to codify E_n-structures. An n-level tree is a planar level tree with n levels. We will use categories of planar level trees in order to gain a description of E_n-homology as functor homology. If $\mathcal{C}$ is a small category we denote by $N\mathcal{C}$ the nerve of $\mathcal{C}$.

Definition 3.1. Let $n \geqslant 1$ be a natural number. The category Epi_n has as objects the elements of $N_{n-1}(\Delta^{\mathrm{epi}})$, *i.e.*, sequences

$$[r_n] \xrightarrow{f_n} [r_{n-1}] \xrightarrow{f_{n-1}} \dots \xrightarrow{f_2} [r_1] \tag{3.1}$$

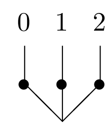
with $[r_i] \in \Delta^{\mathrm{epi}}$ and surjective order-preserving maps f_i. A morphism in Epi_n from the above object to an object $[r_n'] \xrightarrow{f_n'} [r_{n-1}'] \xrightarrow{f_{n-1}'} \dots \xrightarrow{f_2'} [r_1']$ consists of surjective maps $\sigma_i\colon [r_i] \to [r_i']$ for $1 \leqslant i \leqslant n$ such that

$\sigma_1 \in \Delta^{\text{epi}}$ and for all $2 \leqslant i \leqslant n$ the map σ_i is order-preserving on the fibres $f_i^{-1}(j)$ for all $j \in [r_{i-1}]$ and such that the diagram

$$\begin{array}{ccccccc} [r_n] & \xrightarrow{f_n} & [r_{n-1}] & \xrightarrow{f_{n-1}} & \cdots & \xrightarrow{f_2} & [r_1] \\ \downarrow{\scriptstyle \sigma_n} & & \downarrow{\scriptstyle \sigma_{n-1}} & & & & \downarrow{\scriptstyle \sigma_1} \\ [r'_n] & \xrightarrow{f'_n} & [r'_{n-1}] & \xrightarrow{f'_{n-1}} & \cdots & \xrightarrow{f'_2} & [r'_1] \end{array}$$

commutes.

As an example, consider the object $[2]\xrightarrow{\text{id}}[2]$ in Epi_2 which can be viewed as the 2-level tree



Possible maps from this object to $[2]\xrightarrow{d_0}[1]$ are
$$\begin{array}{ccc} [2] & \xrightarrow{\text{id}} & [2] \\ {\scriptstyle \text{id}}\downarrow & & \downarrow{\scriptstyle d_0} \\ [2] & \xrightarrow{d_0} & [1] \end{array} \quad \text{and} \quad \begin{array}{ccc} [2] & \xrightarrow{\text{id}} & [2] \\ {\scriptstyle (0,1)}\downarrow & & \downarrow{\scriptstyle d_0} \\ [2] & \xrightarrow{d_0} & [1] \end{array}$$
where $(0,1)$ denotes the transposition that permutes 0 and 1. For $\sigma_1 = d_1$ there is no possible σ_2 to fill in the diagram.

If $n = 1$, then Epi_1 coincides with the category Δ^{epi}. Note that there is a functor $\iota_n \colon \Delta^{\text{epi}} = \text{Epi}_1 \to \text{Epi}_n$ for all $n \geqslant 1$ with

$$\iota_n([m]) := [m]\longrightarrow[0]\longrightarrow\ldots\longrightarrow[0].$$

We call trees of the form $\iota_n([m])$ *palm trees with* $m+1$ *leaves.* More generally we have functors connecting the various categories of planar level trees.

Lemma 3.2. *For all* $n > k \geqslant 1$ *there are functors* $\iota_n^k \colon \text{Epi}_k \to \text{Epi}_n$, *with*

$$\iota_n^k(\,[r_k]\xrightarrow{f_k}\ldots\xrightarrow{f_2}[r_1]\,) = [r_k]\xrightarrow{f_k}\ldots\xrightarrow{f_2}[r_1]\longrightarrow[0]\longrightarrow\ldots\longrightarrow[0]$$

on objects, with the canonical extension to morphisms. □

Remark 3.3. The maps ι_n^k correspond to iterated suspension morphisms in [2, 4.1]. There is a different way of mapping a planar tree with n levels to one with $n+1$ levels, by sending $[r_n]\xrightarrow{f_n}\ldots\xrightarrow{f_2}[r_1]$ to $[r_n]\xrightarrow{\text{id}_{[r_n]}}[r_n]\xrightarrow{f_n}\ldots\xrightarrow{f_2}[r_1]$. We call such trees *fork trees* and they will need special attention later when we prove that representable functors are acyclic.

For any Σ_*-cofibrant operad $\mathcal{P}$ there exists a homology theory for $\mathcal{P}$-algebras which is denoted by $H_*^{\mathcal{P}}$ and is called $\mathcal{P}$-homology. Fresse studies the particular case of $\mathcal{P} = E_n$ a differential graded operad quasi-isomorphic to the chain operad of the little n-disks operad. He proves that for any commutative algebra the E_n-homology coincides with the homology of its n-fold bar construction. In fact, his result is more general since he defines an analogous n-fold bar construction for E_n-algebras and proves the result for any E_n-algebra in [6, theorem 7.26].

We consider the n-fold bar construction of a non-unital commutative k-algebra $\bar{A}$, $B^n(\bar{A})$, as an n-complex indexed over the objects in Epi_n, such that

$$B^n(\bar{A})_{(r_n,\ldots,r_1)} = \bigoplus_{[r_n]\xrightarrow{f_n}\ldots\xrightarrow{f_2}[r_1]\in\text{Epi}_n} \bar{A}^{\otimes(r_n+1)}.$$

The differential in $B^n(\bar{A})$ is the total differential associated to n-differentials $\partial_1, \ldots, \partial_n$ such that ∂_n is built out of the multiplication in $\bar{A}$, ∂_{n-1} corresponds to the shuffle multiplication on $B(\bar{A})$ and so on. We describe the precise setting in a slightly more general context.

In order to extend the Tor-interpretation of bar homology of Δ^{epi}-modules to functors from Epi_n to modules (alias Epi_n-modules) we describe the n kinds of face maps for Epi_n in detail by considering diagrams of the form

$$\begin{array}{ccccccccccccc} [r_n] & \xrightarrow{f_n} & [r_{n-1}] & \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_{j+2}} & [r_{j+1}] & \xrightarrow{f_{j+1}} & [r_j] & \xrightarrow{f_j} & [r_{j-1}] & \xrightarrow{f_{j-1}} \cdots \xrightarrow{f_2} & [r_1] \\ \tau_n^{i,j}\downarrow & & \tau_{n-1}^{i,j}\downarrow & & \tau_{j+1}^{i,j}\downarrow & & d_i\downarrow & & \mathrm{id}\downarrow & & \mathrm{id}\downarrow \\ [r_n] & \xrightarrow{g_n} & [r_{n-1}] & \xrightarrow{g_{n-1}} \cdots \xrightarrow{g_{j+2}} & [r_{j+1}] & \xrightarrow{g_{j+1}} & [r_j - 1] & \xrightarrow{g_j} & [r_{j-1}] & \xrightarrow{f_{j-1}} \cdots \xrightarrow{f_2} & [r_1]. \end{array} \tag{3.2}$$

Given the object in the first row, it is not always possible to extend $(d_i\colon [r_j] \to [r_j - 1], \mathrm{id}_{[r_{j-1}]}, \ldots, \mathrm{id}_{[r_1]})$ to a morphism in Epi_n: we have to find order-preserving surjective maps g_k for $j \leqslant k \leqslant n$ and bijections $\tau_k^{i,j}\colon [r_k] \to [r_k]$ that are order-preserving on the fibres of f_k for $j+1 \leqslant k \leqslant n$ such that the diagram commutes.

By convention we denote the constant map $[r_1] \to [0]$ by f_1.

Lemma 3.4.

(a) *There is a unique order-preserving surjection $g_j\colon [r_j - 1] \to [r_{j-1}]$ with $g_j \circ d_i = f_j$ if and only if $f_j(i) = f_j(i+1)$. When it exists, g_j is denoted by $f_j|_{i=i+1}$.*

(b) *If $f_j(i) = f_j(i+1)$ then we can extend the diagram to one of the form* (3.2) *so that $\tau_{j+1}^{i,j}$ is a shuffle of the fibres $f_{j+1}^{-1}(i)$ and $f_{j+1}^{-1}(i+1)$. Each choice of a $\tau_{j+1}^{i,j}$ uniquely determines the maps $\tau_k^{i,j}$ for all $j+1 < k \leqslant n$.*

(c) *If $f_j(i) = f_j(i+1)$ then each choice of a $\tau_{j+1}^{i,j}$ uniquely determines the maps g_k for $k \geqslant j$. The diagram* (3.2) *takes the following form*

$$\begin{array}{ccccccccccccc} [r_n] & \xrightarrow{f_n} & [r_{n-1}] & \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_{j+2}} & [r_{j+1}] & \xrightarrow{f_{j+1}} & [r_j] & \xrightarrow{f_j} & [r_{j-1}] & \xrightarrow{f_{j-1}} \cdots \xrightarrow{f_2} & [r_1] \\ \tau_n^{i,j}\downarrow & & \tau_{n-1}^{i,j}\downarrow & & \tau_{j+1}^{i,j}\downarrow & & d_i\downarrow & & \mathrm{id}\downarrow & & \mathrm{id}\downarrow \\ [r_n] & \xrightarrow{g_n^\tau} & [r_{n-1}] & \xrightarrow{g_{n-1}^\tau} \cdots \xrightarrow{g_{j+2}^\tau} & [r_{j+1}] & \xrightarrow{d_i f_{j+1}} & [r_j - 1] & \xrightarrow{f_j|_{i=i+1}} & [r_{j-1}] & \xrightarrow{f_{j-1}} \cdots \xrightarrow{f_2} & [r_1]. \end{array} \tag{3.3}$$

Proof. If there is such a map g_j, then $f_j(i+1) = g_j \circ d_i(i+1) = g_j \circ d_i(i) = f_j(i)$. As f_j is order-preserving, it is determined by the cardinalities of its fibres. The decomposition of morphisms in the simplicial category then ensures that we can factor f_j in the desired way.

For the third claim, assume that g_j exists with the properties mentioned in (a). As g_{j+1} and $d_i \circ f_{j+1}$ are both order-preserving maps from $[r_{j+1}]$ to $[r_j - 1]$, they are determined by the cardinalities of the fibres and thus they have to agree. Then $\tau_{j+1}^{i,j} = \mathrm{id}_{[r_{j+1}]}$ extends the diagram up to layer $j+1$. For the higher layers we then have to choose $g_k = f_k$ and $\tau_k^{i,j} = \mathrm{id}_{[r_k]}$.

In general, $\tau_{j+1}^{i,j}$ has to satisfy the conditions that it is order-preserving on the fibres of f_{j+1}. If $A_i = f_{j+1}^{-1}(i)$ then this implies that $\tau_{j+1}^{i,j}$ is an $(A_0, \ldots, A_{r_j})$-shuffle. Furthermore we have that

$$(d_i \circ f_{j+1})^{-1}(k) = \begin{cases} A_k & \text{if } k < i, \\ A_i \cup A_{i+1} & \text{if } k = i, \\ A_{k+1} & \text{if } k > i. \end{cases}$$

Therefore $\tau_{j+1}^{i,j}$ has to map $A_0, \ldots, A_{i-1}, A_{i+2}, \ldots, A_{r_j}$ identically and is hence an (A_i, A_{i+1})-shuffle.

If we fix a shuffle $\tau_{j+1}^{i,j}$, then the next permutation $\tau_{j+2}^{i,j}$ has to be order-preserving on the fibres of f_{j+2}, thus it is at most a shuffle of the fibres. In addition, it has to satisfy

$$g_{j+2} \circ \tau_{j+2}^{i,j} = \tau_{j+1}^{i,j} \circ f_{j+2}. \tag{3.4}$$

Again, as g_{j+2} is order-preserving we have no choice but to take the order-preserving map satisfying $|g_{j+2}^{-1}(k)| = |f_{j+2}^{-1}(\tau_{j+1}^{i,j}(k))|$, for all $x \in [r_{j+1}]$. By (3.4) we know that $\tau_{j+2}^{i,j}$ has to send $f_{j+2}^{-1}(k)$ to $f_{j+2}^{-1}(\tau_{j+1}^{i,j}(k))$ and this determines $\tau_{j+2}^{i,j}$. A proof by induction shows the general claim in (b). ∎

In the following we will extend the notion of E_n-homology for commutative non-unital k-algebras to Epi_n-modules. Again thanks to Fresse's theorem [6, theorem 7.26], the E_n-homology and the homology of the n-fold bar construction of a commutative algebra coincide.

Definition and Notation 3.5. Let $t:[r_n]\xrightarrow{f_n}\dots\xrightarrow{f_{j+1}}[r_j]\xrightarrow{f_j}\dots\xrightarrow{f_2}[r_1]$ be an n-level tree.

The degree of t, denoted by $d(t)$, is the number of its edges, that is $\sum_{i=1}^n(r_i+1)$.

For fixed j and $i\in[r_j]$ let $t_{j,i}$ be the $(n-j)$-level tree defined by the j-fibre of t over $i\in[r_j]$:

$$(f_{j+1}f_{j+2}\dots f_n)^{-1}(i)\xrightarrow{f_n}\dots\xrightarrow{f_{k+1}}(f_{j+1}\dots f_k)^{-1}(i)\xrightarrow{f_k}\dots\xrightarrow{f_{j+2}}f_{j+1}^{-1}(i).$$

Conversely a tree t can be recovered by its 1-fibres, that is $t=[t_{1,0},\dots,t_{1,r_1}]$ and $d(t)=\sum_{i=0}^{r_1}(d(t_{1,i})+1)=r_1+1+\sum_{i=0}^{r_1}d(t_{1,i})$.

Let F be an Epi_n-module. For a fixed j and an n-tree t as in (3.1) with the condition that $f_j(i)=f_j(i+1)$ or $j=1$ we define

$$d_i^j\colon F([r_n]\xrightarrow{f_n}\dots\xrightarrow{f_{j+1}}[r_j]\xrightarrow{f_j}\dots\xrightarrow{f_2}[r_1])\longrightarrow\bigoplus_{t'=[r_n]\xrightarrow{g_n}\dots\xrightarrow{g_{j+2}}\dots\xrightarrow{d_if_{j+1}}[r_j-1]\xrightarrow{f_j|_{i=i+1}}\dots\xrightarrow{f_2}[r_1]\in\mathrm{Epi}_n}F(t')$$

as

$$d_i^j=\sum_{\tau_{j+1}^{i,j}\in\mathrm{Sh}(f_{j+1}^{-1}(i),f_{j+1}^{-1}(i+1))}\varepsilon(\tau_{j+1}^{i,j};t_{j,i},t_{j,i+1})F(\tau_n^{i,j},\dots,\tau_{j+1}^{i,j},d_i,\mathrm{id},\dots,\mathrm{id}).\tag{3.5}$$

where the sign $\epsilon(\tau_{j+1}^{i,j};t_{j,i};t_{j,i+1})$ is defined as follows: one writes the $(n-j)$-level trees $t_{j,i}$ and $t_{j,i+1}$ as a sequence of $(n-j-1)$-level trees $t_{j,i}=[t_1,\dots,t_p]$ and $t_{j,i+1}=[t_{p+1},\dots t_{p+q}]$; the shuffle $\sigma=\tau_{j+1}^{i,j}$ is indeed a (p,q)-shuffle and acts on t by replacing the fibres $t_{j,i}$ and $t_{j,i+1}$ by the fiber $u_{j,i}=[t_{\sigma(1)},\dots,t_{\sigma(p+q)}]$. The sign $\varepsilon(\sigma;[t_1,\dots,t_p],[t_{p+1},\dots,t_{p+q}])$ picks up a factor of $(-1)^{(d(t_a)+1)(d(t_b)+1)}$ whenever $\sigma(a)>\sigma(b)$ but $a<b$.

Definition 3.6.

- If F is an Epi_n-module, then the *E_n-chain complex of F* is the n-fold chain complex whose $(r_n,\dots,r_1)$ spot is

$$C^{E_n}_{(r_n,\dots,r_1)}(F)=\bigoplus_{[r_n]\xrightarrow{f_n}\dots\xrightarrow{f_2}[r_1]\in\mathrm{Epi}_n}F([r_n]\xrightarrow{f_n}\dots\xrightarrow{f_2}[r_1]).\tag{3.6}$$

  The differential in the j-th coordinate is

$$\partial_j\colon C^{E_n}_{(r_n,\dots,r_j,\dots,r_1)}(F)\to C^{E_n}_{(r_n,\dots,r_j-1,\dots,r_1)}(F)$$

  with

$$\partial_j:=\sum_{i|f_j(i)=f_j(i+1)}(-1)^{s_{j,i}}d_i^j,$$

  where $s_{j,i}$ is obtained as follows: drawing the tree t on a plane with its root at the bottom, one can label its edges – from 1 to $d(t)$ – from bottom to top and left to right; the integer $s_{j,i}$ is the label of the right most top edge of the tree $t_{j,i}$. For $j=n$ we use the convention that $s_{n,i}$ is the label of the i-th leaf of t for $0\leqslant i\leqslant r_n$.
- The *E_n-homology of F*, $H_*^{E_n}(F)$ is defined to be the homology of the total complex associated to (3.6).

Lemma 3.7. *The k-modules $C^{E_n}_{(r_n,\dots,r_1)}(F)$ constitute an n-fold chain complex.*

The proof of the lemma is postponed after exemple 3.10.

In order to prove the main theorem, we need categories of n-trees depending on a fixed finite ordered set X of graded elements denoted Epi_n^X. For any $x\in X$, $d(x)\in\mathbb{N}_0$ will denote its degree. For any subset A of X the degree $d(A)$ is the sum of the degrees of the elements of A. For instance $d(X)=\sum_{x\in X}d(x)$. For any disjoint subsets A,B one defines $\epsilon(A;B)=\prod_{a\in A;b\in B;a>b}(-1)^{d(a)d(b)}$. One has

$$\epsilon(A;B)\epsilon(B;A)=(-1)^{d(A)d(B)}\tag{3.7}$$

An object in the category Epi_n^X is an n-level tree t together with a surjection $\phi : X \to [r_n]$. Any such element is written (t, ϕ) and is called an (X, n)-level tree. A morphism from (t, ϕ) to (t', ϕ') is a morphism $\sigma : t \to t'$ in the category Epi_n satisfying $\phi' = \sigma_n \phi$. The following should be considered as a graded version of 3.5 and 3.6.

Definition 3.8. Let $(t, \phi) : X \xrightarrow{\phi} [r_n] \xrightarrow{f_n} \dots \xrightarrow{f_{j+1}} [r_j] \xrightarrow{f_j} \dots \xrightarrow{f_2} [r_1]$ be an (X, n)-level tree in Epi_n^X.

The degree of (t, ϕ), denoted by $d(t)$, is the sum of the number of its edges and the degrees of elements of X,

$$d(t) = \sum_{i=1}^{n} (r_i + 1) + d(X).$$

For a fixed j and $i \in [r_j]$ let $t_{j,i}$ be the $(X_{j,i}, n-j)$-level tree defined by the j-fibre of t over $i \in [r_j]$:

$$X_{j,i} = (f_{j+1} f_{j+2} \dots f_n \phi)^{-1}(i) \xrightarrow{\phi} (f_{j+1} f_{j+2} \dots f_n)^{-1}(i) \xrightarrow{f_n} \dots \xrightarrow{f_{k+1}} (f_{j+1} \dots f_k)^{-1}(i) \xrightarrow{f_k} \dots \xrightarrow{f_{j+2}} f_{j+1}^{-1}(i).$$

Conversely an (X, n)-level tree (t, ϕ) can be recovered by its 1-fibres, that is $t = [t_{1,0}, \dots, t_{1,r_1}]$ and $d(t) = \sum_{i=0}^{r_1} (d(t_{1,i}) + 1) = r_1 + 1 + \sum_{i=0}^{r_1} d(t_{1,i})$.

Let F be an Epi_n^X-module. For a fixed j and an (X, n)-level tree (t, ϕ) with t as in (3.1) with the condition that $f_j(i) = f_j(i+1)$ or $j = 1$ we define the map d_i^j

$$F(t, \phi) \xrightarrow{d_i^j} \bigoplus_{(t',\phi') = (X \xrightarrow{\phi'} [r_n] \xrightarrow{g_n} \dots \xrightarrow{g_{j+2}} [r_{j+1}] \xrightarrow{d_i f_{j+1}} [r_j - 1] \xrightarrow{f_j|_{i=i+1}} \dots \xrightarrow{f_2} [r_1]) \in \mathrm{Epi}_n^X} F(t', \phi')$$

as

$$d_i^j = \sum_{\tau_{j+1}^{i,j} \in \mathrm{Sh}(f_{j+1}^{-1}(i), f_{j+1}^{-1}(i+1))} \varepsilon(\tau_{j+1}^{i,j}; t_{j,i}, t_{j,i+1}) F(\tau_n^{i,j}, \dots, \tau_{j+1}^{i,j}, d_i, \mathrm{id}, \dots, \mathrm{id}). \tag{3.8}$$

Note that for $j = n$ the $(X_{n,i}, 0)$-tree $t_{n,i}$ is the subset $X_{n,i} = \phi^{-1}(i)$ of X and the equation reads

$$d_i^n = \epsilon(t_{n,i}; t_{n,i+1}) F(d_i, \mathrm{id}, \dots, \mathrm{id}).$$

- If F is an Epi_n^X-module, then the (E_n, X)-*chain complex of* F is the n-fold chain complex whose $(r_n, \dots, r_1)$ spot is

$$C_{(r_n,\dots,r_1)}^{E_n,X}(F) = \bigoplus_{X \xrightarrow{\phi} [r_n] \xrightarrow{f_n} \dots \xrightarrow{f_2} [r_1] \in \mathrm{Epi}_n^X} F(X \xrightarrow{\phi} [r_n] \xrightarrow{f_n} \dots \xrightarrow{f_2} [r_1]). \tag{3.9}$$

  The differential in the j-th coordinate is

$$\partial_j : C_{(r_n,\dots,r_j,\dots,r_1)}^{E_n,X}(F) \to C_{(r_n,\dots,r_j-1,\dots,r_1)}^{E_n,X}(F)$$

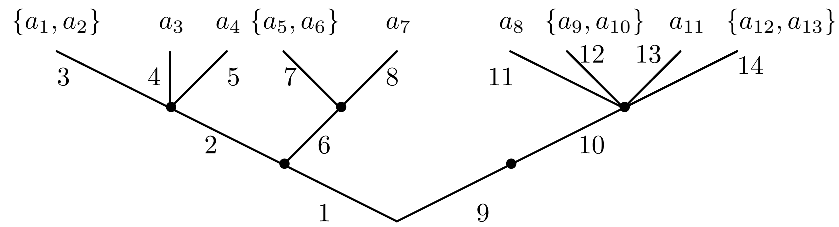
  with

$$\partial_j := \sum_{i | f_j(i) = f_j(i+1)} (-1)^{s_{j,i}} d_i^j,$$

  where the $s_{j,i}$ are obtained as follows: drawing the tree t on a plane with its root at the bottom, one can label its edges from bottom to top and left to right; the integer $s_{j,i}$ is the sum of the label of the right most top edge of the tree $t_{j,i}$ and the degrees of the elements in X which are in the fiber of the leaves that are to the left of the top edge so defined.
- The (E_n, X)-*homology of the* Epi_n^X-*module* F, $H_*^{E_n,X}(F)$ is defined to be the homology of the total complex associated to (3.9).

Lemma 3.9. *The k-modules $C_{(r_n,\dots,r_1)}^{E_n,X}(F)$ constitute an n-fold chain complex.*

Example 3.10. Let t be the following tree of degree 14 with its edges labelled, $X = \{a_1, \ldots, a_{13}\}$ and ϕ can be read off the picture.

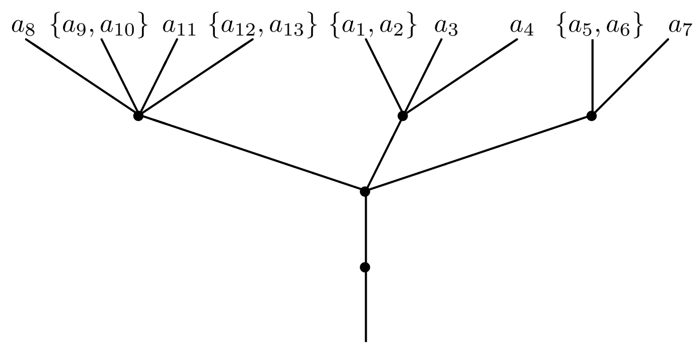


This tree represents the object $X \xrightarrow{\phi} [8] \xrightarrow{f_3} [2] \xrightarrow{f_2} [1] \in \mathrm{Epi}_3^X$, where the map f_3 maps $0, 1, 2$ to 0, it sends $3, 4$ to 1 and $5, 6, 7, 8$ to 2 and f_2 is d_0.

With our notation the tree $t_{1,0}$ is the 2-level tree whose root is the vertex above the edge labelled by 1, the tree $t_{1,1}$ is the subtree above the edge with label 9, the tree $t_{2,0}$ is the 1-level tree above the label 2, $t_{2,1}$ the one above the label 6 and $t_{2,2}$ the one above the label 10.

We have to determine the differentials ∂_1, ∂_2 and ∂_3.

In our example the differential ∂_1 glues the edges labelled by 1 and 9 and shuffles the subtrees $t_{1,0} = [t_{2,0}, t_{2,1}]$ and $t_{1,1} = [t_{2,2}]$. One has $\partial_1 = (-1)^{8+d(\{a_1,\ldots,a_7\})} d_0^1$ where 8 is the label of the right most edge of $t_{1,0}$. In addition we have the shuffle signs. One has $d(t_{2,0}) = 3 + d(\{a_1, \ldots, a_4\})$, $d(t_{2,1}) = 2 + d(a_5) + d(a_6) + d(a_7)$, $d(t_{1,0}) = 7 + d(\{a_1, \ldots, a_7\})$ and $d(t_{2,2}) = 4 + d(\{a_8, \ldots, a_{13}\})$. In the expansion of d_0^1 there are 3 shuffles involved: id, (132), and (312). The first coming with sign $+1$, the second one with sign $(-1)^{(d(t_{2,1})+1)(d(t_{2,2})+1)}$ and the third one with sign $(-1)^{(d(t_{2,0})+1+d(t_{2,1})+1)(d(t_{2,2})+1)}$. For instance the image of the latter shuffle is in $F((t', \phi'))$ where (t', ϕ') is the following tree:



The differential ∂_2 is $(-1)^{5+d(\{a_1,\ldots,a_4\})} d_0^2$ where 5 is the label of the right most top edge of $t_{2,0}$. The shuffles involved in the computation of d_0^2 are the $(3,2)$-shuffles. For such a $(3,2)$-shuffle τ the associated sign is given by $\epsilon(\tau; t_{2,0}, t_{2,1})$ where $t_{2,0} = [t_{3,0}, t_{3,1}, t_{3,2}]$ and $t_{2,1} = [t_{3,3}, t_{3,4}]$.

The differential ∂_3 is given by $\partial_3 = (-1)^{3+d(a_1)+d(a_2)} d_0^3 + (-1)^{4+d(a_1)+d(a_2)+d(a_3)} d_1^3 + (-1)^{7+d(\{a_1,\ldots,a_6\})} d_3^3 + (-1)^{11+d(\{a_1,\ldots,a_8\})} d_5^3 + (-1)^{12+d(\{a_1,\ldots,a_{10}\})} d_6^3 + (-1)^{13+d(\{a_1,\ldots,a_{11}\})} d_7^3$.

Proof of 3.7 and 3.9. The proof that $d = \sum_j \partial_j$ satisfies $d^2 = 0$ is done by induction on n. Since the expression of d in 3.6 coincides with the one in 3.8 when $d(X) = 0$ it is enough to prove 3.8.

The case $n = 1$ has been treated in the previous section, in the non-graded case and the same kind of proof holds in the graded case.

We base our proof on the construction of the iterated bar construction given by Eilenberg and Mac Lane in [5, sections 7–9]: if (A, ∂) is a differential graded commutative algebra then BA is a differential graded commutative algebra with a differential that is the sum of a residual boundary

$$\partial_r([a_1, \ldots, a_k]) = \sum_{i=1}^{k} (-1)^{i+d(a_1)+\ldots+d(a_{i-1})} [a_1, \ldots, \partial a_i, \ldots, a_k]$$

and a simplicial boundary

$$\partial_s([a_1,\dots,a_k]) = \sum_{i=1}^{k-1}(-1)^{i+d(a_1)+\dots+d(a_i)}[a_1,\dots,a_i\cdot a_{i+1},\dots,a_k].$$

The graded commutative product of $a=[a_1,\dots,a_k]$ and $b=[a_{k+1},\dots,a_{k+l}]$ is given by the shuffle product

$$[a_1,\dots,a_k]*[a_{k+1},\dots,a_{k+l}] = \sum_{\sigma\in \mathrm{Sh}(k,l)}\varepsilon(\sigma;a,b)[a_{\sigma(1)},\dots,a_{\sigma(k+l)}]$$

where $\varepsilon(\sigma;a,b)$ picks up a factor $(-1)^{(d(a_i)+1)(d(a_j)+1)}$ whenever $\sigma(i)>\sigma(j)$ but $i<j$. An n-level tree $t=[t_{1,0},\dots,t_{1,r_1}]$ can be considered formally as an element of BT_{n-1} where T_{n-1} is the set of $(n-1)$-level trees. For $n>1$, the differential ∂_1 corresponds to the simplicial boundary, where we view the shuffles $\tau_2^{i,1}$ as the summands in the multiplication $t_{1,i}*t_{1,i+1}$ of two $(n-1)$-level trees. The differential $d_{n-1}=\partial_2+\dots+\partial_n$ corresponds to the residual boundary. Following the proof of [5] one gets that $(d_{n-1}+\partial_1)^2=0$.

For $n=1$ the differential ∂_1 corresponds to the simplicial boundary, where 0-level trees are subsets of X. To an ordered finite set X one can associate an algebra $A=\oplus_{I\subset X}e_I$ with the multiplication

$$e_Ie_J = \begin{cases}\epsilon(I;J)e_{I\sqcup J} & \text{if } \mathrm{I}\cap\mathrm{J}=\emptyset\\ 0 & \text{if not.}\end{cases}$$

The algebra A is graded commutative thanks to relation (3.7). Hence ∂_1 is the simplicial boundary in BA. □

As an example, we will determine the zeroth E_n-homology of an Epi_n-functor F. In total degree zero there is just one summand, namely $F([0]\xrightarrow{\mathrm{id}_{[0]}}\dots\xrightarrow{\mathrm{id}_{[0]}}[0])$. The modules $C^{E_n}_{(0,1,0,\dots,0)}(F)$, …, $C^{E_n}_{(0,\dots,0,1)}(F)$ are all trivial, so the only boundary term that can occur is caused by the unique map

$$C^{E_n}_{(1,0,\dots,0)}(F)\longrightarrow C^{E_n}_{(0,\dots,0)}(F).$$

Therefore

$$H_0^{E_n}(F)\cong F([0]\xrightarrow{\mathrm{id}_{[0]}}\dots\xrightarrow{\mathrm{id}_{[0]}}[0])/\mathrm{image}(F([1]\xrightarrow{d_0}[0]\xrightarrow{\mathrm{id}_{[0]}}\dots\xrightarrow{\mathrm{id}_{[0]}}[0])). \tag{3.10}$$

We can view an Epi_n-module F as an Epi_k-module for all $k\leqslant n$ via the functors ι_n^k.

Proposition 3.11. *For every* Epi_n*-module* F *there is a map of chain complexes* $\mathrm{Tot}(C_*^{E_k}(F\circ\iota_n^k))\longrightarrow \mathrm{Tot}(C_*^{E_n}(F))$ *and therefore a map of graded* k*-modules*

$$H_*^{E_k}(F\circ\iota_n^k)\longrightarrow H_*^{E_n}(F).$$

Proof. There is a natural identification of the module $C^{E_k}_{(r_k,\dots,r_1)}(F\circ\iota_n^k)$ with the module $C^{E_n}_{(r_k,\dots,r_1,0,\dots,0)}(F)$ and this includes $\mathrm{Tot}(C_*^{E_k}(F\circ\iota_n^k))$ as a subcomplex into $\mathrm{Tot}(C_*^{E_n}(F))$. □

3.1. **Relationship to higher order Hochschild homology.** For a non-unital commutative k-algebra $\bar{A}$ we define $\mathcal{L}^n(\bar{A})\colon \mathrm{Epi}_n\to k\text{-mod}$ as

$$\mathcal{L}^n(\bar{A})([r_n]\xrightarrow{f_n}\dots\xrightarrow{f_2}[r_1]) = \bar{A}^{\otimes(r_n+1)}.$$

A morphism

$$\begin{array}{ccccccc}
[r_n] & \xrightarrow{f_n} & [r_{n-1}] & \xrightarrow{f_{n-1}} & \cdots & \xrightarrow{f_2} & [r_1]\\
\downarrow{\scriptstyle\sigma_n} & & \downarrow{\scriptstyle\sigma_{n-1}} & & & & \downarrow{\scriptstyle\sigma_1}\\
[r'_n] & \xrightarrow{f'_n} & [r'_{n-1}] & \xrightarrow{f'_{n-1}} & \cdots & \xrightarrow{f'_2} & [r'_1]
\end{array}$$

induces a map $\bar{A}^{\otimes(r_n+1)}\to\bar{A}^{\otimes(r'_n+1)}$ via

$$a_0\otimes\dots\otimes a_{r_n}\mapsto(\sigma_n)_*(a_0\otimes\dots\otimes a_{r_n}) = b_0\otimes\dots\otimes b_{r'_n}$$

with $b_i = \prod_{\sigma_n(j)=i} a_j$. The E_n-homology of the functor $\mathcal{L}^n(\bar{A})$ coincides with the homology of the n-fold bar construction of the non-unital algebra $\bar{A}$, hence with the E_n-homology of $\bar{A}$. The total complex has been described in [6, Appendix] and it coincides with ours.

There is a correspondence between augmented commutative k-algebras and non-unital k-algebras that sends an augmented k-algebra A to its augmentation ideal $\bar{A}$. Under this correspondence, the $(m+n)$-th homology group of the n-fold bar construction $B^n(A)$ is isomorphic to the m-th homology group of the n-fold iterated bar construction of $\bar{A}$, $B^n(\bar{A})$. As the chain complex $B(A)$ is the chain complex for the Hochschild homology of A with coefficients in k (compare [5, (7.5)]), we can express $B(A)$ as $A\tilde{\otimes}\mathbb{S}^1$. Here, $\mathbb{S}^1$ is the simplicial model of the 1-sphere, which has $n+1$ elements in simplicial degree n and $(A\tilde{\otimes}\mathbb{S}^1)_n = k \otimes A^{\otimes n}$. Therefore

$$B^n(A) \cong (\ldots(A\tilde{\otimes}\mathbb{S}^1)\ldots)\tilde{\otimes}\mathbb{S}^1 \cong A\tilde{\otimes}((\mathbb{S}^1)^{\wedge n}) \cong A\tilde{\otimes}\mathbb{S}^n$$

which gives rise to higher order Hochschild homology of order n of A with coefficients in k, $HH_*^{[n]}(A;k)$, in the sense of Pirashvili [9]. Thus, $HH_{*+n}^{[n]}(A;k) \cong H_*^{E_n}(\bar{A})$.

For the case $F = \mathcal{L}^n(\bar{A})$, $\mathcal{L}^n(\bar{A})([1] \xrightarrow{\mathrm{id}_{[0]}} \ldots \xrightarrow{\mathrm{id}_{[0]}} [0]) = \bar{A}^{\otimes 2}$ and hence for all $n \geqslant 1$ the zeroth E_n-homology group is

$$H_0^{E_n}(\bar{A}) \cong \bar{A}/\bar{A}\cdot\bar{A}.$$

By proposition 3.11 there is a sequence of maps

$$HH_{*+1}(A;k) \cong H_*^{\mathrm{bar}}(\bar{A}) = H_*^{E_1}(\bar{A}) \to H_*^{E_2}(\bar{A}) \to H_*^{E_3}(\bar{A}) \to \ldots \tag{3.11}$$

and the map from $H_*^{E_1}(\bar{A})$ to the higher E_n-homology groups is given on chain level by the inclusion of $C_m^{\mathrm{bar}}(\bar{A})$ into $C_{(m,0,\ldots,0)}^{E_n}(\bar{A})$.

Suspension induces maps

$$\begin{array}{ccccc} HH_\ell(A;k) = \pi_\ell\mathcal{L}(A;k)(\mathbb{S}^1) & \longrightarrow & HH_{\ell+1}^{[2]}(A;k) = \pi_{\ell+1}\mathcal{L}(A;k)(\mathbb{S}^2) & \longrightarrow & \cdots \\ & \searrow & \downarrow & & \\ & & H\Gamma_{\ell-1}(A;k) \cong \pi_\ell^s(\mathcal{L}(A;k)). & & \end{array}$$

For the last isomorphism see [10]. Fresse proves a comparison [6, 8.6] between Gamma homology of A and E_∞-homology of $\bar{A}$. Using the isomorphisms above this sequence gives rise to a sequence of maps involving graded vector spaces that are isomorphic to the ones in (3.11).

The explicit form of the suspension maps is described in [5, (7.9)]: an element $a \in \bar{A}$ is sent to $[a]$ in the bar construction. The iterations of this map correspond precisely to the maps $\iota_n^{n-1}\colon B^{n-1}(\bar{A}) \to B^n(\bar{A})$. Therefore we actually have an isomorphism of sequences, *i.e.*, the suspension maps $HH_{\ell+n}^{[n]}(A;k) \to HH_{\ell+n+1}^{[n+1]}(A;k)$ are related to the natural maps $H_\ell^{E_n}(\bar{A}) \to H_\ell^{E_{n+1}}(\bar{A})$ via the isomorphisms $HH_{*+n}^{[n]}(A;k) \cong H_*^{E_n}(\bar{A})$.

Our description of E_2-homology leads to the following result.

Proposition 3.12. *If $\bar{A}$ and $H_*^{\mathrm{bar}}(\bar{A})$ are k-flat, then there is a spectral sequence*

$$E_{p,q}^1 = \bigoplus_{\ell_0+\ldots+\ell_q=p-q} H_{\ell_0}^{\mathrm{bar}}(\bar{A}) \otimes \ldots \otimes H_{\ell_q}^{\mathrm{bar}}(\bar{A}) \Rightarrow H_{p+q}^{E_2}(\bar{A})$$

where the d_1-differential is induced by the shuffle differential.

Proof. The double complex for E_2-homology looks as follows:

$$
\begin{array}{ccccccc}
 & & & & \vdots & & \\
 & & & & \downarrow & & \\
 & & & & \bar{A}^{\otimes 3} & \longleftarrow & \cdots \\
 & & & & \downarrow & & \\
 & & \bar{A}^{\otimes 2} & \longleftarrow & \bar{A}^{\otimes 3} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
\bar{A} & \longleftarrow & \bar{A}^{\otimes 2} & \longleftarrow & \bar{A}^{\otimes 3} & \longleftarrow & \cdots
\end{array}
$$

The horizontal maps are induced by the b'-differential whereas the vertical maps are induced by the shuffle maps. The horizontal homology of the bottom row is precisely $H_*^{\mathrm{bar}}(\bar{A})$. We can interpret the second row as the total complex associated to the following double complex:

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow{\scriptstyle \mathrm{id}\otimes b'} & & \downarrow{\scriptstyle \mathrm{id}\otimes b'} & & \downarrow{\scriptstyle \mathrm{id}\otimes b'} & & \\
\bar{A}\otimes\bar{A}^{\otimes 3} & \xleftarrow{b'\otimes\mathrm{id}} & \bar{A}^{\otimes 2}\otimes\bar{A}^{\otimes 3} & \xleftarrow{b'\otimes\mathrm{id}} & \bar{A}^{\otimes 3}\otimes\bar{A}^{\otimes 3} & \xleftarrow{b'\otimes\mathrm{id}} & \cdots \\
\downarrow{\scriptstyle \mathrm{id}\otimes b'} & & \downarrow{\scriptstyle \mathrm{id}\otimes b'} & & \downarrow{\scriptstyle \mathrm{id}\otimes b'} & & \\
\bar{A}\otimes\bar{A}^{\otimes 2} & \xleftarrow{b'\otimes\mathrm{id}} & \bar{A}^{\otimes 2}\otimes\bar{A}^{\otimes 2} & \xleftarrow{b'\otimes\mathrm{id}} & \bar{A}^{\otimes 3}\otimes\bar{A}^{\otimes 2} & \xleftarrow{b'\otimes\mathrm{id}} & \cdots \\
\downarrow{\scriptstyle \mathrm{id}\otimes b'} & & \downarrow{\scriptstyle \mathrm{id}\otimes b'} & & \downarrow{\scriptstyle \mathrm{id}\otimes b'} & & \\
\bar{A}\otimes\bar{A} & \xleftarrow{b'\otimes\mathrm{id}} & \bar{A}^{\otimes 2}\otimes\bar{A} & \xleftarrow{b'\otimes\mathrm{id}} & \bar{A}^{\otimes 3}\otimes\bar{A} & \xleftarrow{b'\otimes\mathrm{id}} & \cdots
\end{array}
$$

Therefore the horizontal homology groups of the second row are the homology of the tensor product of the $C^{\mathrm{bar}}(\bar{A})$-complex with itself. Our flatness assumptions guarantee that we obtain $H_*^{\mathrm{bar}}(\bar{A})^{\otimes 2}$ as homology. An induction then finishes the proof. □

4. Tor interpretation of E_n-homology

Let X be a fixed finite ordered set. The following notation will be helpful for the sequel: for an object t in Epi_n (resp. (t,ϕ) in Epi_n^X) let Epi_n^t (resp. ${\mathrm{Epi}_n^X}^{t,\phi}$) denote the representable functor $k[\mathrm{Epi}_n(t,-)]$ (resp. $k[\mathrm{Epi}_n^X((t,\phi),-)]$) and similarly, let $\mathrm{Epi}_{n,t}$ (resp. $\mathrm{Epi}_{n,(t,\phi)}^X$) denote the contravariant representable functor $k[\mathrm{Epi}_n(-,t)]$ (resp. $k[\mathrm{Epi}_n^X(-,(t,\phi))]$). The E_n-homology of an Epi_n-module F (resp. the (E_n,X)-homology of an Epi_n^X-module F) can be computed in different ways, since it is the homology of the total complex associated to an n-complex. The notation $H_*(F,\partial_i)$ stands for the homology of the complex $C_*^{E_n}(F)$ (resp. $C_*^{E_n,X}(F)$) with respect to the differential ∂_i. The complex $(C_*^{E_n}(F),\partial_i)$ splits into subcomplexes

$$
C^{E_n}_{(s_n,s_{n-1},\ldots,s_{i+1},*,s_{i-1},\ldots,s_1)}(F) = \bigoplus_{t=[s_n]\xrightarrow{g_n}\ldots\xrightarrow{g_{i+2}}[s_{i+1}]\xrightarrow{g_{i+1}}[*]\xrightarrow{g_i}[s_{i-1}]\xrightarrow{g_{i-1}}\ldots\xrightarrow{g_2}[s_1]} F(t), \tag{4.1}
$$

whose homology is denoted by $H_{(s_n,s_{n-1},\ldots,s_{i+1},*,s_{i-1},\ldots,s_1)}(F,\partial_i)$. There is an analogous splitting for the complex $(C_*^{E_n,X}(F),\partial_i)$.

Theorem 4.1. *For any* Epi_n*-module* F

$$H_p^{E_n}(F) \cong \mathrm{Tor}_p^{\mathrm{Epi}_n}(b_n^{\mathrm{epi}},F), \text{ for all } p \geqslant 0$$

where

$$b_n^{\mathrm{epi}}(t) \cong \begin{cases} k & \text{for } t = [0] \xrightarrow{\mathrm{id}_{[0]}} \dots \xrightarrow{\mathrm{id}_{[0]}} [0], \\ 0 & \text{for } t \neq [0] \xrightarrow{\mathrm{id}_{[0]}} \dots \xrightarrow{\mathrm{id}_{[0]}} [0]. \end{cases}$$

Proof. Similar to the proof of proposition 2.3, we have to show that $H_*^{E_n}(-)$ maps short exact sequences of Epi_n-modules to long exact sequences, that $H_*^{E_n}(-)$ vanishes on projectives in positive degrees and that $H_0^{E_n}(F)$ and $b_n^{\mathrm{epi}} \otimes_{\mathrm{Epi}_n} F$ agree for all Epi_n-modules F. The homology $H_*^{E_n}(-)$ is the homology of a total complex $C_*^{E_n}(-)$ sending short exact sequences as in (2.2) to short exact sequences of chain complexes and therefore the first claim is true. Note that the left Epi_n-module b_n^{epi} is the cokernel of the map between contravariant representables

$$(d_0)_*\colon \mathrm{Epi}_{n,[1]\longrightarrow[0]\longrightarrow\dots\longrightarrow[0]} \to \mathrm{Epi}_{n,[0]\longrightarrow[0]\dots\longrightarrow[0]}.$$

This remark together with the computation of $H_0^{E_n}(F)$ in relation (3.10) implies the last claim, similar to the proof of proposition 2.3.

In order to show that $H_*^{E_n}(P)$ is trivial in positive degrees for any projective Epi_n-module P it suffices to show that the representables Epi_n^t are acyclic for any planar tree $t = [r_n] \xrightarrow{f_n} \dots \xrightarrow{f_2} [r_1]$.

Let t be such an n-level tree, let X be a finite ordered set and let $\phi : X \to [r_n]$ be a fixed surjection. Assume that every element in X has degree 0. Then we claim that the complexes $C_*^{E_n}(\mathrm{Epi}_n^t)$ and $C_*^{E_n,X}(\mathrm{Epi}_n^{X\,t,\phi})$ are isomorphic. One has

$$k[\mathrm{Epi}_n(t,t')] \cong \bigoplus_{\phi'\colon X\to[r'_n]} k[\mathrm{Epi}_n^X((t,\phi),(t',\phi'))],$$

because any morphism of n-trees $\sigma\colon t \to t'$ determines a component $\phi' = \sigma_n \circ \phi$. This defines an injective map $k[\mathrm{Epi}_n(t,t')] \to \bigoplus_{\phi'\colon X\to[r'_n]} k[\mathrm{Epi}_n^X((t,\phi),(t',\phi'))]$. As every morphism from (t,ϕ) to (t',ϕ') is a morphism of n-trees $\sigma\colon t\to t'$ with $\sigma_n\circ\phi = \phi'$, the map is surjective. By relations (3.6) and (3.9) one has

$$(4.2)\qquad C_*^{E_n}(\mathrm{Epi}_n^t) = \oplus_{t'\in\mathrm{Epi}_n}\mathrm{Epi}_n(t,t') = \oplus_{(t',\phi')\in\mathrm{Epi}_n^X}\mathrm{Epi}_n^X((t,\phi),(t',\phi')) = C_*^{E_n,X}(\mathrm{Epi}_n^{X\,t,\phi})$$

and as every element of X has degree zero, the differentials ∂_j coincide for all j.

In the sequel, we will prove that for any (X,n)-level tree (t,ϕ) the representable $\mathrm{Epi}_n^{X\,t,\phi}$ is acyclic. In particular, if every element in X has degree zero, then this implies that Epi_n^t is acyclic for any n-level tree t.

The case $n=1$ has been proved in proposition 2.3 in the non-graded case and the proof goes the same in the graded case. For $n=2$ we study the bicomplex $C_{(*,*)}^{E_2,X}(\mathrm{Epi}_2^{X\,t,\phi})$. In proposition 4.2 we give the k-module structure of the homology with respect to the differential ∂_2 and give its generators in propositions 4.4 and 4.5. Corollaries 4.3 and 4.6 state the result for $n=2$. For the general case, one uses induction on n and proposition 4.7. As a consequence $H_*^{E_n}(\mathrm{Epi}_n^t) = 0$ for all $* \geqslant 0$ if $t \neq [0] \longrightarrow [0]\dots\longrightarrow[0]$ and in that case

$$H_*^{E_n}(\mathrm{Epi}_n^{[0]\longrightarrow[0]\dots\longrightarrow[0]}) = \begin{cases} 0 & \text{for } * > 0 \\ k & \text{for } * = 0. \end{cases}$$

□

Proposition 4.2. *Let $(t,\phi) = X \xrightarrow{\phi} [r_2] \xrightarrow{f} [r_1]$ be an $(X,2)$-level tree in Epi_2^X.*

$$H_{(*,s)}(\mathrm{Epi}_2^{X\,t,\phi},\partial_2) = 0, \qquad \text{if } r_2 \neq r_1$$

$$H_{(*,s)}(\mathrm{Epi}_2^{X\,t,\phi},\partial_2) \cong \begin{cases} 0 & \text{for } * \neq r_2 \\ k^{\oplus|\Delta^{\mathrm{epi}}([r_2],[s])|} & \text{for } s \leqslant * = r_2. \end{cases}, \qquad \text{if } r_2 = r_1$$

Proof. Let F denote the covariant functor $\mathrm{Epi}_2^{X\,t,\phi}$.

Assume $s=0$. We first prove that the chain complex $\partial_2\colon C_{(*,0)}^{E_2,X}(F) \to C_{(*-1,0)}^{E_2,X}(F)$ is the chain complex associated to a labelled poset, in a sense we will describe now. In fact, we first explain how we prove the proposition in case every element in X has degree 0 and then we show how we can adapt the proof to the general case.

Recall from Wachs [18] and Vallette [17] that a chain complex $\Pi_*(P)$ can be associated to a graded poset P with minimal element β_0 and maximal element β_M. The k-module $\Pi_u(P)$ is the free k-module generated by chains of the form $\beta_0 < \beta_1 < \ldots < \beta_u < \beta_M$, with the differential given by $d = \sum_{i=1}^{u}(-1)^i d_i$ where d_i omits β_i. We define $\Pi_0(P)$ to be the k-module of rank one generated by the chain $\beta_0 < \beta_M$. Indeed, $\Pi_u(P)$ is the order complex associated to the proper part $\overline{P} = P \setminus \{\beta_0, \beta_M\}$ of the poset P, denoted by $\Delta(\overline{P})$. More precisely, $\Pi_u(P) = \Delta_{u-1}(\overline{P})$ where we consider the augmented order complex.

The chain complex $(C^{E_2,X}_{(*,0)}(F), \partial_2)$ has the following form, for $0 < u \leqslant r_2$

$$\bigoplus_\psi k[\mathrm{Epi}_2^X((t,\phi); X \xrightarrow{\psi} [u] \to [0])] \xrightarrow{\sum_{i=0}^{u-1}(-1)^{s_{2,i}} d_i^2} \bigoplus_\psi k[\mathrm{Epi}_2^X((t,\phi); X \xrightarrow{\psi} [u-1] \to [0])],$$

If the elements of X all have degree 0 then $s_{2,i} = i+2$. Let $(A_0, \ldots, A_{r_1})$ be the sequence of preimages of f, and a_i the number of elements in A_i.

The set $\mathrm{Epi}_2^X((t,\phi); X \xrightarrow{\psi} [u] \to [0])$ is either empty or has only one element uniquely determined by a surjective map $\sigma\colon [r_2] \to [u]$ which is order-preserving on A_i. In that case we recall that $\psi = \sigma\phi$. The map σ can be described by the sequence of its preimages $(S_0, \ldots, S_u)$ with the condition (C_S): if $a < b \in A_i$ then $i_a \leqslant i_b$ where i_α is the unique index for which $\alpha \in S_{i_\alpha}$. Let us consider the poset P_f whose objects are elements $(\beta_0, \ldots, \beta_{r_2})$ of $\{0,1\}^{r_2+1}$ satisfying the condition

$$\begin{aligned} \beta_0 \geqslant \beta_1 \geqslant \ldots \geqslant \beta_{a_0-1}, \\ \beta_{a_0} \geqslant \beta_{a_0+1} \geqslant \ldots \geqslant \beta_{a_0+a_1-1}, \\ \ldots \\ \beta_{a_0+\ldots+a_{r_1-1}} \geqslant \ldots \geqslant \beta_{r_2}. \end{aligned} \tag{4.3}$$

The order is the lexicographic one, thus the minimal element is $B_0 = (0, \ldots, 0)$ and the maximal element is $B_M = (1, \ldots, 1)$. An element in $\Pi_u(P_f)$ is a family of (r_2+1)-tuples $B_i = (\beta_0^i, \ldots, \beta_{r_2}^i)$ of P_f with $B_0 < B_1 < \ldots < B_u < B_{u+1} = B_M$. A chain in $\Pi_u(P_f)$ is encoded by a sequence of sets $(S_0, \ldots, S_u)$ where $S_i = \{j | \beta_j^{i+1} > \beta_j^i\}$. This sequence is an ordered partition of $[r_2]$ by non-empty subsets, and condition (4.3) amounts to condition (C_S). As a consequence the two complexes $(C^{E_2,X}_{*,0}(F), \partial_2)$ and $\Pi_*(P_f)$ coincide. The poset P_f is the product of the posets $L_{a_i}, 0 \leqslant i \leqslant r_1$ where L_{a_i} is the linear poset

$$\underbrace{(0, \ldots, 0)}_{a_i \text{ times}} < (1, 0, \ldots, 0) < (1, 1, \ldots, 0) < \ldots < (1, 1, \ldots, 1).$$

The complex $\Pi_*(L_{a_i})$ is acyclic but for $a_i = 1$ where it is free of rank one, concentrated in degree 0. It remains to compute the homology of $\Pi_*(P \times Q) = \Delta_{*-1}(\overline{P \times Q})$ for any graded poset P and Q. Since the order complex $\Delta(P)$ of a poset P is a simplicial complex, its homology coincides with the homology of its geometric realization denoted by $|P|$.

The first step relies on Quillen's and Walker's results. In his PhD thesis [19], Walker proved, following methods of Quillen in [12], that the geometric realization of $\Delta(\overline{P \times Q})$ is homeomorphic to $|\overline{P}| * |\overline{Q}| * |A_2|$ where A_2 is the discrete poset with 2 points and $*$ denotes the join of topological spaces. Recall that the poset $\overline{P} * \overline{Q}$ is $\overline{P} \sqcup \overline{Q}$ where all elements in $\overline{P}$ are smaller than the elements in $\overline{Q}$. Since joins commute with realization (see e.g. [12, Prop 1.9]), one gets

$$|\overline{P \times Q}| \simeq |\overline{P}| * |\overline{Q}| * |A_2| \simeq |\overline{P} * \overline{Q} * A_2|.$$

The second step computes the homology of the order complex of the join of two posets in terms of the homology of the order complex of the posets. This is referred to as the Künneth formula in [18, 5.1.2], but we prove it here because we need to adapt its proof for a general set of graded elements X.

An m-chain in $\Delta(P * Q)$ is of the form $\beta_0 < \ldots < \beta_m$ where there exists a p with $-1 \leqslant p \leqslant m$ such that for all $j \leqslant p$, $\beta_j \in P$ and for all $j > p$, $\beta_j \in Q$. Recall that $\Delta_{-1}(P)$ is the free k-module generated by the empty chain. As a consequence one gets an isomorphism of complexes

$$\Delta_m(P * Q) \to \oplus_{p+q=m-1}\Delta_p(P) \otimes \Delta_q(Q)$$

with the usual differential on the right hand side: for $x \in \Delta_p(P)$ and $y \in \Delta_q(Q)$, $d(x \otimes y) = d(x) \otimes y + (-1)^p x \otimes d(y)$. As every complex under consideration is free as a k-module, the classical Künneth theorem yields an isomorphism on homology.

As a consequence, and since the homology of $\Delta(A_2)$ is of rank 1 concentrated in degree 0, one gets

$$H_m(\Delta(\overline{P \times Q})) = \bigoplus_{p+q=m-2} H_p(\Delta(\overline{P})) \otimes H_q(\Delta(\overline{Q})).$$

Iterating the formula and using the equality $\Pi_*(P_f) = \Delta_{*-1}(\overline{P_f})$ one gets

$$H_m(\Pi_*(P_f)) = \bigoplus_{p_0+\ldots+p_{r_1}=m-r_1} H_{p_0}(\Pi_*(L_{a_0})) \otimes \ldots \otimes H_{p_{r_1}}(\Pi_*(L_{a_{r_1}})).$$

As a consequence, the complex $\Pi_*(P_f)$ is acyclic but for $f = \mathrm{id}_{[r_2]}$ where its homology is concentrated in top degree r_2 and is free of rank 1. This implies the result for $s = 0$.

Now assume that the elements of X are graded, then $s_{2,i}$ is no longer $i+2$: any map σ is determined by its sequence of preimages $(S_0, \ldots, S_u)$ and it comes equipped with degrees, $d(S_i) = \sum_{\phi(x) \in S_i} d(x)$. Then $s_{2,i} = i + 2 + \sum_{k=0}^{i} d(S_k)$. As a consequence the complex does not coincide with the complex associated to the poset P_f but with a graded version of this complex. To a relation $\alpha = (\alpha_0, \ldots, \alpha_{r_2}) < \beta = (\beta_0, \ldots, \beta_{r_2})$ in P_f we assign the set $S = \{j \in [r_2] | \alpha_j < \beta_j\}$ and a degree $d(\alpha < \beta) = d(S) = \sum_{x \in X | \phi(x) \in S} d(x)$. We define the simplicial complex $\Pi_*(P_f)$ as before, except that $d_i(\beta_0 < \beta_1 < \ldots < \beta_i < \ldots < \beta_u < \beta_M) = (-1)^{\sum_{k=0}^{i-1} d(\beta_k < \beta_{k+1})} \beta_0 < \beta_1 < \ldots < \beta_{i-1} < \beta_{i+1} < \ldots < \beta_u < \beta_M$. In terms of the homology of the geometric realization this corresponds to assigning a system of local coefficients to the simplicial complex $\Delta(\overline{P_f})$. The homeomorphism on the level of geometric realizations still holds and there is a Künneth formula in this context as well.

The computation of the generator of $H_{(r_2,0)}(\mathrm{Epi}_2^{X \xrightarrow{\phi} [r_2] \xrightarrow{\mathrm{id}} [r_2]}, \partial_2) \cong k$ is the subject of proposition 4.4.

Assume $s > 0$.

The complex $(C_{(*,s)}^{E_2,X}(F), \partial_2)$ splits into subcomplexes

$$C_{(*,s)}^{E_2,X}(F) = \bigoplus_{\sigma \in \Delta^{\mathrm{epi}}([r_1],[s])} C_{(*,s)}(F_\sigma) = \bigoplus_{\sigma \in \Delta^{\mathrm{epi}}([r_1],[s])} \bigoplus_{g \in \Delta^{\mathrm{epi}}([*],[s]), \psi} F_\sigma(X \xrightarrow{\psi} [*] \xrightarrow{g} [s])$$

where $F_\sigma(X \xrightarrow{\psi} [u] \xrightarrow{g} [s]) \subset \mathrm{Epi}_2^{t,\phi}(X \xrightarrow{\psi} [u] \xrightarrow{g} [s])$ is the free k-module generated by morphisms of the form

(4.4)
$$\begin{array}{ccccc} X & \xrightarrow{\phi} & [r_2] & \xrightarrow{f} & [r_1] \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \sigma} \\ X & \xrightarrow{\psi} & [u] & \xrightarrow{g} & [s]. \end{array}$$

Let $(A_0, \ldots, A_s)$ denote the sequence of preimages of σf and $(B_0, \ldots, B_s)$ the one of g. The latter has to satisfy the condition $|B_i| \leqslant |A_i|, 0 \leqslant i \leqslant s$. Note that $g \in \Delta^{\mathrm{epi}}([u],[s])$ is also uniquely determined by the sequence $(b_0, \ldots, b_s)$ of the cardinalities of its preimages. The differential $\partial_2 \colon C_{(u,s)}(F_\sigma) \longrightarrow C_{(u-1,s)}(F_\sigma)$ has the following form:

$$\partial_2\left(\begin{array}{ccccc} X & \xrightarrow{\phi} & [r_2] & \xrightarrow{f} & [r_1] \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\sigma} \\ X & \xrightarrow{\tau\phi} & [u] & \xrightarrow{g} & [s]\end{array}\right) = \sum_{i|g(i)=g(i+1)} (-1)^{s_{2,i}} \begin{array}{ccccc} X & \xrightarrow{\phi} & [r_2] & \xrightarrow{f} & [r_1] \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle d_i\tau} & & \downarrow{\scriptstyle\sigma} \\ X & \xrightarrow{d_i\tau\phi} & [u-1] & \xrightarrow{g|_{i=i+1}} & [s]\end{array}$$

$$= \sum_{j=0}^{s}\left(\sum_{i\in B_j|g(i)=g(i+1)} (-1)^{s_{2,i}} \begin{array}{ccccc} X & \xrightarrow{\phi} & [r_2] & \xrightarrow{f} & [r_1] \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle d_i\tau} & & \downarrow{\scriptstyle\sigma} \\ X & \xrightarrow{d_i\tau\phi} & [u-1] & \xrightarrow{g|_{i=i+1}} & [s]\end{array}\right)$$

Define D_j by restricting the sum over indices i such that $g(i) = g(i+1)$ to the sum over indices $i \in B_j$ such that $g(i) = g(i+1)$. One has

$$D_j\colon C_{(u,s)}(F_\sigma) = \bigoplus_{b_0+\ldots+b_s=u+1} C_{((b_0,\ldots,b_j,\ldots,b_s),s)}(F_\sigma) \longrightarrow \bigoplus_{b_0+\ldots+b_s=u+1} C_{((b_0,\ldots,b_j-1,\ldots,b_s),s)}(F_\sigma)$$

and $\partial_2 = D_0 + \ldots + D_s$. We claim that the D_j are anti-commuting differentials:

Let i be in B_j and ℓ be in B_k. For $j < k$ it follows that $i < \ell$ therefore we have the relation $d_i d_\ell = d_{\ell-1} d_i$. In order to calculate the effect of $d_{\ell-1}d_i$ we have to determine $s_{2,\ell-1}$ after the application of d_i. Let $\tilde{S}_j$ denote the preimage $(d_i \circ \tau \circ \phi)^{-1}(j)$ and S_j the preimage $(\tau\circ\phi)^{-1}(j)$ for $j \in [u]$. Then

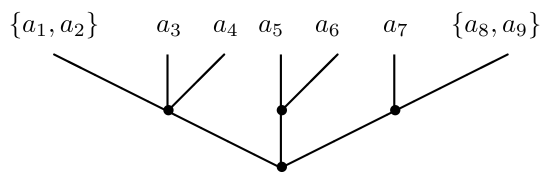
$$d(\tilde{S}_j) = \begin{cases} d(S_j) & j < i \\ d(S_i) + d(S_{i+1}) & j = i \\ d(S_{j+1}) & j > i. \end{cases}$$

Thus $s_{2,\ell-1}$ is $\ell - 1 + k + 2 + \sum_{j=0}^{\ell-1} d(\tilde{S}_j) = \ell + k + 1 + \sum_{j=0}^{\ell} d(S_j)$ whereas $s_{2,\ell} = \ell + k + 2 + \sum_{j=0}^{\ell} d(S_j)$. A similar argument shows that the D_j are differentials.

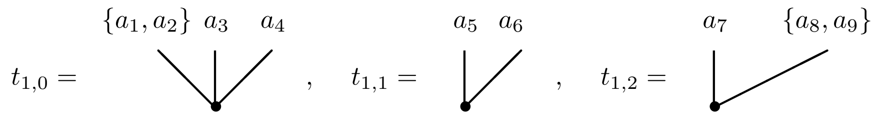
For instance, the complex $(C_{(u,s)}(F_\sigma), D_s)$ splits into subcomplexes $(C_{((b_0,\ldots,b_{s-1}),*)}(F_\sigma), D_s)$ for fixed $b_i \leqslant a_i = |A_i|, i < s$. With the notation of definition 3.8, the tree (t,ϕ) can be written as $t = [t_{1,0},\ldots,t_{1,r_1}]$, with $t_{1,i}$ being an $(X_{1,i},1)$-level tree. Let p be the first integer such that $\sigma(p) = s$. Let $X_{s-1} = \cup_{0\leqslant i\leqslant p-1} X_{1,i}$ and $\tilde{X} = \cup_{p\leqslant i\leqslant r_1} X_{1,i}$. Denote by t_{s-1} the $(X_{s-1},2)$-level tree $t_{s-1} = [t_{1,0},\ldots,t_{1,p-1}]$ and by $\tilde{t}$ the $(\tilde{X},2)$-level tree $\tilde{t} = [t_{1,p},\ldots,t_{1,r_1}]$. Let σ_{s-1} (resp. ϕ_{s-1}) be the map obtained from σ (resp. ϕ) by restriction $\sigma_{s-1}\colon \sigma^{-1}([s-1]) \xrightarrow{\sigma} [s-1]$. Let $u_{s-1} = (\sum_{i<s} b_i) - 1$. The subcomplex $(C_{((b_0,\ldots,b_{s-1}),*)}(F_\sigma), D_s)$ can be expressed as

$$\bigoplus_{\psi;\gamma\in(\mathrm{Epi}_2^{X_{s-1}\,t_{s-1},\phi_{s-1}})_{\sigma_{s-1}}(X_{s-1}\xrightarrow{\psi}[u_{s-1}]\xrightarrow{g|_{[u_s]}}[s-1])} (C^{E_2,\tilde{X}}_{(*,0)}(\mathrm{Epi}_2^{\tilde{X}\,\tilde{t},\tilde{\phi}}), (-1)^{b_0+\ldots+b_{s-1}+s+d(X_{s-1})}\partial_2).$$

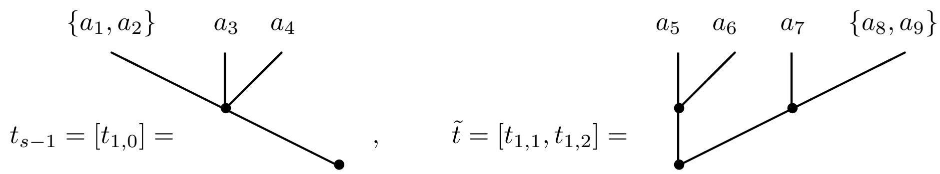
Example. Let $(t,\phi) = X \xrightarrow{\phi} [6] \xrightarrow{f} [2]$ be the following tree



where $(t,\phi) = [t_{1,0}, t_{1,1}, t_{1,2}]$



and $X_{1,0} = \{a_1, a_2, a_3, a_4\}, X_{1,1} = \{a_5, a_6\}$ and $X_{1,2} = \{a_7, a_8, a_9\}$. Let $\sigma : [2] \to [1]$ be the map assigning 0 to 0 and 1 to 1 and 2. One has $s = 1$, $p = 1$, so that $X_{s-1} = \{a_1, \dots, a_4\}$ and $\tilde{X} = \{a_5, \dots, a_9\}$. Moreover



If $f \neq \mathrm{id}$, then there exists $j \in [s]$ such that the restriction of f on $(\sigma \circ f)^{-1}(j) \to \sigma^{-1}(j)$ is different from the identity. Without loss of generality we can assume that $j = s$, hence $\tilde{t}$ is a non-fork tree and the homology of the complex is 0. If $f = \mathrm{id}$, then we deduce from the case $s = 0$ that the complex $(C^{E_2,\tilde{X}}_{(*,0)}(\mathrm{Epi}_2^{\tilde{X}^{\tilde{t},\tilde{\phi}}}), \partial_2)$ has only top homology of rank one; consequently when $t\colon [r_2] \longrightarrow [r_2]$ is the fork tree

$$(H_*(C_{(*,s)}((\mathrm{Epi}_2^{X^{t,\phi}})_\sigma), D_s), D_1 + \ldots + D_{s-1}) \cong (C_{(*,s-1)}((\mathrm{Epi}_2^{X_{s-1}{}^{t_{s-1},\phi_{s-1}}})_{\sigma_{s-1}}), \partial_2).$$

We then have an inductive process to compute the homology of the total complex $(C_{(*,s)}(F_\sigma), \partial_2)$. Consequently, for a fixed $\sigma\colon [r_2] \to [s]$

$$\begin{aligned} &H_{(*,s)}(F_\sigma, \partial_2) = 0, && \text{if } r_2 \neq r_1 \\ &H_{(*,s)}(F_\sigma, \partial_2) \cong \begin{cases} 0 & \text{for } * \neq r_2 \\ k & \text{for } s \leqslant * = r_2 \end{cases}, && \text{if } r_2 = r_1. \end{aligned}$$

Since each $\sigma \in \Delta^{\mathrm{epi}}([r_2], [s])$ contributes to one summand in $H_{r_2,s}(F, \partial_2)$, this proves the claim. The computation of the generators for $s > 0$ is given in proposition 4.5. ☐

Corollary 4.3. *For any non-fork tree* $(t, \phi) = X \xrightarrow{\phi} [r_2] \xrightarrow{f} [r_1], r_2 \neq r_1$, $\mathrm{Epi}_2^{X^{t,\phi}}$ *is acyclic. For any non-fork tree* $t = [r_2] \xrightarrow{f} [r_1], r_2 \neq r_1$, Epi_2^t *is acyclic.*

Proof. The first assertion is a direct consequence of the first equation of proposition 4.2. The second one is a direct consequence of relation (4.2). ☐

Proposition 4.4. *Let* $(t, \phi)\colon X \xrightarrow{\phi} [r] \xrightarrow{\mathrm{id}} [r]$ *be a fork tree and let* $X_i = \phi^{-1}(i)$*. Then the top homology* $H_{(r,0)}(\mathrm{Epi}_2^{X^{t,\phi}}, \partial_2)$ *is freely generated by* $c_{r,X} := \sum_{\sigma \in \Sigma_{r+1}} \mathrm{sgn}(\sigma; X)\sigma$*, where the sign* $\mathrm{sgn}(\sigma; X)$ *picks up a factor* $(-1)^{(d(X_i)+1)(d(X_j)+1)}$ *whenever* $\sigma(i) > \sigma(j)$ *but* $i < j$*.*

In particular, for a fork tree $t\colon [r] \xrightarrow{\mathrm{id}} [r]$*, the top homology* $H_{(r,0)}(\mathrm{Epi}_2{}^t, \partial_2)$ *is freely generated by* $c_r := \sum_{\sigma \in \Sigma_{r+1}} \mathrm{sgn}(\sigma)\sigma$*.*

Proof. The second assertion is a consequence of the first one using relation (4.2). The computation of the top homology amounts to determining the kernel of the map

$$\partial_2\colon \oplus_\psi k[\mathrm{Epi}_2^X(X \xrightarrow{\phi} [r] \xrightarrow{\mathrm{id}} [r]; X \xrightarrow{\psi} [r] \longrightarrow [0])] \longrightarrow \oplus_\psi k[\mathrm{Epi}_2^X(X \xrightarrow{\phi} [r] \xrightarrow{\mathrm{id}} [r]; X \xrightarrow{\psi} [r-1] \longrightarrow [0])].$$

The set $\mathrm{Epi}_2^X(X \xrightarrow{\phi} [r] \xrightarrow{\mathrm{id}} [r]; X \xrightarrow{\psi} [r] \longrightarrow [0])$ is either empty or has only one element uniquely determined by the following diagram

$$\begin{array}{ccccc} X & \xrightarrow{\phi} & [r] & \xrightarrow{\mathrm{id}} & [r] \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \tau} & & \downarrow \\ X & \xrightarrow{\psi=\tau\phi} & [r] & \longrightarrow & [0]. \end{array}$$

Here, the surjection ψ is a permutation of the set $\{X_0, \dots, X_r\}$. We denote such an element by $X \cdot \tau := (X_{\tau(0)}, \dots, X_{\tau(r)})$. As a consequence the computation of the top homology amounts to determining the kernel of the map

$$\partial_2 \colon k[\Sigma_{r+1}] \longrightarrow \oplus_\psi k[\mathrm{Epi}_2^X(X \xrightarrow{\phi} [r] \xrightarrow{\mathrm{id}} [r]; X \xrightarrow{\psi} [r-1] \longrightarrow [0])]$$

where

$$\partial_2(X \cdot \tau) = \sum_{i=0}^{r-1} (-1)^{i+d(X_{\tau(0)})+\ldots+d(X_{\tau(i)})} \epsilon(X_{\tau(i)}; X_{\tau(i+1)})(X_{\tau(0)}, \dots, \{X_{\tau(i)} \cup X_{\tau(i+1)}\}, \dots, X_{\tau(r)}).$$

Therefore, if $x = \sum_{\tau \in \Sigma_{r+1}} \lambda_\tau X \cdot \tau$ is in the kernel of ∂_2, then for all transpositions $(i, i+1)$ and all τ one has $\lambda_{\tau(i,i+1)} = (-1)^{1+d(X_{\tau(i)})+d(X_{\tau(i+1)})+d(X_{\tau(i)})d(X_{\tau(i+1)})} \lambda_\tau$. Since the transpositions generate the symmetric group one has $\lambda_\tau = \mathrm{sgn}(\tau; X)\lambda_{\mathrm{id}}$ and $x = \lambda_{\mathrm{id}} c_{r,X}$. □

For $s > 0$, the computation of the top homology of $(C_{*,s}^{E_2,X}(\mathrm{Epi}_2^{X\,t,\phi}), \partial_2)$ amounts to calculating the kernel of the map ∂_2

$$\bigoplus_{\psi, g \in \Delta^{\mathrm{epi}}([r],[s])} k[\mathrm{Epi}_2^X((t,\phi); X \xrightarrow{\psi} [r] \xrightarrow{g} [s])] \longrightarrow \bigoplus_{\psi, h \in \Delta^{\mathrm{epi}}([r-1],[s])} k[\mathrm{Epi}_2^X((t,\phi); X \xrightarrow{\psi} [r-1] \xrightarrow{h} [s])].$$

We know from proposition 4.2 that it is free of rank equal to the cardinality of $\Delta^{\mathrm{epi}}([r],[s])$. As before, the set $\mathrm{Epi}_2^X((t,\phi); X \xrightarrow{\psi} [r] \xrightarrow{g} [s])$ is either empty or has only one element determined by the commuting diagram

$$\begin{array}{ccccc} X & \xrightarrow{\phi} & [r] & \xrightarrow{\mathrm{id}} & [r] \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle g'} \\ X & \xrightarrow{\psi=\tau\phi} & [r] & \xrightarrow{g} & [s] \end{array}$$

An element g of the latter set is uniquely determined by the sequence $(x_0, \dots, x_s)$ of the cardinalities of its preimages. Furthermore, any map in $\mathrm{Epi}_2([r] \xrightarrow{\mathrm{id}} [r]; [r] \xrightarrow{g} [s])]$ is given by $g' \colon [r] \to [s]$ in Δ^{epi} and $\tau \colon [r] \to [r]$ in Σ_{r+1} such that $g' = g\tau$. This implies that $g' = g$ and $\tau \in \Sigma_{x_0} \times \ldots \times \Sigma_{x_s}$. If there is such a τ satisfying $\psi = \tau\phi$ then the set is non-empty and τ is unique. Let $X_{(x_i)} = (g\phi)^{-1}(i)$. Then $X_{(x_i)}$ is a subset of X and there is a natural partition of it given by $X_{(x_i)} = \sqcup_{j \in g^{-1}(\{i\})} X_j$. The map ψ acts on $X_{(x_i)}$ by permuting the components of the partition.

Let $c_{(x_0,\dots,x_s);X}$ be the element

$$c_{(x_0,\dots,x_s);X} = (\sum_{\sigma^0 \in \Sigma_{x_0}} \mathrm{sgn}(\sigma^0; X_{(x_0)})\sigma^0, \dots, \sum_{\sigma^s \in \Sigma_{x_s}} \mathrm{sgn}(\sigma^s; X_{(x_s)})\sigma^s) \in \Sigma_{x_0} \times \ldots \times \Sigma_{x_s}.$$

If every element of X has degree zero, we denote $(\sum_{\sigma^0 \in \Sigma_{x_0}} \mathrm{sgn}(\sigma^0)\sigma^0, \dots, \sum_{\sigma^s \in \Sigma_{x_s}} \mathrm{sgn}(\sigma^s)\sigma^s)$ by $c_{(x_0,\dots,x_s)}$.

Proposition 4.5. *Let $(t,\phi) \colon X \xrightarrow{\phi} [r] \xrightarrow{\mathrm{id}} [r]$ be a fork tree. The top homology $H_{(r,s)}(\mathrm{Epi}_2^{X\,t,\phi}, \partial_2)$ is freely generated by the elements $c_{(x_0,\dots,x_s);X} = (\sum_{\sigma^0 \in \Sigma_{x_0}} \mathrm{sgn}(\sigma^0; X_{(x_0)})\sigma^0, \dots, \sum_{\sigma^s \in \Sigma_{x_s}} \mathrm{sgn}(\sigma^s; X_{(x_s)})\sigma^s)$, for $g = (x_0, \dots, x_s) \in \Delta^{\mathrm{epi}}([r],[s])$, $X_{(x_k)} = (g\phi)^{-1}(\{k\})$.*

Let $t \colon [r] \xrightarrow{\mathrm{id}} [r]$ be a fork tree. The top homology $H_{(r,s)}(\mathrm{Epi}_2^t, \partial_2)$ is freely generated by the elements $c_{(x_0,\dots,x_s)} = (\sum_{\sigma^0 \in \Sigma_{x_0}} \mathrm{sgn}(\sigma^0)\sigma^0, \dots, \sum_{\sigma^s \in \Sigma_{x_s}} \mathrm{sgn}(\sigma^s)\sigma^s)$, for $(x_0, \dots, x_s) \in \Delta^{\mathrm{epi}}([r],[s])$.

Proof. Similar to the proof of proposition 4.4 we compute the kernel of ∂_2 which decomposes into the sum of anti-commuting differentials $\partial_2 = D_0 + \ldots + D_s$, as in the proof of proposition 4.2. As a consequence $\ker(\partial_2) = \cap_i \ker(D_i)$ which gives the result. ☐

Corollary 4.6. *For any fork tree* $(t,\phi) = X \xrightarrow{\phi} [r] \xrightarrow{\mathrm{id}} [r]$, $\mathrm{Epi}_2^{X\,t,\phi}$ *is acyclic. In particular,* Epi_2^t *is acyclic for any fork tree* $t = [r] \xrightarrow{\mathrm{id}} [r]$.

Proof. It remains to compute the homology of the complex $((H_{(r,*)}(C^{E_2,X}(\mathrm{Epi}_2^{X\,t,\phi}), \partial_2), \partial_1)$ and prove that it vanishes for all $*$ if $r > 0$. Propositions 4.4 and 4.5 give its k-module structure:

$$H_{(r,s)}(C^{E_2,X}(\mathrm{Epi}_2^{X\,t,\phi}), \partial_2) = \bigoplus_{(x_0,\ldots,x_s)\in\Delta^{\mathrm{epi}}([r],[s])} k c_{(x_0,\ldots,x_s);X}.$$

To compute $\partial_1(c_{(x_0,\ldots,x_s);X})$ it is enough to compute $\partial_1(\mathrm{id}_{\Sigma_0\times\ldots\times\Sigma_s})$ in $C^{E_2,X}_{(r,s)}(\mathrm{Epi}_2^{X\,t,\phi})$. We apply relations (3.8) and (3.9):

$$\partial_1\left(\begin{array}{ccccc} X & \xrightarrow{\phi} & [r] & \xrightarrow{\mathrm{id}} & [r] \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle (x_0,\ldots,x_s)} \\ X & \xrightarrow{\phi} & [r] & \xrightarrow{(x_0,\ldots,x_s)} & [s] \end{array}\right)$$

$$= \sum_{i=0}^{s-1} (-1)^{i+x_0+d(X_{(x_0)})+\cdots+x_i+d(X_{(x_i)})} \left(\begin{array}{ccccc} X & \xrightarrow{\phi} & [r] & \xrightarrow{\mathrm{id}} & [r] \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle d_i(x_0,\ldots,x_s)} \\ X & \xrightarrow{\phi} & [r] & \xrightarrow{d_i(x_0,\ldots,x_s)} & [s-1] \end{array}\right.$$

$$\left. \pm \sum_{\xi} \begin{array}{ccccc} X & \xrightarrow{\phi} & [r] & \xrightarrow{\mathrm{id}} & [r] \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle d_i(x_0,\ldots,x_s)} \\ X & \xrightarrow{\xi\phi} & [r] & \xrightarrow{d_i(x_0,\ldots,x_s)} & [s-1] \end{array}\right),$$

with ξ running over the $(X_{(x_i)}, X_{(x_{i+1})})$-shuffles. Thus,

$$\partial_1(c_{(x_0,\ldots,x_s);X}) = \sum_{i=0}^{s-1} (-1)^{i+x_0+d(X_{(x_0)})+\cdots+x_i+d(X_{(x_i)})} c_{(x_0,\ldots,x_i+x_{i+1},\ldots,x_s);X}$$

and the complex $(H_{(r,*)}(C^{E_2,X}(\mathrm{Epi}_2^{X\,t,\phi}), \partial_2), \partial_1)$ agrees with a graded version of the complex $C_*^{\mathrm{bar}}((\Delta^{\mathrm{epi}})^r)$ of definition 2.2. Proposition 2.3 states that it is acyclic, which remains true in the graded case, and that

$$H_0(C_*^{\mathrm{bar}}((\Delta^{\mathrm{epi}})^r) = \begin{cases} 0 & \text{if } r > 0 \\ k & \text{if } r = 0. \end{cases}$$

As a consequence the spectral sequence associated to the bicomplex $(C^{E_2,X}_{(*,*)}, \partial_1+\partial_2)$ collapses at the E^2-stage and one gets $H_p^{E_2,X}(\mathrm{Epi}_2^{X\,t,\phi}) = 0$ for all $p > 0$. ☐

Proposition 4.7. *Let* $(t,\phi) = X \xrightarrow{\phi} [r_n] \xrightarrow{f_n} [r_{n-1}] \xrightarrow{f_{n-1}} \ldots \xrightarrow{f_2} [r_1]$ *be an* (X,n)*-level tree and let* $\bar{t}$ *be its* $(n-1)$*-truncation* $X[1] \xrightarrow{f_n\phi} [r_{n-1}] \xrightarrow{f_{n-1}} \ldots \xrightarrow{f_2} [r_1]$, *where* $X[1]$ *is the ordered set obtained from* X *by increasing the degree of its elements by* 1, *then*

$$H_{(*,s_{n-1},\ldots,s_1)}(\mathrm{Epi}_n^{X\,t,\phi}, \partial_n) = 0, \qquad \text{if } r_n \neq r_{n-1},$$

$$H_{(*,s_{n-1},\ldots,s_1)}(\mathrm{Epi}_n^{X\,t,\phi}, \partial_n) \cong \begin{cases} 0 & \text{for } * \neq r_n \\ C^{E_{n-1},X[1]}_{(s_{n-1},\ldots,s_1)}(\mathrm{Epi}_{n-1}^{X[1]\,\bar{t},f_n\phi}) & \text{for } s_{n-1} \leqslant * = r_n \end{cases}, \qquad \text{if } r_n = r_{n-1}.$$

Furthermore the $(n-1)$*-complex structure induced on* $H_{(r_n,s_{n-1},\ldots,s_1)}(\mathrm{Epi}_n^{X\,t,\phi},\partial_n)$ *by the* n*-complex structure of* $C^{E_n,X}_{(*,\ldots,*)}(\mathrm{Epi}_n^{X\,t,\phi})$ *coincides with the one on* $C^{E_{n-1},X[1]}_{(s_{n-1},\ldots,s_1)}(\mathrm{Epi}_{n-1}^{X[1]\,\bar{t},f_n\phi})$*.*

Proof. Recall from definition 3.8 that

$$\partial_n\left(\begin{array}{ccccccccc} X & \xrightarrow{\phi} & [r_n] & \xrightarrow{f_n} & [r_{n-1}] & \xrightarrow{f_{n-1}} & \cdots & \xrightarrow{f_2} & [r_1] \\ \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle \sigma_n} & & \big\downarrow{\scriptstyle \sigma_{n-1}} & & & & \big\downarrow{\scriptstyle \sigma_1} \\ X & \xrightarrow{\sigma_n\phi} & [s_n] & \xrightarrow{g_n} & [s_{n-1}] & \xrightarrow{g_{n-1}} & \cdots & \xrightarrow{g_2} & [s_1] \end{array}\right)$$

$$= \sum_{i,g_n(i)=g_n(i+1)} (-1)^{s_{n,i}}\epsilon((\sigma_n\phi)^{-1}(i);(\sigma_n\phi)^{-1}(i+1)) \begin{array}{ccccccccc} X & \xrightarrow{\phi} & [r_n] & \xrightarrow{f_n} & [r_{n-1}] & \xrightarrow{f_{n-1}} & \cdots & \xrightarrow{f_2} & [r_1] \\ \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle d_i\sigma_n} & & \big\downarrow{\scriptstyle \sigma_{n-1}} & & & & \big\downarrow{\scriptstyle \sigma_1} \\ X & \xrightarrow{d_i\sigma_n\phi} & [s_n-1] & \xrightarrow{g_n|_{i=i+1}} & [s_{n-1}] & \xrightarrow{g_{n-1}} & \cdots & \xrightarrow{g_2} & [s_1]. \end{array}$$

The same proof as in proposition 4.2 provides the computation of the homology of the complex with respect to the differential ∂_n: if t is not a fork tree, then the homology of the complex vanishes, and if t is the fork tree $f_n = \mathrm{id}_{[r_{n-1}]}$, then its homology groups are concentrated in top degree r_n. Let us describe all the bijections τ of $[r_{n-1}]$ such that the following diagram commutes

$$\begin{array}{ccccccccc} X & \xrightarrow{\phi} & [r_{n-1}] & \xrightarrow{\mathrm{id}} & [r_{n-1}] & \xrightarrow{f_{n-1}} & \cdots & \xrightarrow{f_2} & [r_1] \\ \big\downarrow{\scriptstyle \mathrm{id}} & & {\scriptstyle \tau}\big\downarrow & & \big\downarrow{\scriptstyle \sigma_{n-1}} & & & & \big\downarrow{\scriptstyle \sigma_1} \\ X & \xrightarrow{\tau\phi} & [r_{n-1}] & \xrightarrow{g_n} & [s_{n-1}] & \xrightarrow{g_{n-1}} & \cdots & \xrightarrow{g_2} & [s_1]. \end{array}$$

Let $(x_0,\ldots,x_{s_{n-1}})$ be the sequence of cardinalities of the preimages of σ_{n-1}, which also determines g_n. There exists a bijection of $[r_{n-1}]$ such that $\sigma_{n-1} = g_n\xi$. If ξ,ξ' are bijections of $[r_{n-1}]$ both satisfying the previous equality then $\xi(\xi')^{-1} \in \Sigma_{x_0}\times\ldots\times\Sigma_{x_{s_{n-1}}}$. Any element τ that makes the diagram commute is of the form $\alpha\xi$ for $\alpha \in \Sigma_{x_0}\times\ldots\times\Sigma_{x_{s_{n-1}}}$. As in proposition 4.5, the element $\mathrm{sgn}(\xi)c_{(x_0,\ldots,x_{s_{n-1}},X)}\xi$ does not depend on the choice of ξ and it is a generator of $H_{(r_n,s_{n-1},\ldots,s_1)}(\mathrm{Epi}_n^{X\,t,\phi},\partial_n)$. This gives the desired isomorphism of k-modules between this homology group and $C^{E_{n-1},X[1]}_{(s_{n-1},\ldots,s_1)}(\mathrm{Epi}_{n-1}^{X[1]\,\bar{t},f_n\phi})$. A direct inspection of the signs in 3.8 gives that the induced differential ∂_i coincides with the one on $C^{E_{n-1},X[1]}_{(s_{n-1},\ldots,s_1)}(\mathrm{Epi}_{n-1}^{X[1]\,\bar{t},f_n\phi})$ for $1\leqslant i\leqslant n-1$. For $i=n-1$ the computation has been done in corollary 4.6. □

References

[1] Andrew Baker, Birgit Richter, *Gamma-cohomology of rings of numerical polynomials and E_∞ structures on K-theory*, Commentarii Mathematici Helvetici **80** (4) (2005), 691–723.

[2] Michael A. Batanin, *The Eckmann-Hilton argument and higher operads*, Adv. Math. **217** (2008), 334–385.

[3] Clemens Berger, *Iterated wreath product of the simplex category and iterated loop spaces*, Adv. Math. **213** (2007), 230–270.

[4] J. Michael Boardman, Rainer M. Vogt, *Homotopy-everything H-spaces*, Bull. Amer. Math. Soc. **74** (1968) 1117–1122.

[5] Samuel Eilenberg, Saunders Mac Lane, *On the groups of $H(\Pi,n)$. I*, Ann. of Math. (2) **58** (1953), 55–106.

[6] Benoit Fresse, *The iterated bar complex of E-infinity algebras and homology theories*, preprint arXiv:0810.5147.

[7] Paul G. Goerss, Michael J. Hopkins, Moduli spaces of commutative ring spectra, in 'Structured Ring Spectra', London Math. Lecture Notes **315**, Cambridge University Press (2004), 151–200.

[8] Jean-Louis Loday, *Cyclic homology*, Second edition, Grundlehren der Mathematischen Wissenschaften **301**, Springer-Verlag, Berlin (1998), xx+513 pp.

[9] Teimuraz Pirashvili, *Hodge decompostition for higher order Hochschild homology*, Ann. Scient. École Norm. Sup. **33** (2000), 151–179.

[10] Teimuraz Pirashvili, Birgit Richter, *Robinson-Whitehouse complex and stable homotopy*, Topology **39** (2000), 525–530.

[11] Teimuraz Pirashvili, Birgit Richter, *Hochschild and cyclic homology via functor homology*, K-theory **25** (1) (2002), 39–49.

[12] Daniel Quillen, *Homotopy properties of the poset of non-trivial p-subgroups of a group*, Advances in Math. **28** (1978), 101–128.
[13] Birgit Richter, Alan Robinson, *Gamma-homology of group algebras and of polynomial algebras*, in: Homotopy Theory: Relations with Algebraic Geometry, Group Cohomology, and Algebraic K-Theory, eds.: Paul Goerss and Stewart Priddy, Northwestern University, Cont. Math. **346**, AMS (2004), 453–461.
[14] Alan Robinson, Gamma homology, Lie representations and E_∞ multiplications, Invent. Math. **152** (2003), 331–348.
[15] Alan Robinson, Sarah Whitehouse, *Operads and gamma homology of commutative rings*, Math. Proc. Cambridge Philos. Soc. **132** (2002), 197–234.
[16] Horst Schubert, *Kategorien II*, Heidelberger Taschenbücher, Springer Verlag (1970), viii+148 pp.
[17] Bruno Vallette, *Homology of generalized partition posets*, J. Pure Appl. Algebra, **208** (2) (2007), 699–725.
[18] Michelle L. Wachs, *Poset topology: tools and applications* in: Geometric combinatorics, IAS/Park City Math. Ser., **13**, AMS (2007), 497–615.
[19] James William Walker, *Topology and Combinatorics of Ordered Sets* PhD thesis, MIT, 1981, available at `http://dspace.mit.edu/handle/1721.1/16153`
[20] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics **38**, Cambridge University Press, (1994), xiv+450 pp.

Institut Galilée, Université Paris Nord, 93430 Villetaneuse, France
E-mail address: `livernet@math.univ-paris13.fr`

Department Mathematik der Universität Hamburg, Bundesstrasse 55, 20146 Hamburg, Germany
E-mail address: `richter@math.uni-hamburg.de`