Linear-quadratic infinite time horizon optimal control for differential-algebraic equations - a new algebraic criterion

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1 Introduction

We revisit the linear-quadratic infinite time horizon optimal control problem for linear constant coefficient differential-algebraic systems

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad Ex(t_0) = Ex_0 \tag{1}$$

with $x_0 \in \mathbb{R}^n$, $B \in \mathbb{R}^{n,m}$ and $E, A \in \mathbb{R}^{n,n}$, such that the pencil $sE - A \in \mathbb{R}[s]^{n,n}$ is regular, that is, $\det(sE - A) \neq 0$. For systems governed by ordinary differential-equations (that is, E is the identity matrix), a rigorous analysis of this problem has its origin in the 60s of the 20th century [6, 8–10, 13, 15, 21]. In particular, the article [20] by Willems gives a complete characterization of linear-quadratic optimal control of ordinary systems by means of solvability of an associated algebraic Riccati equation and feasibility of a certain linear matrix inequality.

Mainly two approaches to the generalization of this theory exist for the differential-algebraic case: The articles [11, 12, 14] by Kawamoto et al. and Kurina on the one hand, and [4] by Bender and Laub on the other hand introduce different

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kinds of generalized algebraic Riccati equations. Both theories present only sufficient but not necessary criteria for the existence of an optimal control. In particular, both approaches require the input being fully and positively weighted in the cost functional. This assumption is however very restrictive for linear-quadratic infinite time horizon optimal control arising in the analysis of dissipative systems [7, 19]. In the case of ordinary differential equations, the algebraic Riccati equation is then replaced with a certain linear matrix inequality, which is also known as *Kalman-Yakubovich-Popov lemma* [20]. A straightforward generalization of this lemma to the differential-algebraic case is not readily possible and only leads to sufficient criteria [7, 19].

The aim of this article is to present a suitable generalization of this algebraic criterion to differential-algebraic equations that gives necessary and sufficient criteria for optimizability under only very slight conditions related to controllability of the system (1).

Throughout this article we use the following notation:

$\mathbb{R}$, ($\mathbb{R}_{\geq 0}$, $\mathbb{R}_{\leq 0}$)		set real (non-negative, non-positive) numbers,
$\mathbb{R}[s]$		the ring of real polynomials,
$R^{n,m}$		the set of $n \times m$ matrices with entries in the ring R,
I_n, $0_{m,n}$		identity and zero matrix of size n and $m \times n$, resp. (subscripts can be omitted, if clear from context),
M^T		transpose of $M \in \mathbb{R}^{m,n}$,
$\dot{f}$		distributional derivative of $f : \mathcal{I} \to \mathbb{R}^n$ with $\mathcal{I} \subseteq \mathbb{R}$,
$L^2_{\text{loc}}(\mathcal{I}; \mathbb{R}^n)$		the set of measurable and locally square integrable functions $f : \mathcal{I} \to \mathbb{R}^n$ on the set $\mathcal{I} \subseteq \mathbb{R}$,
$H^1_{\text{loc}}(\mathcal{I}; \mathbb{R}^n)$	$=$	$\left\{ f \in L^2_{\text{loc}}(\mathcal{I}; \mathbb{R}^n) \;\middle|\; \dot{f} \in L^2_{\text{loc}}(\mathcal{I}; \mathbb{R}^n) \right\}$.

2 Behavior and cost functionals

A trajectory $(x, u) : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m$ is said to be a *solution* of (1) if, and only if, it belongs to the *behavior* of (1):

$$\begin{aligned} \mathfrak{B}_{[E,A,B]} := \{ & (x(\cdot), u(\cdot)) \in L^2_{\text{loc}}(\mathbb{R}; \mathbb{R}^n) \times L^2_{\text{loc}}(\mathbb{R}; \mathbb{R}^m) \mid \\ & Ex(\cdot) \in H^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^n) \text{ and } (x, u) \text{ fulfills} \\ & E\dot{x}(t) = Ax(t) + Bu(t) \; \forall \, t \in \mathbb{R} \} \, . \end{aligned}$$

We first recall different concepts related to controllability for differential-algebraic equations (1).

Definition 1. *Let a system* (1) *with* $E, A \in \mathbb{R}^{n,n}$ *and* $B \in \mathbb{R}^{n,m}$ *be given. Then,*

(1) *is called*

(i) *impulse controllable* $\iff$ $\forall\, x_0 \in \mathbb{R}^n,\ t_0 \in \mathbb{R},\ \exists\,(x,u) \in \mathfrak{B}_{[E,A,B]}$ *with* $Ex(t_0) = Ex_0$,

(ii) *strongly controllable* $\iff$ $\forall\, x_0, x_1 \in \mathbb{R}^n, t_0 < t_1 \in \mathbb{R},\ \exists\,(x,u) \in \mathfrak{B}_{[E,A,B]}$ *with* $Ex(t_0) = Ex_0$ *and* $Ex(t_1) = Ex_1$.

As in the case of standard standard systems, these properties can be equivalently characterized by means of algebraic criteria generalizing the *Hautus test* [17, Chap. 3].

Proposition 2. *[5] Let a system* (1) *with* $E,\ A \in \mathbb{R}^{n,n}$ *and* $B \in \mathbb{R}^{n,m}$ *be given. Further, let* $r = \operatorname{rank}(E)$ *and* $S_\infty \in \mathbb{R}^{n,n-r}$ *be a matrix with* $\operatorname{im} S_\infty = \ker E$*. Then,* (1) *is called*

(i) *impulse controllable* $\iff$ $\operatorname{rank}[E,\ AS_\infty,\ B] = n$,

(ii) *strongly controllable* $\iff$ $\operatorname{rank}[sE - A, B] = n$ *for all* $s \in \mathbb{C}$ *and* $\operatorname{rank}[E,\ AS_\infty,\ B] = n$.

For a set $\mathcal{I} \subset \mathbb{R}$ and matrices $Q \in \mathbb{R}^{n,n}$, $S \in \mathbb{R}^{n,m}$, $R \in \mathbb{R}^{m,m}$ with $Q = Q^T$ and $R = R^T$, consider the *cost functional*

$$\mathcal{J}^{\mathcal{I}}_{[Q,S,R]}(x,u) = \int_{\mathcal{I}} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau.$$

As in [20], we consider the following minimization problems:

a) $$V_f^+(Ex_0) = \inf\left\{\ \mathcal{J}^{\mathbb{R}_{\geq 0}}_{[Q,S,R]}(x,u)\ \middle|\ (x,u) \in \mathfrak{B}_{[E,A,B]} \text{ with } Ex(0) = Ex_0\ \right\},$$

b) $$V^+(Ex_0) = \inf\left\{ \mathcal{J}^{\mathbb{R}_{\geq 0}}_{[Q,S,R]}(x,u) \middle| \ (x,u) \in \mathfrak{B}_{[E,A,B]} \text{ with } Ex(0) = Ex_0 \text{ and } \lim_{t\to\infty} Ex(t) = 0\right\},$$

c) $$V^-(Ex_0) = -\inf\left\{ \mathcal{J}^{\mathbb{R}_{\leq 0}}_{[Q,S,R]}(x,u) \middle| \ (x,u) \in \mathfrak{B}_{[E,A,B]} \text{ with } Ex(0) = Ex_0 \text{ and } \lim_{t\to-\infty} Ex(t) = 0\right\},$$

d) $$V_n^+(Ex_0) = \inf\left\{\ \mathcal{J}^{[0,T]}_{[Q,S,R]}(x,u)\ \middle|\ T \geq 0,\ (x,u) \in \mathfrak{B}_{[E,A,B]} \text{ with } Ex(0) = Ex_0\ \right\}.$$

One aim of this article is to characterize finiteness of $V_f^+(Ex_0)$, $V^+(Ex_0)$, $V^-(Ex_0)$ and $V_n^+(Ex_0)$ for all $x_0 \in \mathbb{R}^n$. It is obvious that impulse controllability of (1) is necessary for $V_f^+(Ex_0)$, $V^+(Ex_0)$, $V^-(Ex_0)$, $V_n^+(Ex_0) \in \mathbb{R}$ for all $x_0 \in \mathbb{R}^n$. Note that the assumption of strong controllability is just for sake of brevity, and can be relaxed to stabilizability together with some slight additional technical condition [16]. To present equivalent criteria on the cost functional and the system, we consider a class of functions $V : \operatorname{im}(E) \to \mathbb{R}$ which satisfies the *dissipation inequality*

$$\begin{gathered} \mathcal{J}_{[Q,S,R]}^{[t_0,t_1]}(x,u) + V(Ex(t_1)) \geq V(Ex(t_0)) \\ \forall (x,u) \in \mathfrak{B}_{[E,A,B]},\ \ t_0, t_1 \in \mathbb{R} \text{ with } t_0 \leq t_1. \end{gathered} \tag{2}$$

This is equivalent to

$$\begin{gathered} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \geq -V'(Ex(t))E\dot{x}(t) = -V'(Ex(t))(Ax(t) + Bu(t)) \\ \forall (x,u) \in \mathfrak{B}_{[E,A,B]},\ \ t \in \mathbb{R}, \end{gathered} \tag{3}$$

where $V'(Ex(t)) \in \mathbb{R}^{1,n}$ is the Jacobian of V in $Ex(t)$.

As we present in the following, the existence of certain functions satisfying the dissipation inequality will be related to finiteness of the above introduced functionals. The proofs are left to a forthcoming article [16].

Theorem 3. *Let E, A, $Q \in \mathbb{R}^{n,n}$, B, $S \in \mathbb{R}^{n,m}$ and $R \in \mathbb{R}^{m,m}$ such that $sE - A$ is regular and $Q = Q^T$, $R = R^T$. Assume that the system (1) is strongly controllable. Then the following statements are equivalent:*

(i) $\mathcal{J}_{[Q,S,R]}^{[0,T]}(x,u) \geq 0$ *for all* $T \in \mathbb{R}_{\geq 0}$, $(x,u) \in \mathfrak{B}_{[E,A,B]}$ *with* $Ex(0) = 0$*;*

(ii) $V^-(Ex_0) \in \mathbb{R}_{\leq 0}$ *for all* $x_0 \in \mathbb{R}^n$*;*

(iii) $V_f^+(Ex_0) > -\infty$ *for all* $x_0 \in \mathbb{R}^n$*;*

(iv) $V_n^+(Ex_0) > -\infty$ *for all* $x_0 \in \mathbb{R}^n$*;*

(v) There exists some functional $V : \operatorname{im}(E) \to \mathbb{R}_{\leq 0}$ *that satisfies the dissipation inequality* (2).

Furthermore, if the above conditions are fulfilled, then for all $x_0 \in \mathbb{R}^n$*, there holds*

$$-\infty < V^-(Ex_0) \leq V_n^+(Ex_0) \leq V_f^+(Ex_0) \leq V^+(Ex_0) < \infty.$$

Property (i) is often called *dissipativity* [18]. The most important special cases of dissipativity are *bounded realness* and *positive realness* [1–3].

We now present criteria which are equivalent to V^+ and V^- being finite.

Theorem 4. *Let E, A, $Q \in \mathbb{R}^{n,n}$, B, $S \in \mathbb{R}^{n,m}$ and $R \in \mathbb{R}^{m,m}$ such that $sE - A$ is regular and $Q = Q^T$, $R = R^T$. Assume that the system (1) is strongly controllable. Then the following statements are equivalent:*

(i) $\mathcal{J}^{[0,T]}_{[Q,S,R]}(x,u) \geq 0$ *for all* $T \in \mathbb{R}_{\geq 0}$, $(x,u) \in \mathfrak{B}_{[E,A,B]}$ *with* $Ex(0) = 0$ *and* $Ex(T) = 0$*;*

(ii) $V^+(Ex_0) > -\infty$ *for all* $x_0 \in \mathbb{R}^n$*;*

(iii) $V^-(Ex_0) < \infty$ *for all* $x_0 \in \mathbb{R}^n$*;*

(iv) There exists some functional $V : \mathrm{im}(E) \to \mathbb{R}$ *that satisfies the dissipation inequality* (2)*.*

Moreover, if the above conditions are fulfilled, then for all $x_0 \in \mathbb{R}^n$*, there holds*

$$-\infty < V^-(Ex_0) \leq V(Ex_0) \leq V^+(Ex_0) < \infty.$$

3 Algebraic criteria

As for the case of standard systems, one can show that, in case of existence, the function $V : \mathrm{im}(E) \to \mathbb{R}$ which satisfies the dissipation inequality (2) can be chosen quadratically. More precisely, we can make the ansatz

$$V(Ex_0) = x_0^T X^T E x_0,$$

where $X \in \mathbb{R}^{n,n}$ is a matrix with $X^T E = E^T X$ (the latter property makes $V(Ex_0)$ well-defined). The dissipation inequality (3) is now equivalent to

$$\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^T \begin{bmatrix} A^T X + X^T A + Q & X^T B + S \\ B^T X + S^T & R \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \geq 0 \; \forall \, (x,u) \in \mathfrak{B}_{[E,A,B]}, \; t \in \mathbb{R}.$$

By using impulse controllability of (1), one can show that

$$\left\{ \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \;\middle|\; (x,u) \in \mathfrak{B}_{[E,A,B]}, \; t \in \mathbb{R} \right\} = \mathcal{V}_{[E,A,B]},$$

where

$$\mathcal{V}_{[E,A,B]} = \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R}^m \;\middle|\; Ax + Bu \in \mathrm{im}(E) \right\}.$$

This leads to the new matrix inequality criterion

$$\begin{pmatrix} x \\ u \end{pmatrix}^T \begin{bmatrix} A^T X + X^T A + Q & X^T B + S \\ B^T X + S^T & R \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \geq 0 \; \forall \, (x,u) \in \mathcal{V}_{[E,A,B]}, \tag{4}$$

that is sufficient for the validity of the equivalent statements in Theorem 3 and Theorem 4. The following results state that these criteria are necessary as well.

Theorem 5. *Let* $E, A, Q \in \mathbb{R}^{n,n}$*,* $B, S \in \mathbb{R}^{n,m}$ *and* $R \in \mathbb{R}^{m,m}$ *such that* $sE - A$ *is regular and* $Q = Q^T$*,* $R = R^T$*. Assume that the system* (1) *is strongly controllable. Then the assertions (i)-(v) in Theorem 3 are fulfilled, if, and only if, there* *exists some* $X \in \mathbb{R}^{n,n}$ *such that* $X^T E$ *is symmetric and negative semi-definite and, moreover,* (4) *is fulfilled.*

Theorem 6. *Let* $E, A, Q \in \mathbb{R}^{n,n}$, $B, S \in \mathbb{R}^{n,m}$ *and* $R \in \mathbb{R}^{m,m}$ *such that* $sE - A$ *is regular and* $Q = Q^T$, $R = R^T$. *Assume that the system* (1) *is strongly controllable. Then the assertions (i)-(iv) in Theorem 4 are fulfilled, if, and only if, there exists some* $X \in \mathbb{R}^{n,n}$ *such that* $X^T E$ *is symmetric and, moreover,* (4) *is fulfilled.*

One can further show that the set of matrices $X \in \mathbb{R}^{n,n}$ with $E^T X = X^T E$ and (4) has the property of containing *extremal elements*, that is, there are solutions $X_-, X_+ \in \mathbb{R}^{n,n}$ such that for all other solutions X, there holds

$$x_0^T X_-^T E x_0 \le x_0^T X^T E x_0 \le x_0^T X_+^T E x_0 \qquad \forall x_0 \in \mathbb{R}^n.$$

In particular, there holds

$$V^-(Ex_0) = x_0^T X_-^T E x_0, \quad V^-(Ex_0) = x_0^T X_+^T E x_0 \qquad \forall x_0 \in \mathbb{R}^n.$$

4 Conclusions

We have presented a step-by-step generalization of some results on dissipativity and linear-quadratic optimal control to differential-algebraic equations. Under the assumption of strong controllability, new equivalent criteria for dissipativity and finiteness of the the optimal value have been presented.

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