A structured doubling algorithm for Lur'e Equations

Federico Poloni[1*], Timo Reis[2]

[1] *Dipartimento di Informatica, Università di Pisa, Largo B. Pontecorvo 3, 56127 Pisa, Italy*
[2] *Fachbereich Mathematik, Universität Hamburg, Bundesstraße 55, 22083 Hamburg, Germany*

SUMMARY

We introduce a numerical method for the numerical solution of the *Lur'e equations*, a system of matrix equations that arises, for instance, in linear-quadratic infinite time horizon optimal control. Via a Cayley transformation, the problem is transformed to the discrete-time case, and the structural infinite eigenvalues of the associated matrix pencil are deflated. This gives a symplectic problem with several Jordan blocks of eigenvalue 1 and even size, which arise from the nontrivial Kronecker chains at infinity of the original problem. For the solution of this modified problem, we use the *structure-preserving doubling algorithm* (SDA). Implementation issues such as the choice of the parameter γ in the Cayley transform are discussed. The numerical examples presented confirm the effectiveness of this method.

Received ...

KEY WORDS: matrix equation; Lur'e equations; structured doubling algorithm; matrix pencil; optimal control

1. INTRODUCTION

Several problems in control theory, such as linear-quadratic optimal control, dissipativity analysis [1–4], model reduction [5–9], $\mathcal{H}_\infty$ control [10], differential games [11], lead to the computation of the semi-stable Lagrangian deflating subspace of a matrix pencil of the form

$$s\mathcal{E} - \mathcal{A} = \begin{bmatrix} 0 & -sI + A & B \\ sI + A^T & Q & C \\ B^T & C^T & R \end{bmatrix} \tag{1}$$

with $A, Q \in \mathbb{R}^{n,n}$, $B, C \in \mathbb{R}^{n,m}$, $R \in \mathbb{R}^{m,m}$ and $Q = Q^T$, $R = R^T$. The word *semi-stable* here means that all the associated eigenvalues are in the closed left half-plane, and a subspace $\mathcal{U} \subset \mathbb{R}^{2n}$ is called *Lagrangian* if $\dim \mathcal{U} = n$ and for every pair of vectors $v, w \in \mathbb{U}$ holds $v^T J w = 0$, where

$$\mathcal{J} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}. \tag{2}$$

When R is nonsingular, this problem (under a mild rank assumption) is equivalent to solving the *algebraic Riccati equation* (ARE) [10, 12, 13]

$$A^T X + XA - (XB + C)R^{-1}(XB + C)^T + Q = 0. \tag{3}$$

*Correspondence to: Dipartimento di Informatica, Università di Pisa, Largo B. Pontecorvo 3, 56127 Pisa, Italy. Email: fpoloni@di.unipi.it

Its maximal solution $X \in \mathbb{R}^{n,n}$ is related to the semi-stable Lagrangian invariant subspace through

$$\mathcal{U} = \operatorname{im} \begin{bmatrix} I \\ X \end{bmatrix}.$$

While there is abundant literature on AREs including textbooks and survey articles [13–16], the case of singular R has been treated more sporadically in journal articles [17–21]. The singularity or R is however a structural property in several applications [22] and can therefore not be excluded by arguments of genericity.

The closest analogous to (3) when R is singular are the *Lur'e equations* [23, 24]

$$\begin{aligned} A^T X + XA + Q &= K^T K, \\ XB + C &= K^T L, \\ R &= L^T L, \end{aligned} \tag{4}$$

to be solved for the triple $(X, K, L) \in \mathbb{R}^{n,n} \times \mathbb{R}^{p,n} \times \mathbb{R}^{p,m}$ with $X = X^T$ and p as small as possible.

Let us briefly review the known approaches for solving them: basically, these can be divided into elimination and perturbation approaches:

a) The works [25, 26] present an iterative technique for the elimination of variables corresponding to $\ker R$: By performing an orthogonal transformation of R, and an accordant transformation of L, the equations can be divided into a 'regular part' and a 'singular part'. The latter leads to an explicit equation for a part of the matrix K. Plugging this part into (4), one obtains Lur'e equations of slightly smaller size. After a finite number of steps this leads to an algebraic Riccati equation. This also gives an equivalent solvability criterion that is obtained by the feasibility of this iteration.

b) In [21] a deflation technique is proposed. A "critical deflating subspace" of the even matrix pencil (1) is determined. Thereafter, matrices which are spanning this critical subspace are used to eliminate certain parts of the Lur'e equation, such that a projected algebraic Riccati equation is obtained. This projected ARE is accordantly solved by a Newton-Kleinman iteration. The deflation has been done by numerical computation of so-called "$\mathcal{E}$-neutral Wong sequences", a successive nullspace computation.

c) In the engineering practice, the most common approach to the solution of Lur'e equations is the perturbation of R by εI_m for some $\varepsilon > 0$. Then, by using the invertibility of $R + \varepsilon I$, the corresponding perturbed Lur'e equations are now equivalent to the Riccati equation

$$A^T X_\varepsilon + X_\varepsilon A - (X_\varepsilon B + C)(R + \varepsilon I)^{-1}(X_\varepsilon B + C)^T + Q = 0. \tag{5}$$

It is shown in [27, 28] that the corresponding maximal solutions X_ε then converge to the maximal solution of (4).

The big problem of the perturbation approach c) is that, so far, there exist no bounds for the perturbation error $\|X - X_\varepsilon\|$. On top of that, the numerical condition of the Riccati equation (5) increases drastically as ε tends to 0.

The approaches in a) and b) also have certain numerical drawbacks: they rely on successive nullspace computations, which may be an arbitrarily ill-conditioned problem. In a) it is necessary to identify, for several matrices M_k (starting from $M_0 = R$), two complementary subspaces $\mathcal{U}_{1,k}$ and $\mathcal{U}_{2,k}$ such that M_k is invertible when restricted to $\mathcal{U}_{1,k}$ and zero when restricted to $\mathcal{U}_{2,k}$. In practice, often this choice is not clear-cut, since the singular values of the matrices M_k may not have a large gap in magnitude. One needs to choose an arbitrary threshold under which they are set to zero; it is possible to end up with matrices that are ill-conditioned on $\mathcal{U}_{1,k}$ and "not quite zero" on $\mathcal{U}_{2,k}$. Similarly, In b) one has to determine spanning matrices for the $\mathcal{E}$-neutral (see Def. 5) deflating subspace $\mathcal{V}$ of the even matrix (1); by the same reasons, this is numerically ill-conditioned.

Type	Size	$\mathcal{C}_j(s)$	Parameters
W1	$k_j \times k_j$	$(s-\lambda)I_{k_j} - N_{k_j}$	$k_j \in \mathbb{N}$, $\lambda \in \mathbb{C}$
W2	$k_j \times k_j$	$sN_{k_j} - I_{k_j}$	$k_j \in \mathbb{N}$

Table I. Block types in Weierstrass canonical form

We present here a numerical method based on a modification of the *structure preserving doubling algorithm* (SDA), an iterative scheme for continuous- and discrete-time algebraic Riccati equations [29]. It is shown in [30] that, unlike other iterative schemes, this algorithm has good convergence properties also when the pencil has eigenvalues (of even multiplicity) on the unit circle, as is the case in our problem.

The method works directly on the unperturbed problem, without the need for regularization, and has the distinctive advantage that no rank decisions are needed. This feature sets it apart from most algorithms for singular control problems that appeared in the literature.

As a byproduct of this analysis, we obtain some auxiliary results that are interesting in the context of the SDA literature: we derive a formula for its initial values that is more compact than the known one, and discuss how we can use it to improve the heuristics to choose the parameter γ in the required Cayley transform.

2. CONTROL AND MATRIX THEORETIC PRELIMINARIES

The symbols $\|\cdot\|$, $\|\cdot\|_F$ stand for the spectral and Frobenius matrix norms, respectively. For Hermitian matrices $P, Q \in \mathbb{C}^{n,n}$, we write $P > Q$ ($P \geq Q$) if $P - Q$ is positive (semi-)definite. The symbol $\mathbb{R}(s)$ stands for the field of real rational functions.

For every positive k, we define the matrices $J_k, M_k, N_k \in \mathbb{R}^{k,k}$ as

$$J_k = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}, \quad M_k = \begin{bmatrix} & & 1 & 0 \\ & \iddots & \iddots & \\ 1 & \iddots & & \\ 0 & & & \end{bmatrix}, \quad N_k = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}.$$

Definition 1
Let $s\mathcal{E} + \mathcal{A}$ be a matrix pencil with $\mathcal{E}, \mathcal{A} \in \mathbb{R}^{m,n}$. Then $s\mathcal{E} + \mathcal{A}$ is called *regular* if $m = n$ and $\operatorname{rank}_{\mathbb{R}(s)}(s\mathcal{E} + \mathcal{A}) = n$. A pencil $s\mathcal{E} + \mathcal{A}$ is called *even* if $\mathcal{E} = -\mathcal{E}^T$ and $\mathcal{A} = \mathcal{A}^T$. A pencil with $\mathcal{E}, \mathcal{A} \in \mathbb{R}^{2n,2n}$ is called *symplectic* if $\mathcal{E}J\mathcal{E}^T = \mathcal{A}J\mathcal{A}^T$, with J as in (2).

Many properties of a regular matrix pencil can be characterized in terms of the *Weierstrass canonical form (WCF)*.

Theorem 2 ([31])
For any regular matrix pencil $s\mathcal{E} + \mathcal{A}$ with $\mathcal{E}, \mathcal{A} \in \mathbb{R}^{n,m}$, there exist matrices $U_l \in \mathrm{Gl}_n(\mathbb{C})$, $U_r \in \mathrm{Gl}_m(\mathbb{C})$, such that

$$U_l(s\mathcal{E} + \mathcal{A})U_r = \operatorname{diag}(\mathcal{C}_1(s), \ldots, \mathcal{C}_k(s)), \tag{6}$$

where each of the pencils $\mathcal{C}_j(s)$ is of one of the types presented in Table I.

The numbers λ appearing in the blocks of type W1 are called the *(generalized) eigenvalues* of $s\mathcal{E} + \mathcal{A}$. Blocks of type W2 are said to be corresponding to infinite eigenvalues.

A special modification of the WCF for even matrix pencils, the so-called *even Weierstrass canonical form (EWCF)*, is presented in [32]. Note that there is also a 'realness-preserving version' of this result [33].

Type	Size	$\mathcal{D}_j(s)$	Parameters
E1	$2k_j \times 2k_j$	$\begin{bmatrix} 0_{k_j,k_j} & (\lambda-s)I_{k_j}-N_{k_j} \\ (\overline{\lambda}+s)I_{k_j}-N_{k_j}^T & 0_{k_j,k_j} \end{bmatrix}$	$k_j \in \mathbb{N}$, $\lambda \in \mathbb{C}^+$
E2	$k_j \times k_j$	$\epsilon_j((-is-\mu)J_{k_j}+M_{k_j})$	$k_j \in \mathbb{N}, \mu \in \mathbb{R}$, $\epsilon_j \in \{-1,1\}$
E3	$k_j \times k_j$	$\epsilon_j(isM_{k_j}+J_{k_j})$	$k_j \in \mathbb{N}$, $\epsilon_j \in \{-1,1\}$

Table II. Block types in even Weierstrass canonical form

Theorem 3 ([32])
For any even matrix pencil $s\mathcal{E}+\mathcal{A}$ with $\mathcal{E},\mathcal{A} \in \mathbb{R}^{n,n}$, there exists a matrix $U \in \mathrm{Gl}_n(\mathbb{C})$ such that

$$U^*(s\mathcal{E}+\mathcal{A})U = \mathrm{diag}(\mathcal{D}_1(s),\dots,\mathcal{D}_k(s)), \tag{7}$$

where each of the pencils $\mathcal{D}_j(s)$ is of one of the types presented in Table II.

The numbers ϵ_j in the blocks of type E2 and E3 are called the *block signatures*. The blocks of type E1 contains pairs $(\lambda,-\overline{\lambda})$ of generalized eigenvalues. Together with realness of $\mathcal{E}$ and $\mathcal{A}$, this implies that non-imaginary eigenvalues occur in quadruples $(\lambda,\overline{\lambda},-\lambda,-\overline{\lambda})$. The blocks of type E2 and E3 respectively correspond to the purely imaginary and infinite eigenvalues.

Definition 4
An eigenvalue λ of a matrix pencil is called *c-stable*, *c-critical* or *c-anti-stable* respectively if $\mathrm{Re}(\lambda)$ is smaller than, equal to, or greater than 0. A right deflating subspace is called *c-stable* (resp. *c-anti-stable*) if it contains only c-stable (resp. c-anti-stable) eigenvalues, and *c-semi-stable* (resp. *c-semi-anti-stable*) if it contains only c-stable or c-critical (resp. c-anti-stable or c-critical) eigenvalues. The same definitions with the prefix c- replaced by d- hold if we change the expression $\mathrm{Re}(\lambda)$ to $|\lambda|-1$.

Definition 5
Let $\mathcal{M} \in \mathbb{C}^{k,k}$ be given. A subspace $\mathcal{V} \subset \mathbb{C}^k$ is called $\mathcal{M}$*-neutral* if $x^*\mathcal{M}y = 0$ for all $x,\ y \in \mathcal{V}$.

Definition 6
Given $\gamma \in \mathbb{R}$, $\gamma \neq 0$, the *Cayley transform* of a regular pencil $s\mathcal{E}-\mathcal{A}$ is the pencil

$$s\mathcal{E}_\gamma - \mathcal{A}_\gamma, \qquad \mathcal{E}_\gamma = A+\gamma\mathcal{E}, \quad \mathcal{A}_\gamma = A-\gamma\mathcal{E}.$$

This is the extension to matrix pencils of the scalar map

$$\begin{aligned} \mathcal{C}: \quad \mathbb{C}\cup\{\infty\} &\to \mathbb{C}\cup\{\infty\}, \\ \lambda &\mapsto \frac{\lambda-\gamma}{\lambda+\gamma}. \end{aligned}$$

We have $|\mathcal{C}(\lambda)| = 1$ if, and only if, λ is infinity or on the imaginary axis. Moreover, in the case $\gamma > 0$, we have $|\mathcal{C}(\lambda)| < 1$ if, and only if, $\mathrm{Re}(\lambda) > 0$, whereas, in the case $\gamma > 0$, there holds $|\mathcal{C}(\lambda)| < 1$ if, and only if, $\mathrm{Re}(\lambda) < 0$.

Via transformation into (even) Kronecker form, it can be seen that the Cayley transform of a matrix pencil preserves left and right eigenvectors and Jordan chains, while the eigenvalues are transformed according to $\lambda \mapsto \mathcal{C}(\lambda)$.

We recall from [20] the following theoretical results on Lur'e equations and their solvability that are needed in our article.

Theorem 7 ([20])
Let the Lur'e equations (4) with $A,Q \in \mathbb{R}^{n,n}$, $B,C \in \mathbb{R}^{n,m}$ and $R \in \mathbb{R}^{m,m}$ be given and assume that the associated even pencil (1) is regular and the pair (A,B) is stabilizable. Then, the following statements are equivalent.

(i) There exists a solution (X, K, L) of the Lur'e equations.

(ii) For all $\omega \in \mathbb{R}$ with $\imath\omega \notin \sigma(A)$ it holds $\Phi(\imath\omega) \geq 0$, where

$$\begin{aligned}\Phi(s) = &\begin{bmatrix}(sI-A)^{-1}B\\ I_m\end{bmatrix}^* \begin{bmatrix}Q & C\\ C^T & R\end{bmatrix}\begin{bmatrix}(sI-A)^{-1}B\\ I_m\end{bmatrix}\\ =&R - \begin{bmatrix}B\\ C\end{bmatrix}^T \begin{bmatrix}0 & A-\gamma I\\ A^T-\gamma I & Q\end{bmatrix}^{-1}\begin{bmatrix}B\\ C\end{bmatrix} \in \mathbb{R}(s)^{m,m}.\end{aligned} \tag{8}$$

is the *spectral density function* or *Popov function* of the system.

(iii) In the EWCF of $s\mathcal{E}+\mathcal{A}$, all blocks of type E2 have positive signature and even size, and all blocks of type E3 have negative sign and odd size.

Moreover, if the above conditions hold,

a) $\Phi(s) \in \mathbb{R}(s)^{m,m}$ is invertible (as a matrix with entries in the field $\mathbb{R}(s)$).

b) Among the solutions there is one, called the *stabilizing* solution and denoted by (X_+, K_+, L_+), such that $X \leq X_+$ for each other solution (X, K, L), in the positive definite ordering.

c) If (X_+, K_+, L_+) is the stabilizing solution, then the matrix

$$\mathcal{V} = \operatorname{span}\begin{bmatrix}X_+ & 0\\ I_n & 0\\ 0 & I_m\end{bmatrix} \tag{9}$$

spans the unique $n+m$-dimensional semi-c-stable $\mathcal{E}$-neutral subspace of the pencil (1).

d) If

$$\begin{bmatrix}Q & C\\ C^T & R\end{bmatrix} \geq 0, \tag{10}$$

then X_+ is the only (Hermitian) positive semidefinite solution of (4).

e) Let U be a matrix such that (7) holds, and partition it as $U = [\,U_1\,, \ldots, U_k\,]$, with block sizes compatible with the right-hand side of (7). The subspace $\mathcal{V}$ is spanned by

$$V = \begin{bmatrix}V_1 & \ldots & V_k\end{bmatrix} \in \mathbb{C}^{2n+m,n+m} \quad \text{for } V_j = U_j Z_j, \tag{11}$$

where

$$Z_j = \begin{cases}[\,I_{k_j}\,,\,0_{k_j}\,]^T, & \text{if } \mathcal{D}_j \text{ is of type E1,}\\ [\,I_{k_j/2}\,,\,0_{k_j/2}\,]^T, & \text{if } \mathcal{D}_j \text{ is of type E2,}\\ [\,I_{(k_j+1)/2}\,,\,0_{(k_j-1)/2}\,]^T, & \text{if } \mathcal{D}_j \text{ is of type E3.}\end{cases}$$

In other words, the subspace (9) contains all the vectors belonging to the Kronecker chains relative to c-stable eigenvalues, no vectors from the Kronecker chains relative to c-anti-stable eigenvalues, the first $k_j/2$ vectors from the chains relative to c-critical eigenvalues, and the first $(k_j+1)/2$ from the chains relative to eigenvalues at infinity.

Moreover, in the following we need these two elementary lemmas.

Lemma 8
Let $X, Y, G \in \mathbb{R}^{n,n}$ be symmetric matrices with $0 \leq X \leq Y$ and $G \leq 0$; then

$$X(I-GX)^{-1} \leq Y(I-GY)^{-1}.$$

Proof
Let $X_\varepsilon = X + \varepsilon I$ and $Y_\varepsilon = Y + \varepsilon I$; then, $Y_\varepsilon^{-1} \leq X_\varepsilon^{-1}$ and both inverses exist. Thus, we have

$$(I - GY_\varepsilon)Y_\varepsilon^{-1} = Y_\varepsilon^{-1} - G \leq X_\varepsilon^{-1} - G = (I - GX_\varepsilon)X_\varepsilon^{-1}.$$

Inverting the leftmost and rightmost term of the above inequality and letting $\varepsilon \to 0$ yields the desired result. □

Lemma 9
Let a nonsingular symmetric matrix and its inverse be partitioned as

$$\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}^{-1} = \begin{bmatrix} S & T \\ T^T & U \end{bmatrix}$$

with $X, S \in \mathbb{R}^{n_1,n_1}$, $Z, U \in \mathbb{R}^{n_2,n_2}$ and $Y, T \in \mathbb{R}^{n_1,n_2}$, such that, moreover, $X \leq 0$, $Z \geq 0$. Then $S \leq 0$, $U \geq 0$.

Proof
In the case where Z is nonsingular, the Schur complement formula yields $S^{-1} = X - YZ^{-1}Y^T \leq 0$, and similarly for U if X is nonsingular. As above, a continuity argument can be used to obtain the thesis when these blocks are singular. □

3. THE STRUCTURED DOUBLING ALGORITHM AND ITS CONVERGENCE PROPERTIES

The structure-preserving doubling algorithm (SDA) [29,30,34] is a matrix iteration which computes two special deflating subspaces of a matrix pencil, one semi-stable and one semi-anti-stable. It is directly related to several other types of algorithms that based on performing a "repeated squaring" in a matrix pencil setting [35–37].

A pencil $s\mathcal{L} - \mathcal{M}$ with $\mathcal{L}, \mathcal{M} \in \mathbb{R}^{N+M,N+M}$ is said to be in *standard symplectic-like form (SSF)* if

$$\mathcal{L} = \begin{bmatrix} I_N & -G \\ 0 & F \end{bmatrix}, \quad \mathcal{M} = \begin{bmatrix} E & 0 \\ -H & I_M \end{bmatrix}, \tag{12}$$

where the block sizes are chosen such that $E \in \mathbb{R}^{N,N}$ and $F \in \mathbb{R}^{M,M}$.

Theorem 10
[38] Suppose that $s\mathcal{L} - \mathcal{M}$ is an SSF pencil such that both matrices $I_N - GH$ and $I_M - HG$ are nonsingular. Then, the deflating subspaces of the pencil

$$s \begin{bmatrix} I_N & -G' \\ 0 & F' \end{bmatrix} - \begin{bmatrix} E' & 0 \\ -H' & I_M \end{bmatrix}, \tag{13}$$

$$\begin{aligned} E' &= E(I_N - GH)^{-1}E \quad & G' &= G + E(I_N - GH)^{-1}GF \\ F' &= F(I_M - HG)^{-1}F \quad & H' &= H + F(I_M - HG)^{-1}HE \end{aligned} \tag{14}$$

coincide with those of $s\mathcal{L} - \mathcal{M}$, and its eigenvalues are the squares of the corresponding eigenvalues of $s\mathcal{L} - \mathcal{M}$.

The structured doubling algorithm (see [39] for more details) consists in iterating the transformation (14), producing sequences (E_k, F_k, G_k, H_k) from a starting (E_0, F_0, G_0, H_0) defining a pencil in SSF.

Notice that, when $N = M$, a pencil in SSF is symplectic if and only if $E^T = F$, $G = G^T$ and $H = H^T$. SDA preserves symplecticity, i.e., at each step k we have $E_k^T = F_k$, $G_k = G_k^T$, $H_k = H_k^T$. Some computational savings can be obtained by exploiting this property in the algorithm; namely, one needs to compute only one of E_{k+1} and $F_{k+1} = E_{k+1}^T$ and only one of $I_N - G_kH_k$ and $I_M - H_kG_k = (I_N - G_kH_k)^T$.

Upon repeated squaring, eigenvalues with $|\lambda| < 1$ converge to zero and eigenvalues with $|\lambda| > 1$ to infinity, and this leads to convergence in SDA. In fact, under suitable assumptions, convergence happens also in presence of unimodular eigenvalues. We report here the convergence result for the symplectic case, since this all that we need in the following. More general convergence results obtained with similar techniques can be found in [39].

Definition 11
Let $s\mathcal{L} - \mathcal{M}$ be a matrix pencil with eigenvalues $\lambda_1, \ldots, \lambda_s$ and corresponding partial multiplicities $r_1, \ldots, r_s$. Furthermore, assume that all partial multiplicities corresponding to the unimodular eigenvalues are even. Then the *canonical* semi-stable (resp. semi-unstable) subspace is defined as the unique deflating subspace whose associated eigenvalues have partial multiplicities

$$\begin{cases} r_\ell & \text{if } \lambda_\ell \text{ is d-stable (resp. unstable),} \\ r_\ell/2 & \text{if } \lambda_\ell \text{ is unimodular,} \\ 0 & \text{if } \lambda_\ell \text{ is d-unstable (resp. stable).} \end{cases}$$

Theorem 12 ([30])
Let the SDA be applied to a symplectic pencil (12) such that all its unimodular eigenvalues have even partial multiplicity. Suppose that there exist matrices $G_\infty, H_\infty \in \mathbb{R}^{n,n}$ such that

$$\begin{bmatrix} I \\ H_\infty \end{bmatrix}, \qquad \begin{bmatrix} G_\infty \\ I \end{bmatrix} \tag{15}$$

span respectively the canonical semi-stable and semi-unstable invariant subspaces of (12). Suppose in addition that the sequences $(E_k, F_k = E_k^T, G_k, H_k)$ defined by SDA are well-defined. Then,

- $\|E_k\| = \|F_k\| = \mathcal{O}(2^{-k})$,
- $\|H_\infty - H_k\| = \mathcal{O}(2^{-k})$,
- $\|G_\infty - G_k\| = \mathcal{O}(2^{-k})$.

Well-definedness of the sequence can be proven under suitable hypotheses, which hold true in the optimal control applications.

Theorem 13 ([34])
Suppose that G_0, H_0 are semidefinite, one positive and one negative. Then, SDA is well-defined (i.e., $I - G_k H_k$ and $I - H_k G_k$ are nonsingular), and the sequences $0, G_0, G_1, \ldots$ and $0, H_0, H_1, \ldots$ are monotonic.

One can transform a regular pencil into SSF easily using the following result.

Theorem 14
Let $s\mathcal{E} - \mathcal{A}$ be a matrix pencil with $\mathcal{E}, \mathcal{A} \in \mathbb{R}^{N+M,N+M}$, and partition both matrices as

$$\mathcal{E} = \begin{bmatrix} \mathcal{E}_1 & \mathcal{E}_2 \end{bmatrix} \qquad \mathcal{A} = \begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_2 \end{bmatrix}$$

with $\mathcal{E}_1, \mathcal{A}_1 \in \mathbb{R}^{N+M,N}$ and $\mathcal{E}_2, \mathcal{A}_2 \in \mathbb{R}^{N+M,M}$. An SSF pencil having the same eigenvalues and right deflating subspaces of the original pencil exists if and only if $\begin{bmatrix} \mathcal{E}_1 & \mathcal{A}_2 \end{bmatrix}$ is nonsingular; in this case, it is unique and it holds

$$\begin{bmatrix} E & -G \\ -H & F \end{bmatrix} = \begin{bmatrix} \mathcal{E}_1 & \mathcal{A}_2 \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{A}_1 & \mathcal{E}_2 \end{bmatrix}. \tag{16}$$

Proof
We are looking for a nonsingular matrix Q such that

$$sQ \begin{bmatrix} \mathcal{E}_1 & \mathcal{E}_2 \end{bmatrix} - Q \begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_2 \end{bmatrix} = s \begin{bmatrix} I & -G \\ 0 & F \end{bmatrix} - \begin{bmatrix} E & 0 \\ -H & I \end{bmatrix}.$$

By taking only some of the blocks from the above equation, we get

$$Q\mathcal{E}_1 = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad Q\mathcal{A}_2 = \begin{bmatrix} 0 \\ I \end{bmatrix}, \qquad \text{i.e.,} \qquad Q \begin{bmatrix} \mathcal{E}_1 & \mathcal{A}_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

thus Q must be the inverse of $\begin{bmatrix} \mathcal{E}_1 & \mathcal{A}_2 \end{bmatrix}$. Taking the other two blocks we get

$$Q\mathcal{A}_1 = \begin{bmatrix} E \\ -H \end{bmatrix}, \quad Q\mathcal{E}_2 = \begin{bmatrix} -G \\ F \end{bmatrix},$$

which promptly yields (16). □

Notice that the formula in Theorem 14 can be applied also to SDA for continuous-time Riccati equation [29], where it yields (using the notation of [29])

$$\begin{bmatrix} \widehat{A} & \widehat{G} \\ -\widehat{H} & -\widehat{A}^T \end{bmatrix} = \begin{bmatrix} A_\gamma & -G \\ -H & -A_\gamma^T \end{bmatrix} \begin{bmatrix} \bar{A}_\gamma & -G \\ -H & -\bar{A}_\gamma^T \end{bmatrix}. \tag{17}$$

Rearranging the blocks gives a system $MX = N$, with M and X $2n \times 2n$ symmetric matrices, whose solution costs $8n^3$ flops [40, Appendix C]. This compares favorably with the formulas in [29, Equations (9)–(11)], which require two LU factorizations, the solution of four linear systems of the form $MX = N$, one product and one explicit inversion, all of them involving unsymmetric $n \times n$ matrices, for a total cost of $(13 + \frac{1}{3})n^3$ flops [40]. Moreover, these formulas are simpler to analyze and can be implemented as a single LAPACK call.

The same trick can be applied, with computational advantage, to SDA for nonsymmetric algebraic Riccati equations [41].

4. A REDUCED LUR'E PENCIL

Let $s\mathcal{E} - \mathcal{A}$ be the pencil (1) associated to the Lur'e equations (4). Throughout the remaining part, we employ the following assumptions.

A1 The Lur'e equations (4) are solvable.
A2 The pencil (1) is regular.
A3 The pair (A, B) is stabilizable.

Let $\gamma > 0$ be such that both $\Phi(\gamma)$ as in (8) and $A - \gamma I$ are nonsingular (there exist at least one such γ, since $\operatorname{rank}_{\mathbb{R}(s)} \Phi(s) = m$ by assumption), and define

$$T := \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}, \qquad \mathcal{T} := \begin{bmatrix} T & 0 \\ 0 & 0_m \end{bmatrix}, \qquad \mathcal{A}_\gamma := \mathcal{A} - \gamma\mathcal{T}.$$

We apply Theorem 14 to the Cayley transform of $s\mathcal{E} + \mathcal{A}$, in order to obtain the entries of its SSF, which we denote by

$$s \begin{bmatrix} I_n & -\widehat{G} \\ 0 & \widehat{F} \end{bmatrix} - \begin{bmatrix} \widehat{E} & 0 \\ -\widehat{H} & I_{n+m} \end{bmatrix}. \tag{18}$$

The resulting expression is

$$\begin{bmatrix} \widehat{E} & -\widehat{G} \\ -\widehat{H} & \widehat{F} \end{bmatrix} = \mathcal{A}_\gamma^{-1} \mathcal{A}_{-\gamma} \tag{19}$$

Notice that the inverse exists, since both the leading $2n \times 2n$ principal block of $\mathcal{A}_\gamma$

$$\begin{bmatrix} 0 & A - \gamma I \\ A^T - \gamma I & Q \end{bmatrix}$$

and its Schur complement $\Phi(\gamma)$ are nonsingular, and that the c-stable subspace will become the d-unstable one after the Cayley transform (since $\gamma > 0$).

The quantity in (19) can be expressed as

$$\begin{bmatrix} \widehat{E} & -\widehat{G} \\ -\widehat{H} & \widehat{F} \end{bmatrix} = \mathcal{A}_\gamma^{-1}\left(\mathcal{A}_\gamma + 2\gamma\mathcal{T}\right) = I_{2n+m} + 2\gamma\mathcal{A}_\gamma^{-1}\mathcal{T} = \begin{bmatrix} I_{2n} + 2\gamma ZT & 0 \\ * & I_m \end{bmatrix}, \tag{20}$$

where we denote by Z the leading principal $2n \times 2n$ block of $\mathcal{A}_\gamma^{-1}$.

This block structure translates to the SSF blocks; namely, we can define

$$\widehat{E} = E, \qquad \widehat{F} = \begin{bmatrix} F & 0 \\ * & I_m \end{bmatrix}, \qquad \widehat{G} = \begin{bmatrix} G & 0 \end{bmatrix}, \qquad \widehat{H} = \begin{bmatrix} H \\ * \end{bmatrix}, \tag{21}$$

so that the smaller blocks E, F, G, H have all size $n \times n$. In particular, the pencil (18) is block lower triangular; it follows that a special right deflating subspace is

$$\begin{bmatrix} 0_{2n\times m} \\ I_m \end{bmatrix},$$

whose only eigenvalue is $1 = \mathcal{C}_\gamma(\infty)$ with algebraic and geometric multiplicity m, while the deflating subspaces relative to the other eigenvalues are in the form

$$\begin{bmatrix} V \\ * \end{bmatrix},$$

where V has $2n$ rows and is a deflating subspace of the reduced pencil

$$s\begin{bmatrix} I_n & -G \\ 0 & F \end{bmatrix} - \begin{bmatrix} E & 0 \\ -H & I_n \end{bmatrix}. \tag{22}$$

Moreover, as Z is Hermitian in (20), it follows that the leading $2n \times 2n$ block $\widehat{A} := I_{2n} + 2\gamma ZT$ is such that $\widehat{A}T$ is Hermitian. This means that $E^T = F$ and $G = G^T$, $H = H^T$, that is, the pencil (22) is symplectic.

The pencil (22) is given by $P^T(s\mathcal{E} - \mathcal{A})P$, where P is the projection on

$$\left(\operatorname{span}\left(\begin{bmatrix} 0 \\ 0 \\ I_m \end{bmatrix}\right)\right)^\perp = (\ker \mathcal{E})^\perp.$$

With this characterization, it is easy to derive the WCF of (22) from that of the Cayley transform of (1). We see that $\ker\mathcal{E}$ is the space spanned by the first column of each block of type W2 (as a corollary, we see that there are exactly $m = \dim\ker E$ such blocks). These blocks are transformed into blocks of type W1 with $\lambda = 1$ by the Cayley transform. Thus projecting on their orthogonal complement corresponds to dropping the first row and column from each of the blocks of type W1 corresponding to $\lambda = 1$. In particular, it follows that if the criteria in Theorem 7 hold, then in the WCF of the pencil (22) every block of type W1 corresponding to a d-critical eigenvalue has even size. Therefore, all the unimodular eigenvalues of the reduced pencil (22) have even partial multiplicities. By considering which vectors are needed from each vector chain corresponding to blocks in the WCF to form the subspace in (11) we get therefore the following result.

Theorem 15

Suppose that for $A, Q \in \mathbb{R}^{n,n}$, $B, C \in \mathbb{R}^{n,m}$ and $R \in \mathbb{R}^{m,m}$ the Lur'e equations (4) satisfy Assumptions **A1**, **A2** and **A3**. Let V span an invariant subspace of the reduced pencil (22) with E, F, G and H as in (20), (21). Then,

$$\widetilde{V} = \begin{bmatrix} V & 0 \\ 0 & I_m \end{bmatrix} \tag{23}$$

spans a deflating subspace of (1). In particular, if V spans the canonical n-dimensional d-semi-unstable invariant subspace of (22), then $\widetilde{V}$ spans the subspace $\mathcal{V}$ in (9). Moreover, under our assumptions the Lur'e equations have a stabilizing solution, and thus G_∞ exists in (15).

Remark 1
From the above discussion, one also obtains that the matrix X_+ appearing in (9) is the canonical weakly stabilizing solution of the discrete-time algebraic Riccati equation (DARE)

$$X = EX(I - HX)^{-1}E^T + G. \tag{24}$$

If the matrix H_∞ in (15) exists as well, then we can apply Theorem 12 to show that the sequence G_k generated by SDA converges to X_+. Unfortunately, this hypothesis is too restrictive for an important class of Lur'e equations, namely those corresponding to optimal control problems with positive semidefinite cost functional, i.e., when (10) holds true. Indeed, in all cases in which R is singular but the even pencil (1) is regular, there is an E3 block of length $k_j \geq 3$ in the EWCF of (1), and thus using [20, Lemma A.2], one can show that all solutions to the Lur'e equations are singular. In particular, the canonical anti-stabilizing subspace of the pencil is spanned by

$$\begin{bmatrix} X_- \\ I \end{bmatrix},$$

for a suitable solution X_-, and thus the topmost block is singular.

However, in numerical experiments, we observe that G_k converges to X_+ nevertheless, while H_k diverges and G_kH_k and H_kG_k are bounded. The same phenomenon was observed also in [30, Example 5.5] without a full proof. We now prove here a convergence result that covers these cases.

Theorem 16
Suppose that $G_0 \leq 0$ and $H_0 \geq 0$ in SDA, and that there exists at least one X satisfying

$$X \geq 0, \text{ and } \begin{bmatrix} I \\ X \end{bmatrix} \text{ is an invariant subspace of (12).} \tag{25}$$

Then, there is an X_* satisfying (25) such that $X_* \leq X$ for each other X satisfying it, and the sequence (H_k) converges to X_*.

Proof
An early result in the theory of doubling methods [35] shows that $H_k = X_{2^k}$, where X_k is the sequence defined by

$$X_0 = 0, \qquad X_{k+1} = H_0 + E_0^T X_k (I - G_0 X_k)^{-1} E_0. \tag{26}$$

Therefore, we may reduce the problem to computing the limit of (26). Notice that this is a fixed-point iteration for the DARE associated with the pencil (12). Using Lemma 8, we can easily prove by induction that $X_{k+1} \geq X_k$ and that $X - X_k \geq 0$ for each positive semidefinite solution X of the Riccati equation. The sequence X_k is bounded and increasing, and therefore it converges; its limit X_* is a positive semidefinite solution of the DARE, as obtained by passing (26) to the limit, and satisfies $X_* \leq X$ for every other solution $X \geq 0$. □

Remark 2
The same results hold with all the inequalities reversed (proof: if we change sign to G_k and H_k for each k, the formulas in (14) are unchanged).

Remark 3
A corresponding result holds for G_k, namely: suppose that $G_0 \geq 0$, $H_0 \leq 0$, and there exists at least one Y such that

$$Y \geq 0, \text{ and } \begin{bmatrix} Y \\ I \end{bmatrix} \text{ is an invariant subspace of (12).} \tag{27}$$

Then, there is a minimal Y_* satisfying it, and $G_k \to Y_*$ (proof: apply the previous remark to the dual equation $Y = G_0 + E_0 Y (I - H_0 Y)^{-1} E_0^T$). As above, we may also reverse all inequalities and replace "minimal" with "maximal".

Remark 3 is the one that applies to our setting. We can prove the following convergence result by showing that its hypotheses are satisfied for the SSF pencil produced by Lur'e equations under condition (10).

Theorem 17
Let the solvable Lur'e equations (4) with $A, Q \in \mathbb{R}^{n,n}$, $B, C \in \mathbb{R}^{n,m}$ and $R \in \mathbb{R}^{m,m}$ be given and assume that the associated even pencil (1) is regular and the pair (A, B) is stabilizable. Furthermore, assume that

$$\begin{bmatrix} Q & C \\ C^T & R \end{bmatrix} \geq 0$$

and let $\gamma > 0$ be such that $\Phi(\gamma)$ with $\Phi(s) \in \mathbb{R}(s)^{m,m}$ as in (8) and $\gamma I - A$ are nonsingular. Then, for the matrices E, F, G and H as in (20), (21), the SDA iteration is well-defined and the sequence (G_k) converges to the maximal solution X_+ of the Lur'e equations (4).

Proof
By Theorem 7 e), there is exactly one positive semidefinite solution X_+ to the Lur'e equations, and thus, by Theorem 13, there is only one $Y = X_+$ satisfying (27). In view of the modification of Theorem 16 given in Remark 3, we now only need to show that the matrices E, F, G and H as in (20), (21) fulfill $G \geq 0$ and $H \leq 0$. The former statement follows by Lemma 9. For the latter one, we first prove positive semidefiniteness by additionally assuming that R is nonsingular, and then invoke a continuity argument again as in the proof of Lemmas 8 and 9. When R is invertible, the leading $2n \times 2n$ block of $\mathcal{A}_\gamma^{-1}$ is the inverse of the Schur complement of R

$$\begin{aligned} &\begin{bmatrix} 0 & A - \gamma I \\ A^T - \gamma I & Q \end{bmatrix} - \begin{bmatrix} B \\ C \end{bmatrix} R^{-1} \begin{bmatrix} B^T & C^T \end{bmatrix} \\ &= \begin{bmatrix} -BR^{-1}B^T & A - BR^{-1}C^T - \gamma I \\ (A - BR^{-1}C^T - \gamma I)^T & Q - CR^{-1}C^T \end{bmatrix}. \end{aligned} \tag{28}$$

Notice that $-BR^{-1}B^T \leq 0$ and $Q - CR^{-1}C^T \geq 0$, as the latter is a Schur complement in a positive semidefinite matrix, and thus the matrix in (28) satisfies the hypotheses of Lemma 9. In particular, the matrix H, which is the lower right block of the inverse of the matrix in (28), is negative semidefinite. □

5. IMPLEMENTATION OF SDA FOR LUR'E EQUATIONS

Based on the results of the previous sections, we can use the SDA-I algorithm to compute the solution to a Lur'e equation. The resulting algorithm is reported as Algorithm 1.

As we saw in Section 3, the symplecticity of the pencil is preserved during the SDA iterations, and helps reducing the computational cost of the iteration. Moreover, in this way we can preserve the eigenvalue symmetry of the original pencil along the iteration.

Algorithm 1 produces a sequence G_k of approximations of the maximal solution X. Corresponding sequences K_k, L_k of L and K satisfying (4) can be constructed by performing an eigenvalue decomposition

$$\begin{bmatrix} A^T X_k + X_k A + Q & X_k B + C \\ B^T X_k + C^T & R \end{bmatrix} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} U_1 & U_2 \end{bmatrix}^T,$$
$$\Sigma_1 \in \mathbb{R}^{m,m}, \quad U_1 \in \mathbb{R}^{m+n,m}, \quad \Sigma_2 \in \mathbb{R}^{n,n}, \quad U_2 \in \mathbb{R}^{m+n,n},$$

ordered such that Σ_1 contains the largest diagonal elements, and taking

$$\begin{bmatrix} K_k & L_k \end{bmatrix} = \Sigma_1^{1/2} U_1^T.$$

Notice, though, that K and L are non-unique and are typically not needed in applications; we use them here only to check the residual of the Lur'e equations *a posteriori*. Namely, with this choice

Algorithm 1: A structured doubling algorithm for the maximal solution of a Lur'e equation

input : A, B, C, Q, R defining Lur'e equations (4) fulfilling **A1**–**A3**
output: An approximation of the maximal solution X_+
Choose a suitable $\gamma > 0$;
Compute

$$T \longleftarrow \begin{bmatrix} 0 & A-\gamma I & B \\ A^T-\gamma I & Q & C \\ B^T & C^T & R \end{bmatrix}^{-1} \begin{bmatrix} 0 & A+\gamma I \\ A^T+\gamma I & Q \\ B^T & C^T \end{bmatrix};$$

Partition

$$T = \begin{bmatrix} E & -G \\ -H & E^T \\ * & * \end{bmatrix};$$

Use SDA on E, $F = E^T$, G,H to compute G_∞, H_∞;
Return $X_+ = G_\infty$;

of K_k and L_k, we can define the *relative Lur'e residual* as

$$\frac{\left\| \begin{bmatrix} A^TX_k + X_kA + Q & X_kB + C \\ B^TX_k + C^T & R \end{bmatrix} - \begin{bmatrix} K_k^T \\ L_k^T \end{bmatrix} \begin{bmatrix} K_k & L_k \end{bmatrix} \right\|_F}{\left\| \begin{bmatrix} A^TX_k + X_kA + Q & X_kB + C \\ B^TX_k + C^T & R \end{bmatrix} \right\|_F}. \tag{29}$$

A delicate choice which affects the accuracy of the computed solution is the choice of γ in the Cayley transform. A heuristic strategy to this purpose is presented in [29]. The authors perform an error analysis in the ∞ norm for their version of the formulas that give the initial values E_0, G_0, H_0 of SDA, obtaining a first-order upper bound $F(\gamma)$ for the absolute error, and then apply an univariate optimization method to approximate $\arg\min F(\gamma)$. This heuristic is not always satisfactory, as it minimizes the error in the first step of the algorithm only; in particular, the objective function $F(\gamma)$ has a qualitatively different behavior from the actual error attainable by SDA in the limit $\gamma \to 0$: the former typically converges to a finite limit, while the latter diverges. However, up to our knowledge, it is the only such heuristic available.

The simpler expression for the SDA initial values given in (17) allows one to apply the standard accuracy theory for linear systems in order to give a simpler error bound for their computation; namely, the forward error is bounded by

$$\hat{F}(\gamma) = \kappa_\infty \left(\begin{bmatrix} A_\gamma & -G \\ -H & -A_\gamma^T \end{bmatrix} \right) \left\| \begin{bmatrix} \bar{A}_\gamma & -G \\ -H & -\bar{A}_\gamma^T \end{bmatrix} \right\|_\infty.$$

This formula gives a tighter bound than the one in [29], for instance in cases in which the block A_γ is ill-conditioned but the full matrix is well-conditioned. The new approach can be extended easily to Lur'e equations: the equation for the initial values is (19), and thus we have the error estimate

$$f(\gamma) = \kappa_\infty(\mathcal{A}_\gamma) \left\| \mathcal{A}_{-\gamma} \right\|_\infty.$$

Hence in our experiments we use the same optimization method as [29] (Fibonacci search), but with this new objective function $f(\gamma)$.

6. NUMERICAL EXPERIMENTS

We have implemented Algorithm 1 (SDA-L) using MATLAB®, and applied it to the following test problems.

Figure 1. Relative residual for **P1**

n	m	SDA-L	R+S $\varepsilon = 10^{-6}$	R+S $\varepsilon = 10^{-8}$	R+S $\varepsilon = 10^{-12}$	R+N $\varepsilon = 10^{-8}$
10	3	$5 \cdot 10^{-15}$	$2 \cdot 10^{-08}$	$8 \cdot 10^{-10}$	$10 \cdot 10^{-06}$	$3 \cdot 10^{-10}$
50	5	$4 \cdot 10^{-15}$	$8 \cdot 10^{-09}$	$2 \cdot 10^{-08}$	$2 \cdot 10^{-04}$	$4 \cdot 10^{-10}$
500	10	$2 \cdot 10^{-14}$	$8 \cdot 10^{-10}$	$2 \cdot 10^{-08}$	$2 \cdot 10^{-04}$	$8 \cdot 10^{-10}$

Figure 2. Relative residual for **P2**

Problem #	SDA-L	R+S $\varepsilon = 10^{-6}$	R+S $\varepsilon = 10^{-8}$	R+S $\varepsilon = 10^{-12}$	R+N $\varepsilon = 10^{-8}$
3	$2 \cdot 10^{-15}$	$6 \cdot 10^{-02}$	$6 \cdot 10^{-02}$	$6 \cdot 10^{-02}$	$1 \cdot 10^{-09}$
4	$4 \cdot 10^{-15}$	$6 \cdot 10^{-07}$	$6 \cdot 10^{-09}$	$9 \cdot 10^{-08}$	$6 \cdot 10^{-09}$
5	$7 \cdot 10^{-13}$	$3 \cdot 10^{-07}$	$1 \cdot 10^{-09}$	$2 \cdot 10^{-08}$	$1 \cdot 10^{-09}$
6	$1 \cdot 10^{-15}$	$7 \cdot 10^{-12}$	$2 \cdot 10^{-13}$	$4 \cdot 10^{-13}$	$2 \cdot 10^{-09}$

P1 a Lur'e equation with a random stable matrix $A \in \mathbb{R}^{n,n}$, a random $C = B$, $Q = 0$ and R the $m \times m$ matrix with all the entries equal to 1, with $\mathrm{rank}(R) = 1$. Namely, B was generated with the command

```
B=rand(n,m);
```

To generate a stable A, we used the following sequence of commands:

```
V=randn(n);
W=randn(n);
A=-V*V'-W+W';
```

P2 a set of problems motivated from real-world examples, taken with some modifications from the benchmark set CAREX [42]. Namely, we took Examples 3 to 6 (the real-world applicative problems) of that paper, which are a set of real-world problems varying in size and numerical characteristics, and changed the value of R to get a singular problem. In the original versions of all examples, R is the identity matrix of appropriate size; we simply replaced its $(1,1)$ entry with 0, in order to get a singular problem.

P3 a highly ill-conditioned high-index problem with $m = 1$, $A = I_n + N_n$, $B = e_n$ (the last column of the $n \times n$ identity matrix), $C = -B$, $R = 0$ and

$$Q = \begin{bmatrix} -2 & -1 & & & \\ -1 & -2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & -2 & -1 \\ & & & -1 & -2 \end{bmatrix}.$$

Such a problem corresponds to a Kronecker chain of length $2n+1$ associated to an infinite eigenvalue, and its canonical semi-stable solution is $X = I$. Notice that the conditioning of the invariant subspace problem in this case is $\epsilon^{1/(2n+1)}$, for an unstructured perturbation of the input data of the order of the machine precision ϵ [43, section 16.5].

The results of SDA-L are compared to those of a regularization method as the one described in (5), for different values of the regularization parameter ε. After the regularization, the equations are solved using SDA after a Cayley transform with the same parameter γ (R+S), or with the matrix sign method with norm scaling [40, 44] (R+N). We point out that the control toolbox of Matlab contains a command `gcare` that solves a so-called *generalized continuous-time algebraic Riccati equation*; this is equivalent to finding X_+ for a pencil in the form (1). However, this command is not designed to deal with a singular R, nor with eigenvalues numerically on the imaginary axis. Therefore, when

Figure 3. Forward error for **P3**

n	SDA-L	R+S $\varepsilon = 10^{-6}$	R+S $\varepsilon = 10^{-8}$	R+S $\varepsilon = 10^{-12}$	R+N $\varepsilon = 10^{-8}$
1	$1 \cdot 10^{-08}$	$1 \cdot 10^{-03}$	$10 \cdot 10^{-05}$	$1 \cdot 10^{-06}$	$1 \cdot 10^{-04}$
2	$8 \cdot 10^{-05}$	$3 \cdot 10^{-02}$	$1 \cdot 10^{-02}$	$4 \cdot 10^{-02}$	$1 \cdot 10^{-02}$
3	$4 \cdot 10^{-03}$	$1 \cdot 10^{-01}$	$6 \cdot 10^{-02}$	$1 \cdot 10^{+01}$	$6 \cdot 10^{-02}$
4	$3 \cdot 10^{-02}$	$4 \cdot 10^{-01}$	$2 \cdot 10^{-01}$	$10 \cdot 10^{-01}$	$5 \cdot 10^{-01}$
5	$8 \cdot 10^{-02}$	$1 \cdot 10^{+00}$	$5 \cdot 10^{-01}$	$2 \cdot 10^{+00}$	$6 \cdot 10^{-01}$

applied to nearly all these experiments, this command fails reporting the presence of eigenvalues too close to the imaginary axis.

For the problem **P3**, where an analytical solution $X = I$ is known, we reported in Figure 3 the values of the relative forward error

$$\frac{\left\|\tilde{X} - X\right\|_F}{\|X\|_F}.$$

For **P1** and **P2**, for which no analytical solution is available, we computed instead the relative Lur'e residual (29), which are in Figures 1 and 2 respectively.

We see that in all the experiments our solution method obtains a better result than the ones based on regularization. The reader may wonder why the residual for problem 5 in **P2** is two order of magnitude larger than for the other problems. It turns out that the culprit is the choice of γ in the Cayley transform: with a hand-picked value, the error drops to $9 \cdot 10^{-16}$. This shows that the heuristic for the choice of γ is still not perfect; as far as we know, finding the optimal value of the parameter γ is still an open problem in all applications of Cayley transforms.

7. CONCLUSION AND OPEN ISSUES

In this work we have introduced a new numerical method for the solution of Lur'e matrix equations. Unlike previous methods based on regularization, this approach allows one to solve the original equation without introducing any artificial perturbation and without relying on possibly ill-posed rank problems on the block R.

The first step of this approach is applying a Cayley transform to convert the problem to an equivalent discrete-time pencil. In this new form, the infinite eigenvalues can be easily deflated, reducing the problem to a discrete-time algebraic Riccati equation with eigenvalues on the unit circle. For the solution of this latter equation, the structured-preserving doubling algorithm was chosen, due to its good convergence properties in presence of eigenvalues on the unit circle, as proved in [30]. Direct methods, such as the symplectic eigensolvers presented in [45], can also be used for the solution of the deflated DARE.

Moreover, we derive a novel, simpler formula (16) for the initial values of SDA, and, correspondingly, a simplification of the heuristic criterion in [29] for the choice of the parameter γ of the Cayley transform.

The numerical experiments confirm the effectiveness of our new approach for regular matrix pencils. It is not clear whether a similar method can be adapted to work in cases in which the pencil (1) is singular, a situation which may indeed happen in the context of Lur'e equations. Another issue is finding a method to exploit the low-rank structure of Q (when present). These further developments are currently under our investigation.

REFERENCES

1. Anderson B. Algebraic description of bounded real matrices. *Electronics Letters* 1966; **2**(12):464–465.
2. Anderson B, Newcomb R. Impedance synthesis via state-space techniques. *Proc. IEE* 1968; **115**(7):928–936.
3. Anderson B, Vongpanitlerd S. *Network Analysis and Synthesis.* Prentice Hall: Englewood Cliffs, NJ, 1973.

4. Willems J. Dissipative dynamical systems. II: Linear systems with quadratic supply rates. *Arch. Ration. Mech. Anal.* 1972; **45**:352–393.
5. Chen X, Wen J. Positive realness preserving model reduction with $\mathcal{H}_\infty$ norm error bounds. *Systems Control Lett.* 2005; **54**(4):361–374.
6. Gugercin S, Antoulas A. A survey of model reduction by balanced truncation and some new results. *Internat. J. Control* 2004; **77**(8):748–766.
7. Opdenacker P, Jonckheere E. A contraction mapping preserving balanced reduction scheme and its infinity norm error bounds. *IEEE Trans. Circuits Syst. I Regul. Pap.* 1988; **35**(2):184–189.
8. Phillips J, Daniel L, Silveira L. Guaranteed passive balancing transformations for model order reduction. *IEEE Trans. Computer-Aided Design Integr. Circuits Syst.* 2003; **22**(8):1027–1041.
9. Reis T, Stykel T. Passive and bounded real balancing for model reduction of descriptor systems. *Internat. J. Control* 2010; **83**(1):74–88.
10. Zhou K, Doyle J, Glover K. *Robust and Optimal Control*. Prentice-Hall: Princeton, 1996.
11. Basar T, Bernhard P. *$\mathcal{H}_\infty$-Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach*. Birkhäuser: Boston-Basel-Berlin, 1995.
12. Willems J. Least squares stationary optimal control and the algebraic Riccati equation. *IEEE Trans. Automat. Control* 1971; **16**:621–634.
13. Lancaster P, Rodman L. *Algebraic Riccati equations*. Clarendon Press: Oxford, 1995.
14. Abou-Kandil H, Freiling G, Ionescu V, Jank G. *Matrix Riccati equations. In control and systems theory*. Systems & Control: Foundations & Applications, Birkhäuser Verlag: Basel, 2003.
15. Benner P, Saak J. Numerical solution of large and sparse continuous time algebraic matrix Riccati and Lyapunov equations: a state of the art survey. *GAMM-Mitt.* 2013; **36**(1):32–52.
16. Bini D, Iannazzo B, Meini B. *Numerical solution of algebraic Riccati equations*, *Fundamentals of Algorithms*, vol. 9. SIAM publishing: Philadelphia, PA, 2012.
17. Clements D, Anderson B. *Singular optimal control: the linear-quadratic problem*, *Lecture Notes in Control and Information Sciences*, vol. 5. Springer-Verlag: Berlin-New York, 1978.
18. Yakubovich V. Singular problem in optimal control of linear stationary system with quadratic functional. *Siberian Math. J.* 1985; **26**(1):148–158.
19. Ionescu V, Oară C, Weiss M. *Generalized Riccati theory and robust control. A Popov function approach.* John Wiley & Sons Ltd.: Chichester, 1999.
20. Reis T. Lur'e equations and even matrix pencils. *Linear Algebra Appl.* 2010; **434**(1):152–173.
21. Poloni F, Reis T. A deflation approach for large-scale Lur'e equations. *SIAM. J. Matrix Anal. & Appl.* 2012; **33**(4):1339–1368.
22. Reis T, Stykel T. Pabtec: Balanced truncation for electrical circuits. *IEEE Trans. Computer-Aided Design Integr. Circuits Syst.* 2010; **29**(9):1354–1367.
23. Lur'e A. *Certain Nonlinear Problems in the Theory of Automatic Control*. Gostekhizat: Moscow, Leningrad, 1951. Translated into English, H.M. Stationery, 1957.
24. Barabanov N. Kalman-Yakubovich lemma in general finite-dimensional case. *Int. J. Robust Nonlinear Control* 2007; **17**:369–386.
25. Jacobson D, Speyer J. Necessary and sufficient conditions for optimality for singular control problems: a transformation approach. *J. Math. Anal. Appl.* 1971; **33**:163–187.
26. Weiss H, Wang Q, Speyer J. System characterization of positive real conditions. *IEEE Trans. Automat. Control* 1994; **39**(3):540–544.
27. Jacobson D, Speyer J. Necessary and sufficient conditions for singular control problems: a limit approach. *J. Math. Anal. Appl.* 1971; **34**:239–266.
28. Trentelman H. Families of linear-quadratic problems: Continuity properties. *IEEE Trans. Automatic Control* 1987; **AC-32**(4):323–329.
29. Chu EKW, Fan HY, Lin WW. A structure-preserving doubling algorithm for continuous-time algebraic Riccati equations. *Linear Algebra Appl.* 2005; **396**:55–80, doi:10.1016/j.laa.2004.10.010. URL http://dx.doi.org/10.1016/j.laa.2004.10.010.
30. Huang TM, Lin WW. Structured doubling algorithms for weakly stabilizing Hermitian solutions of algebraic Riccati equations. *Linear Algebra Appl.* 2009; **430**(5-6):1452–1478, doi:10.1016/j.laa.2007.08.043. URL http://dx.doi.org/10.1016/j.laa.2007.08.043.
31. Gantmacher F. *Theory of Matrices*. Chelsea Publishing Company: New York, 1959.
32. Thompson R. The characteristic polynomial of a principal subpencil of a Hermitian matrix pencil. *Linear Algebra Appl.* 1976; **14**:135–177.
33. Thompson R. Pencils of complex and real symmetric and skew matrices. *Linear Algebra Appl.* 1991; **147**:323–371.
34. Lin WW, Xu SF. Convergence analysis of structure-preserving doubling algorithms for Riccati-type matrix equations. *SIAM J. Matrix Anal. Appl.* 2006; **28**(1):26–39 (electronic), doi:10.1137/040617650. URL http://dx.doi.org/10.1137/040617650.
35. Anderson B. Second-order convergent algorithms for the steady-state Riccati equation. *Internat. J. Control* 1978; **28**(2):295–306.
36. Bai Z, Demmel J, Gu M. An inverse free parallel spectral divide and conquer algorithm for nonsymmetric eigenproblems. *Numer. Math.* 1997; **76**(3):279–308, doi:10.1007/s002110050264. URL http://dx.doi.org/10.1007/s002110050264.
37. Benner P. Contributions to the numerical solution of algebraic Riccati equations and related eigenvalue problems. Doctoral dissertation, Fakultät für Mathematik, TU Chemnitz-Zwickau, Chemnitz 1997. Published by Logos-Verlag, Berlin.
38. Chu EW, Fan HY, Lin WW, Wang CS. Structure-preserving algorithms for periodic discrete-time algebraic Riccati equations. *Internat. J. Control* 2004; **77**(8):767–788.

39. Chiang CY, Chu EKW, Guo CH, Huang TM, Lin WW, Xu SF. Convergence analysis of the doubling algorithm for several nonlinear matrix equations in the critical case. *SIAM J. Matrix Anal. Appl.* 2009; **31**(2):227–247, doi: 10.1137/080717304. URL http://dx.doi.org/10.1137/080717304.
40. Higham N. *Functions of matrices*. Society for Industrial and Applied Mathematics (SIAM): Philadelphia, PA, 2008. Theory and computation.
41. Guo XX, Lin WW, Xu S. A structure-preserving doubling algorithm for nonsymmetric algebraic Riccati equation. *Numer. Math.* 2006; **103**(3):393–412, doi:10.1007/s00211-005-0673-7. URL http://dx.doi.org/10.1007/s00211-005-0673-7.
42. Benner P, Laub A, Mehrmann V. A collection of benchmark examples for the numerical solution of algebraic Riccati equations I: the continuous-time case. *Technical Report SPC 95-22*, Forschergruppe 'Scientific Parallel Computing', Fakultät für Mathematik, TU Chemnitz-Zwickau 1995. Version dated February 28, 1996.
43. Gohberg I, Lancaster P, Rodman L. *Invariant subspaces of matrices with applications*, *Classics in Applied Mathematics*, vol. 51. Society for Industrial and Applied Mathematics (SIAM): Philadelphia, PA, 2006. Reprint of the 1986 original.
44. Mehrmann V. *The autonomous linear quadratic control problem: theory and numerical solution*, *Lecture Notes in Control and Information Sciences*, vol. 163. Springer-Verlag: Berlin, 1991, doi:10.1007/BFb0039443. URL http://dx.doi.org/10.1007/BFb0039443.
45. Fassbender H. *Symplectic methods for the symplectic eigenproblem*. Kluwer Academic/Plenum Publishers, New York, 2000.