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The ADI method for bounded real and positive real Lur'e equations

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Abstract We propose an algorithm for the numerical solution of the Lur'e equations in the bounded real and positive real lemma for stable systems. The recently developed ADI iteration for algebraic Riccati equations is generalized to Lur'e equations. The algorithm provides approximate solutions in low-rank factored form. We prove that the sequence of approximate solutions is monotonically increasing with respect to definiteness. If the shift parameters are chosen appropriately, the sequence is proven to be convergent to the minimal solution of the Lur'e equations.

Keywords Lur'e equation · ADI iteration · numerical method in control theory · linear-quadratic optimal control · bounded real lemma · positive real lemma

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1 Introduction

We consider an algorithm for the approximation of the minimal solutions of the bounded real and positive real Lur'e equations. In this introduction we focus on the bounded real case.

Consider the bounded real Lur'e equation

$$\begin{aligned} A^*X + XA + C^*C &= -K^*K, \\ B^*X + D^*C &= -J^*K, \\ D^*D - I &= -J^*J, \end{aligned} \tag{1}$$

where $A \in \mathbb{C}^{n\times n}$ is stable (i.e. all its eigenvalues are in the open left half-plane), $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{p\times n}$ and $D \in \mathbb{C}^{p\times m}$ are given; the unknowns in this equation are the Hermitian matrix

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$X \in \mathbb{C}^{n\times n}$ and the further matrices $K \in \mathbb{C}^{q\times n}$, $J \in \mathbb{C}^{q\times m}$ with $q \leq m$. We will call X a *solution of* (1), if there exist $q \in \mathbb{N}_0$ and $K \in \mathbb{C}^{q\times n}$, $J \in \mathbb{C}^{q\times m}$ such that (1) holds true. A solution X is called *minimal*, if $X \leq Y$ (i.e., $Y - X$ is positive semi-definite) for all other solutions Y of (1). Note that if $D^*D - I$ is invertible, then J and K can be eliminated and (1) becomes equivalent to the algebraic Riccati equation

$$A^*X + XA + C^*C + (XB + C^*D)(I - D^*D)^{-1}(B^*X + D^*C) = 0.$$

An important application of the bounded real Lur'e equations is *bounded real balanced truncation* [11, 12], a model reduction method which preserves contractivity of a system. In particular in this application there is a need for an efficient numerical method for the large-scale case (i.e., n is large). This large-scale case arises for example when considering discretizations of partial differential equations (see Section 5 for a typical example). In the large scale case it is unfeasible to even store the dense matrix $X \in \mathbb{C}^{n\times n}$. Our algorithm provides a sequence (X_k) of approximate solutions of the form $X_k = R_k^* R_k$ for some $R_k \in \mathbb{C}^{\ell_k \times n}$ with, typically, $\ell_k \ll n$ (i.e., X_k is given in "low-rank factored form"). For a "shift parameter sequence" $(\alpha_j)_{j=1}^k$ with $\alpha_j \in \mathbb{C}$ with $\mathrm{Re}(\alpha_j) > 0$, the main computational cost in the algorithm consists of, for each α_j ($j = 1, \dots, k$), solving a linear system of the form $(\alpha_j - A)x = v$, where $v \in \mathbb{C}^{n\times p}$. The above features make the proposed algorithm attractive for the case where n is large, p is small and A is sparse. This situation is typical when considering discretizations of partial differential equations.

The proposed algorithm is an extension of the recently developed *ADI method* for algebraic Riccati equations of the type $A^*X + XA + C^*C - XBB^*X = 0$ [8, 10], which in turn is an extension of the ADI method for Lyapunov equations [7, 9, 19].

For the convergence analysis of the algorithm, we use the following connection between the minimal solution of the bounded real Lur'e equation and an optimal control problem. It is well-known that the quadratic form defined by the minimal solution of the bounded real Lur'e equation (1) expresses the *available storage* [23]. Namely, for all $x_0 \in \mathbb{C}^n$ there holds

$$x_0^* X x_0 = \sup_{u \in L^2(0,\infty;\mathbb{C}^m)} \int_0^\infty \|y(t)\|^2 - \|u(t)\|^2 \, \mathrm{d}t, \tag{2}$$

where

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0, \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{3}$$

see [21–23]. Thereby we follow the ideas in [10], which gives an interpretation of the ADI method for the algebraic Riccati equation [8] in terms of the underlying optimal control problem.

The theoretical foundation for our algorithm is a sequence of subspaces

$$\mathscr{V}_k := \mathrm{span}\{\mathrm{e}^{-\alpha_1 t}, \dots, \mathrm{e}^{-\alpha_k t}\} \subset L^2(0,\infty). \tag{4}$$

In this introduction we assume for notational simplicity that the "shift parameters" α_j are distinct (in the main part of the article we drop this assumption; the definition of $\mathscr{V}_k$ has to be modified in case of non-distinct parameters). Let $P_k : L^2(0,\infty;\mathbb{C}^p) \to L^2(0,\infty;\mathbb{C}^p)$ denote the orthogonal projection onto $\mathscr{V}_k \otimes \mathbb{C}^p$. The matrix X_k produced by our algorithm is proven to represent the optimal cost for the following control problem (see Theorem 3)

$$x_0^* X_k x_0 = \sup_{u \in L^2(0,\infty;\mathbb{C}^m)} \int_0^\infty \|(P_k y)(t)\|^2 - \|u(t)\|^2 \, \mathrm{d}t, \tag{5}$$

subject to (3). Since the spaces $\mathscr{V}_k$ are nested, this representation shows that the sequence (X_k) is monotonically increasing with respect to monotonicity, that is $X_k \geq X_{k-1}$ for all $k \in \mathbb{N}$. In the case where

$$\overline{\bigcup_{k\in\mathbb{N}} \mathscr{V}_k} = L^2(0,\infty), \tag{6}$$

we immediately see that we will have convergence of (X_k) to X. The property (6) is proven in [17] to be equivalent to the *non-Blaschke condition*

$$\sum_{j=1}^{\infty} \frac{\mathrm{Re}(\alpha_j)}{1+|\alpha_j|^2} = \infty. \tag{7}$$

We note that this non-Blaschke condition is for example satisfied if the parameters all belong to a fixed compact set contained in the open right half-plane (in particular, if the shift parameters are periodic).

We further consider the ADI method for positive real Lur'e equation

$$\begin{aligned} A^*X + XA &= -K^*K, \\ B^*X - C &= -J^*K, \\ -(D^* + D) &= -J^*J, \end{aligned} \tag{8}$$

where $A \in \mathbb{C}^{n\times n}$ is stable, and $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{m\times n}$, $D \in \mathbb{C}^{m\times m}$. These equations arise in passivity characterization of linear systems [1, 2] and in the passivity-preserving model reduction method of *positive real balanced truncation* [12,18]. Our considerations are based on the fact that the minimal solution expresses the available storage for passivity, that is

$$x_0^*Xx_0 = \sup_{u\in L^2(0,\infty;\mathbb{C}^m)} -2\mathrm{Re}\int_0^\infty y(t)^*u(t)\,\mathrm{d}t \tag{9}$$

subject to (3).

At this point, we briefly summarize existing approaches to the solution of bounded real and positive real Lur'e equations. If $I - D^*D$ (resp. $D + D^*$) is invertible, then, of course, the huge variety of existing methods for algebraic Riccati equations (see [3] for an overview) can be used. In the case where this matrix is however singular, there are only few methods available: The *structured doubling algorithm* was recently developed for Lur'e equations [15]. In contrast to our method, the structured doubling algorithm does not provide factorizations of low rank form and is therefore memory consuming in the large-scale case. Another approach to numerical solution was presented in [15], where some "critical part" of the Lur'e equation is extracted such that an algebraic Riccati equation is obtained. The latter is then solved by Newton-Kleinman iteration [3]. This method can be formulated such that approximate low rank factors are obtained. A drawback of this approach is that the extraction of the critical part consists of successive nullspace computations which may be numerically unstable.

This article is organized as follows. In the forthcoming Section 2, we introduce the systems theoretic and functional analytic framework. Some fundamentals on singular optimal control, spectral factorization and their relations to the minimal solutions of positive real and bounded real Lur'e equations are presented. Thereafter, in Section 3, we consider (generalizations of) the spaces $\mathscr{V}_k$ from (4). In particular we consider an orthonormal basis for these spaces (the Takenaka–Malmquist system) and provide matrix representations of the solution maps associated with the dynamical system (3) with respect to this basis. In Section 4 we apply these findings to the optimal control problem. In particular, we show that the matrix representations from Section 3 can be used to determine the solution X_k of (5). This gives rise to an iterative algorithm for the determination of the minimal solutions of the bounded real (and positive real) Lur'e equations. In this section we also prove convergence of the algorithm. In Section 5 we demonstrate our results by means of a numerical example.

2 Linear systems and singular optimal control

We present the connection between the minimal solutions of the Lur'e equations (1) and (8) to the optimization problems (2) and (9) respectively. We give an explicit formula of the minimal solution of the Lur'e equation in terms of operators associated to the linear system (3). This will be the theoretical basis for our algorithm.

Definition 1 (Output map, input-output map) Assume that $A\in\mathbb{C}^{n\times n}$ is stable, $B\in\mathbb{C}^{n\times m}$, $C\in\mathbb{C}^{p\times n}$ and $D\in\mathbb{C}^{p\times m}$. Consider the following maps associated to the system (3):

a) the *output map* $\Psi:\mathbb{C}^n\to L^2(0,\infty;\mathbb{C}^p)$ which maps the initial state x_0 to the output y (for control $u=0$),
$$\Psi x_0 = t\mapsto C\mathrm{e}^{At}x_0; \tag{10}$$

b) the *input-output map* $\mathbb{F}:L^2(0,\infty;\mathbb{C}^m)\to L^2(0,\infty;\mathbb{C}^p)$ which maps the input u to the output y (for initial condition $x_0=0$);
$$\mathbb{F}u = t\mapsto \int_0^t C\mathrm{e}^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau + Du(t). \tag{11}$$

The adjoints $\Psi^*:L^2(0,\infty;\mathbb{C}^p)\to\mathbb{C}^n$, $\mathbb{F}^*:L^2(0,\infty;\mathbb{C}^p)\to L^2(0,\infty;\mathbb{C}^m)$ are given by
$$\begin{aligned}\Psi^*z &= \int_0^\infty \mathrm{e}^{A^*\tau}C^*z(\tau)\,\mathrm{d}\tau,\\ \mathbb{F}^*z &= t\mapsto \int_t^\infty B^*\mathrm{e}^{A^*(\tau-t)}C^*z(\tau)\,\mathrm{d}\tau + D^*z(t).\end{aligned} \tag{12}$$

With the above introduced mappings, the supremized expression in (2) is $\|\Psi x_0+\mathbb{F}u\|_{L^2}^2 - \|u\|_{L^2}^2$; the supremized expression in (9) becomes $-2\mathrm{Re}\langle u,\Psi x_0+\mathbb{F}u\rangle_{L^2}$.

Outer systems play a crucial role in linear-quadratic optimal control.

Definition 2 (Outer system) Assume that $A\in\mathbb{C}^{n\times n}$ is stable, $B\in\mathbb{C}^{n\times m}$, $C\in\mathbb{C}^{p\times n}$ and $D\in\mathbb{C}^{p\times m}$. The system (3) is called *outer*, if the operator $\mathbb{F}$ in (11) has dense range.

The property of a system being outer can be characterized by an algebraic criterion on the matrices A, B, C and D.

Proposition 1 *Assume that $A\in\mathbb{C}^{n\times n}$ is stable, $B\in\mathbb{C}^{n\times m}$, $C\in\mathbb{C}^{p\times n}$ and $D\in\mathbb{C}^{p\times m}$. The system* (3) *is outer, if, and only if,*
$$\mathrm{rk}\begin{bmatrix}-\lambda I+A & B\\ C & D\end{bmatrix} = n+p \quad \forall\lambda\in\mathbb{C} \text{ with } \mathrm{Re}(\lambda)>0. \tag{13}$$

Proof The proof will make use of the following two facts:

a) A bounded operator that maps between Hilbert spaces has dense range, if, and only if, its adjoint has a trivial nullspace.

b) By the representation of $\mathbb{F}^*$ in (12) and the variations of constants formula, the following holds true: For $z \in L^2(0,\infty;\mathbb{C}^p)$ there holds $v = \mathbb{F}^* z \in L^2(0,\infty;\mathbb{C}^m)$, if, and only if, there exists some absolutely continuous $w \in L^2(0,\infty;\mathbb{C}^n)$ with

$$\begin{aligned} -\dot{w} &= A^* w + C^* z, \\ v &= B^* w + D^* z. \end{aligned} \tag{14}$$

Note that, by the fact that A is stable, $w \in L^2(0,\infty;\mathbb{C}^n)$ is uniquely determined by $z \in L^2(0,\infty;\mathbb{C}^p)$.

First we prove that, if $\mathbb{F}$ has dense range, then (13) holds true: Assuming the converse of the latter, then there exists some $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > 0$ and some $\tilde{w} \in \mathbb{C}^n$, $\tilde{z} \in \mathbb{C}^p$ with $\left[\begin{smallmatrix} \tilde{w} \\ \tilde{z} \end{smallmatrix}\right] \neq 0$, such that

$$\begin{bmatrix} -\lambda I + A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} \tilde{w} \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Then we have $\tilde{z} \neq 0$, since, otherwise, $(-\lambda I + A^*)\tilde{w} = 0$ with $\tilde{w} \neq 0$, which is a contradiction to the stability of A. Now define

$$\begin{bmatrix} w \\ z \end{bmatrix} = t \mapsto \mathrm{e}^{-\lambda t} \begin{bmatrix} \tilde{w} \\ \tilde{z} \end{bmatrix}.$$

These functions fulfill (14). By b), we have $0 = \mathbb{F}^* z$. Since $\tilde{z} \neq 0$, we have $z \neq 0$, whence $\ker \mathbb{F}^*$ is nontrivial. By a), this is a contradiction to $\mathbb{F}$ having dense range.
Finally we assume (13) and aim to prove that $\mathbb{F}$ has dense range. To this end, we prove that $\mathbb{F}^*$ has a trivial nullspace. Assume that $v \in \ker \mathbb{F}^*$. Then, by b), there exists some absolutely continuous $w \in L^2(0,\infty;\mathbb{C}^n)$, such that $\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]$ solves the differential-algebraic equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix}.$$

Then it follows by a transformation of the matrix pencil $\left[\begin{smallmatrix} sI + A^* & C^* \\ B^* & D^* \end{smallmatrix}\right]$ into Kronecker form [5, Chap. XII, §7] that z has the form

$$z(t) = \sum_{k=1}^{\ell} p_k(t) \mathrm{e}^{-\lambda_k t},$$

where $p_1, \dots, p_\ell : [0,\infty) \to \mathbb{C}^m$ are vector-valued complex polynomials, and $\lambda_1, \dots, \lambda_\ell$ are distinct complex numbers with

$$\mathrm{rk} \begin{bmatrix} \lambda_k I + A^* & C^* \\ B^* & D^* \end{bmatrix} < n + p \quad \text{for } k = 1, \dots, \ell. \tag{15}$$

Numbers with the latter property are called *generalized eigenvalues* of the matrix pencil $\left[\begin{smallmatrix} sI - A & -B \\ -C & -D \end{smallmatrix}\right]$. Eqs. (13) & (15) implies that $\lambda_1, \dots, \lambda_\ell$ have positive real part. The property $z \in L^2(0,\infty;\mathbb{C}^p)$ then gives rise to $p_1 = \ldots = p_k = 0$, and thus $z = 0$.

Remark 1 By taking the Schur complement, we see that (13) holds true, if, and only if, the transfer function $G(s) := D + C(sI - A)^{-1} B$ has full row rank on the open right complex half plane.

Now we study solvability of Lur'e equations and characterize the properties of the solutions. We study the more general case of Lur'e equations

$$\begin{aligned} A^*X + XA - C^*QC &= -K^*K, \\ B^*X - (D^*QC + S^*C) &= -J^*K, \\ -(D^*QD + S^*D + D^*S + R) &= -J^*J, \end{aligned} \tag{16}$$

where $A \in \mathbb{C}^{n\times n}$ is stable, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{p\times n}$, $D \in \mathbb{C}^{p\times m}$ and $Q \in \mathbb{C}^{p\times p}$, $S \in \mathbb{C}^{p\times m}$, $R \in \mathbb{C}^{m\times m}$ with $R = R^*$ and $Q = Q^*$. Note that we obtain the bounded real Lur'e equation by setting $Q = -I$, $S = 0$ and $R = I$; the positive real Lur'e equation is given by (16) with $p = m$, $Q = R = 0$ and $S = I$.

The following concepts are crucial for the existence of minimal solutions of Lur'e equations and their relation to optimization problems.

Definition 3 (Popov function, Popov operator) Assume that $A \in \mathbb{C}^{n\times n}$ is stable, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{p\times n}$, $D \in \mathbb{C}^{p\times m}$ and $Q \in \mathbb{C}^{p\times p}$, $S \in \mathbb{C}^{p\times m}$, $R \in \mathbb{C}^{m\times m}$ with $R = R^*$ and $Q = Q^*$. Then, for $G(s) = C(sI - A)^{-1}B + D$, the *Popov function* $\Pi : \imath\mathbb{R} \to \mathbb{C}^{m\times m}$ is defined by

$$\Pi(\imath\omega) := \begin{bmatrix} G(\imath\omega)^* & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} G(\imath\omega) \\ I \end{bmatrix}.$$

Let the operator $\mathbb{F}$ be defined as in Definition 1. The *Popov operator* $\mathscr{R} : L^2(0,\infty;\mathbb{C}^m) \to L^2(0,\infty;\mathbb{C}^m)$ is

$$\mathscr{R} := \begin{bmatrix} \mathbb{F}^* & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} \mathbb{F} \\ I \end{bmatrix}. \tag{17}$$

Remark 2 (Popov operator, Popov function, Lur'e equations)

a) The Popov operator is positive semi-definite, if, and only if, the Popov function fulfills $\Pi(\imath\omega) \geq 0$ for all $\omega \in \mathbb{R}$ [4].
b) If the Lur'e equation (16) is solvable, then the Popov function fulfills $\Pi(\imath\omega) \geq 0$ for all $\omega \in \mathbb{R}$ [16].
c) If the Popov function fulfills $\Pi(\imath\omega) \geq 0$ for all $\omega \in \mathbb{R}$ and the system (3) is controllable, then there exists a minimal solution of the Lur'e equation (16). This follows from the results in [16] and the substitutions

$$\begin{aligned} X &\rightsquigarrow -X, & C^*QC &\rightsquigarrow Q, \\ C^*QD + C^*S &\rightsquigarrow C, & D^*QD + S^*D + D^*S + R &\rightsquigarrow R, \\ &\text{"minimal solution"} \rightsquigarrow \text{"maximal solution"}. \end{aligned} \tag{18}$$

d) In the bounded real case, the Popov operator reads $I - \mathbb{F}^*\mathbb{F}$. Solvability of the bounded real Lur'e equation (1) therefore implies $\|\mathbb{F}\| \leq 1$. This property is called *contractivity*. The Popov function now reads $\imath\omega \mapsto I - G^*(\imath\omega)G(\imath\omega)$. By b) the solvability of (1) implies $\|G(\imath\omega)\| \leq 1$ for all $\omega \in \mathbb{R}$. Further using stability of A, the maximum principle yields that the $\mathscr{H}_\infty$-norm of G does not exceed one. By b), under the assumption of controllability, the converse implications are also true.
e) In the positive real case, the Popov operator is given by $\mathscr{R} = \mathbb{F}^* + \mathbb{F}$. Positive semidefiniteness of this operator is called *passivity*. Here the Popov function is given by $\imath\omega \mapsto G^*(\imath\omega) + G(\imath\omega)$. If the positive real Lur'e equation (8) is solvable, the stability of A and the maximum principle give rise to $G(s) + G(s)^* \geq 0$ for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$. The latter property is called *positive realness*.

Remark 3 (Dissipation inequality)

a) Solutions of the Lur'e equation (16) are special solutions of the *dissipation inequality*

$$\begin{bmatrix} A^*X + XA - C^*QC & XB - (C^*QD + C^*S) \\ B^*X - (D^*QC + S^*C) & -(D^*QD + S^*D + D^*S + R) \end{bmatrix} \leq 0, \quad X = X^*. \tag{19}$$

The solutions of (3) fulfill

$$x(t_1)^*Xx(t_1) - x(t_2)^*Xx(t_2) \geq -\int_{t_1}^{t_2} \begin{bmatrix} y(\tau) \\ u(\tau) \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} y(\tau) \\ u(\tau) \end{bmatrix} \mathrm{d}\tau \quad \forall t_1, t_2 \in \mathbb{R} \text{ s.t. } t_1 \leq t_2, \tag{20}$$

see [23]. More precisely, if the left hand side in (19) equals $-\begin{bmatrix} \widetilde{K} & \widetilde{J} \end{bmatrix}^* \begin{bmatrix} \widetilde{K} & \widetilde{J} \end{bmatrix}$ for some $\widetilde{K} \in \mathbb{C}^{\ell \times n}$, $\widetilde{J} \in \mathbb{C}^{\ell \times m}$, then the solutions of (3) fulfill

$$\begin{aligned} &x(t_1)^*Xx(t_1) - x(t_2)^*Xx(t_2) \\ = &-\int_{t_1}^{t_2} \begin{bmatrix} y(\tau) \\ u(\tau) \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} y(\tau) \\ u(\tau) \end{bmatrix} \mathrm{d}\tau + \int_{t_1}^{t_2} \|\widetilde{K}x(\tau) + \widetilde{J}u(\tau)\|^2 \mathrm{d}\tau \quad \forall t_1, t_2 \in \mathbb{R} \text{ s.t. } t_1 \leq t_2, \end{aligned} \tag{21}$$

see [21].

b) If the Popov function fulfills $\Pi(\imath\omega) \geq 0$ and the system (3) is controllable, then the Lur'e equation (16) has a minimal solution. If the system (3) is stabilizable, and the dissipation inequality (19) has a solution, then the Lur'e equation (16) has a minimal solution [16].

Now we present the relation between the minimal solutions and optimization problems subject to the linear system (3). The following result is only a slight modification and specialization of those presented in [4].

Theorem 1 *Assume that $A \in \mathbb{C}^{n \times n}$ is stable, $B \in \mathbb{C}^{n \times m}$, $C \in \mathbb{C}^{p \times n}$, $D \in \mathbb{C}^{p \times m}$ and $Q \in \mathbb{C}^{p \times p}$, $S \in \mathbb{C}^{p \times m}$, $R \in \mathbb{C}^{m \times m}$ with $R = R^*$ and $Q = Q^*$. Let $\mathbb{F}$ be the input-output operator and Ψ be the output operator of the system* (3). *Assume that the dissipation inequality* (19) *has a solution. Let X be the minimal solution of the Lur'e equations* (16) *and let $K \in \mathbb{C}^{q \times n}$, $J \in \mathbb{C}^{q \times m}$ be such that* (16) *holds true. Then the following hold true:*

a) The system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0, \\ y_\Xi(t) &= Kx(t) + Ju(t), \end{aligned} \tag{22}$$

with output map $\Psi_\Xi : \mathbb{C}^n \to L^2(0,\infty;\mathbb{C}^q)$ and input-output map $\mathbb{F}_\Xi : L^2(0,\infty;\mathbb{C}^m) \to L^2(0,\infty;\mathbb{C}^q)$ is outer.

b) The operator $\mathbb{F}_\Xi$ and the Popov operator (17) *are related by*

$$\mathscr{R} = \mathbb{F}_\Xi^* \mathbb{F}_\Xi. \tag{23}$$

c) The operators $\mathbb{F}_\Xi$, Ψ_Ξ, the output map Ψ, and the input-output map $\mathbb{F}$ of the system (3) *are related by*

$$\mathbb{F}_\Xi^* \Psi_\Xi = (\mathbb{F}^* Q + S^*)\Psi. \tag{24}$$

d) The minimal solution fulfills

$$X = \Psi_\Xi^* \Psi_\Xi - \Psi^* Q \Psi. \tag{25}$$

e) *For all* $u \in L^2(0,\infty;\mathbb{C}^m)$ *there holds*

$$-\left\langle \begin{bmatrix} \mathbb{F}u + \Psi x_0 \\ u \end{bmatrix}, \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} \mathbb{F}u + \Psi x_0 \\ u \end{bmatrix} \right\rangle_{L^2} = x_0^* X x_0 - \|\mathbb{F}_\Xi u + \Psi_\Xi x_0\|_{L^2}^2 .$$

Proof

a) Let X be the minimal solution. Then, by using the substitutions in (18), it has been shown in [16, Sec. 5] that

$$\operatorname{im} \begin{bmatrix} -\lambda I + A & B \\ K & J \end{bmatrix} = \mathbb{C}^{n+q} \quad \forall \lambda \in \mathbb{C} \text{ with } \operatorname{Re}(\lambda) > 0.$$

Then it follows from Proposition 1 that (22) is outer.

b) Let Γ be the input-output operator of the system $\dot{x}(t) = Ax(t) + Bu(t)$, $y_{\text{ext}}(t) = x(t)$. Then we have $\mathbb{F} = C\Gamma + D$ and $\mathbb{F}_\Xi = K\Gamma + J$. Let $u \in L^2(0,\infty;\mathbb{C}^m)$ be continuous. Then by integration by parts, we obtain for $t \in [0,\infty)$ that

$$\begin{aligned}
&(\Gamma^* A^* X \Gamma u)(t) \\
&= \int_t^\infty B^* \mathrm{e}^{A^*(\tau - t)} A^* X \int_0^\tau \mathrm{e}^{A(\tau - s)} Bu(s)\, \mathrm{d}s \mathrm{d}\tau \\
&= -\int_t^\infty B^* \mathrm{e}^{A^*(\tau - t)} X \frac{\mathrm{d}}{\mathrm{d}\tau} \int_0^\tau \mathrm{e}^{A(\tau - s)} Bu(s)\, \mathrm{d}s \mathrm{d}\tau + B^* \mathrm{e}^{A^*(\tau - t)} X \int_0^\tau \mathrm{e}^{A(\tau - s)} Bu(s)\, \mathrm{d}s \Big|_t^\infty \\
&= -\int_t^\infty B^* \mathrm{e}^{A^*(\tau - t)} X \frac{\mathrm{d}}{\mathrm{d}\tau} \int_0^\tau \mathrm{e}^{A(\tau - s)} Bu(s)\, \mathrm{d}s \mathrm{d}\tau - B^* X \int_0^t \mathrm{e}^{A(t - s)} Bu(s)\, \mathrm{d}s \\
&= -\int_t^\infty B^* \mathrm{e}^{A^*(\tau - t)} XA \int_0^\tau \mathrm{e}^{A(\tau - s)} Bu(s)\, \mathrm{d}s \mathrm{d}\tau - \int_t^\infty B^* \mathrm{e}^{A^*(\tau - t)} XBu(\tau) \mathrm{d}\tau \\
&\quad - B^* X \int_0^t \mathrm{e}^{A(t - s)} Bu(s)\, \mathrm{d}s \\
&= -(\Gamma^* X A \Gamma u)(t) - (\Gamma^* X B u)(t) - (B^* X \Gamma u)(t).
\end{aligned}$$

Making use of this expression, we obtain for all continuous $u \in L^2(0,\infty;\mathbb{C}^m)$ and $t \in [0,\infty)$ that

$$\begin{aligned}
(\mathbb{F}_\Xi^* \mathbb{F}_\Xi u)(t) &= ((K\Gamma)^*(K\Gamma u))(t) + (J^*(K\Gamma u))(t) + ((K\Gamma)^* Ju)(t) + J^* Ju(t) \\
&= (\Gamma^* K^* K \Gamma u)(t) + (J^* K \Gamma u)(t) + (\Gamma^* K^* Ju)(t) + J^* Ju(t) \\
&= -(\Gamma^*(A^* X + XA - C^* QC)\Gamma u)(t) + (J^* K \Gamma u)(t) + (\Gamma^* K^* Ju)(t) \\
&\quad + (D^* QD + S^* D + D^* S + R) u(t) \\
&= (\Gamma^* XBu)(t) + (B^* X \Gamma u)(t) + (\Gamma^* C^* QC \Gamma u)(t) + (J^* K \Gamma u)(t) \\
&\quad + (\Gamma^* K^* Ju)(t) + (D^* QD + S^* D + D^* S + R) u(t) \\
&= (\Gamma^* C^* QC \Gamma u)(t) + (\Gamma^*(C^* QD + C^* S) u)(t) + ((D^* QC + S^* C)\Gamma u)(t) \\
&\quad + (D^* QD + S^* D + D^* S + R) u(t) \\
&= \begin{bmatrix} C\Gamma + D \\ I \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} C\Gamma + D \\ I \end{bmatrix} u(t) = (\mathscr{R} u)(t).
\end{aligned}$$

The density of the continuous and square integrable functions in L^2 then implies that $\mathscr{R} = \mathbb{F}_\Xi^* \mathbb{F}_\Xi$.

c) By integration by parts, we obtain that for all $x_0 \in \mathbb{C}^n$ there holds

$$\begin{aligned}\int_t^\infty B^* e^{A^*(\tau-t)} A^* X e^{A\tau} x_0 \mathrm{d}\tau &= -\int_t^\infty B^* e^{A^*(\tau-t)} X \frac{\mathrm{d}}{\mathrm{d}\tau} e^{A\tau} x_0 \mathrm{d}\tau + B^* e^{A^*(\tau-t)} X e^{A\tau} x_0 \Big|_t^\infty \\ &= -\int_t^\infty B^* e^{A^*(\tau-t)} X A e^{A\tau} x_0 \mathrm{d}\tau - B^* X e^{At} x_0.\end{aligned}$$

Using the previous formula, we obtain

$$\begin{aligned}(\mathbb{F}^* Q + S^*)\Psi x_0 &= \int_t^\infty B^* e^{A^*(\tau-t)} C^* Q C e^{A\tau} x_0 \,\mathrm{d}\tau + (D^* Q C + S^* C) e^{At} x_0 \\ &= \int_t^\infty B^* e^{A^*(\tau-t)} (K^* K + A^* X + XA) e^{A\tau} x_0 \,\mathrm{d}\tau + J^* K e^{At} x_0 + B^* X e^{At} x_0 \\ &= (\mathbb{F}_\Xi^* \Psi_\Xi) x_0 + \int_t^\infty B^* e^{A^*(\tau-t)} (A^* X + XA) e^{A\tau} x_0 \,\mathrm{d}\tau + B^* X e^{At} x_0 \\ &= (\mathbb{F}_\Xi^* \Psi_\Xi) x_0.\end{aligned}$$

d) Stability of A combined with $A^*X + XA - C^*QC + K^*K = 0$ implies

$$X = \int_0^\infty e^{A^*t} (-C^* Q C + K^* K) e^{At} \mathrm{d}t = -\Psi^* Q \Psi + \Psi_\Xi^* \Psi_\Xi.$$

e) Let $u \in L^2(0,\infty;\mathbb{C}^m)$. Then, by using b), c) and d), we obtain

$$\begin{aligned}&- \left\langle \begin{bmatrix} \mathbb{F}u + \Psi x_0 \\ u \end{bmatrix}, \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} \mathbb{F}u + \Psi x_0 \\ u \end{bmatrix} \right\rangle_{L^2} \\ &= - \langle u, (\mathbb{F}^* Q \mathbb{F} + \mathbb{F}^* S + S^* \mathbb{F} + R) u \rangle_{L^2} - \langle u, (\mathbb{F}^* Q + S^*) \Psi x_0 \rangle_{L^2} - \langle (\mathbb{F}^* Q + S^*) \Psi x_0, u \rangle_{L^2} \\ &\quad - \langle \Psi^* Q \Psi x_0, u \rangle_{L^2} - x_0^* \Psi^* Q \Psi x_0 \\ &= - \langle u, \mathbb{F}_\Xi^* \mathbb{F}_\Xi u \rangle_{L^2} + \langle u, \mathbb{F}_\Xi^* \Psi_\Xi x_0 \rangle_{L^2} - \langle \mathbb{F}_\Xi^* \Psi_\Xi x_0, u \rangle_{L^2} - \langle \Psi^* Q \Psi x_0, u \rangle_{L^2} - x_0^* \Psi^* Q \Psi x_0 \\ &= - \|\mathbb{F}_\Xi u + \Psi_\Xi x_0\|^2 + x_0^* \Psi_\Xi^* \Psi_\Xi x_0 - x_0^* \Psi^* Q \Psi x_0 \\ &= - \|\mathbb{F}_\Xi u + \Psi_\Xi x_0\|^2 + x_0^* X x_0.\end{aligned}$$

Remark 4 (Lur'e equations)

a) Equation (23) is called *spectral factorization* [4, 24].

b) If the Popov operator is positive definite and boundedly invertible, then $\mathbb{F}_\Xi$ will be boundedly invertible as well. In this case, (24) implies

$$\Psi_\Xi^* \Psi_\Xi = \Psi^* (\mathbb{F}^* Q + S^*)^* \mathbb{F}_\Xi^{-1} \mathbb{F}_\Xi^{-*} (\mathbb{F}^* Q + S^*) \Psi = \Psi^* (\mathbb{F}^* Q + S^*)^* \mathscr{R}^{-1} (\mathbb{F}^* Q + S^*) \Psi.$$

Consequently, the minimal solution reads

$$X = \Psi^* (\mathbb{F}^* Q + S^*)^* \mathscr{R}^{-1} (\mathbb{F}^* Q + S^*) \Psi - \Psi^* Q \Psi,$$

which coincides with [20, Proposition 7.2] (up to a minus sign which is due to a different sign convention).

c) The property of $\mathbb{F}_\Xi$ being outer implies that for all $x_0 \in \mathbb{C}^n$, $\varepsilon > 0$, there exists some $u \in L^2(0,\infty;\mathbb{C}^m)$ with $\|\mathbb{F}_\Xi u + \Psi_\Xi x_0\|^2 < \varepsilon$. As a consequence, we have, from Theorem 1 e), that for all $x_0 \in \mathbb{C}^n$

$$x_0^* X x_0 = \sup_{u \in L^2(0,\infty;\mathbb{C}^m)} - \left\langle \begin{bmatrix} \mathbb{F}u + \Psi x_0 \\ u \end{bmatrix}, \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} \mathbb{F}u + \Psi x_0 \\ u \end{bmatrix} \right\rangle_{L^2}. \tag{26}$$

d) For the bounded real Lur'e equation, the minimal solution reads

$$X = \Psi_{\Xi}^{*}\Psi_{\Xi} + \Psi^{*}\Psi.$$

In particular, X is positive semidefinite in this case.
In the positive real case, we have

$$X = \Psi_{\Xi}^{*}\Psi_{\Xi},$$

so that the minimal solution is again positive semidefinite.

e) It follows from Theorem 1 e) that for $u \in L^2(0,\infty;\mathbb{C}^m)$ there holds

$$-\left\langle \begin{bmatrix} \mathbb{F}u+\Psi x_0 \\ u \end{bmatrix}, \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} \mathbb{F}u+\Psi x_0 \\ u \end{bmatrix} \right\rangle_{L^2} = x_0^* X x_0,$$

(i.e. u is an optimal control) if, and only if, $\mathbb{F}_{\Xi}u + \Psi_{\Xi}x_0 = 0$. Using Theorem 1 a), this means that there exists some $x : [0,\infty) \to \mathbb{R}^n$ such that the differential-algebraic equation

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{u}(t) \end{bmatrix} = \begin{bmatrix} A & B \\ K & L \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad x(0) = x_0 \tag{27}$$

is fulfilled. Then it follows by a transformation of the matrix pencil $\left[\begin{smallmatrix} sI-A & -B \\ -K & -L \end{smallmatrix}\right]$ into Kronecker form [5, Chap. XII, §7] that x and u can be expressed by sums of exponential functions of type $\sum_{k=1}^{\ell} p_k(t)\mathrm{e}^{-\lambda_k t}$ (cf. proof of Proposition 1), where $p_1,\ldots,p_\ell$ are vector-valued complex polynomials, and the distinct numbers $\lambda_1,\ldots,\lambda_\ell$ are the generalized eigenvalues of the pencil $\left[\begin{smallmatrix} sI-A & -B \\ -K & -L \end{smallmatrix}\right]$. By using the substitutions in (18), the latter are shown in [16] to be the negatives of the stable generalized eigenvalues of the *even matrix pencil*

$$s\mathscr{E} - \mathscr{A} = \begin{bmatrix} 0 & -sI+A & B \\ sI+A^* & -C^*QC & -C^*QD - C^*S \\ B^* & -D^*QC - S^*C & -D^*QD - S^*D - D^*S - R \end{bmatrix}.$$

3 Convolution systems and matrix representations

In this section we review results from [10] which give matrix representations of the adjoints of the output map Ψ and the input-output map $\mathbb{F}$ with respect to a certain orthonormal basis of $L^2(0,\infty)$.

Definition 4 Let $(\alpha_j)_{j=1}^{\infty}$ be a complex sequence with $\mathrm{Re}(\alpha_j) > 0$ for all $j \in \mathbb{N}$. We define the corresponding *Takenaka–Malmquist system* $(\psi_j)_{j=1}^{\infty}$, $\psi_j \in L^2(0,\infty)$ by

$$\begin{aligned} \phi_1 &= t \mapsto \mathrm{e}^{-\alpha_1 t}, & \psi_1 &= \sqrt{2\mathrm{Re}(\alpha_1)} \cdot \phi_1, \\ \phi_j &= \phi_{j-1} - (\alpha_j + \overline{\alpha_{j-1}}) \cdot (\mathrm{e}^{-\alpha_j \cdot} * \phi_{j-1}), & \psi_j &= \sqrt{2\mathrm{Re}(\alpha_j)} \cdot \phi_j, \end{aligned} \tag{28}$$

where $*$ denotes the convolution product, i.e., $(g*h)(t) = \int_0^t g(t-\tau)h(\tau)\,\mathrm{d}\tau$.
The space generated by the first k Takenaka–Malmquist functions is denoted by $\mathscr{K}_k(\alpha)$.

Remark 5

a) The Takenaka–Malmquist system is orthonormal (see e.g. [17, Appendix B] for a proof).
b) The spaces $\mathscr{K}_k(\alpha)$ can be interpreted as rational Krylov subspaces [10].

c) The *convolution system* $(\varphi_j)_{j=1}^{\infty}$, $\varphi_j \in L^2(0,\infty)$, which is defined by

$$\varphi_1 := t \mapsto \mathrm{e}^{-\alpha_1 t}, \qquad \varphi_j := \mathrm{e}^{-\alpha_j \cdot} * \varphi_{j-1}, \tag{29}$$

fulfills $\operatorname{span}\{\varphi_1,\dots,\varphi_k\} = \mathscr{K}_k(\alpha)$.

d) Consider the distinct numbers $q_1,\dots,q_J$ with $\{q_1,\dots,q_J\} = \{\alpha_1,\dots,\alpha_k\}$. Let ℓ_j be the number of indices in which q_j appears in $(\alpha_j)_{j=1}^k$ (thus $k = \ell_1 + \ldots + \ell_J$). Then

$$\operatorname{span}\{\varphi_1,\dots,\varphi_k\} = \bigoplus_{j=1}^{J} \operatorname{span}\left\{ t \mapsto t^l \mathrm{e}^{-q_j t} \;\middle|\; l = 0,\dots,\ell_j - 1 \right\},$$

see [10, 17].

The most important property of the above introduced space is that it is $\mathbb{F}^*$-invariant.

Theorem 2 *Let $A \in \mathbb{C}^{n\times n}$ stable and $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{p\times n}$, $D \in \mathbb{C}^{p\times m}$. For $\mathbb{F}$ as in* (11) *and $\mathscr{K}_k(\alpha)$ the sequence of subspaces from Definition 4, we have that*

$$\mathbb{F}^*\left(\mathscr{K}_k(\alpha)\otimes\mathbb{C}^p\right) \subset \mathscr{K}_k(\alpha)\otimes\mathbb{C}^m.$$

Proof The proof is contained in [10] for the case $D = 0$. The general result follows by regarding D as a pointwise multiplication operator $D : L^2(0,\infty;\mathbb{C}^m) \to L^2(0,\infty;\mathbb{C}^p)$. The latter obviously fulfills

$$D^*\left(\mathscr{K}_k(\alpha)\otimes\mathbb{C}^p\right) \subset \mathscr{K}_k(\alpha)\otimes\mathbb{C}^m.$$

The above invariance gives rise to the existence of matrix representations of $\mathbb{F}^*$ with respect to the Takenaka–Malmquist systems. These will be explicitly constructed in the following.

Definition 5 Let $(\alpha_j)_{j=1}^{\infty}$ be such that $\operatorname{Re}(\alpha_j) > 0$ for all $j \in \mathbb{N}$. Let $(\psi_j)_{j=1}^{\infty}$, $\psi_j \in L^2(0,\infty)$ be the corresponding Takenaka–Malmquist system (28). For $k \in \mathbb{N}$, the mapping $\iota_k : \mathbb{C}^k \to L^2(0,\infty)$ is defined by

$$\iota_k x = \sum_{j=1}^{k} x_j \cdot \psi_j. \tag{30}$$

Further, for the identity matrix $I \in \mathbb{C}^{p\times p}$, we identify $\iota_k : \mathbb{C}^{kp} \to L^2(0,\infty;\mathbb{C}^p)$ with the tensor product $\iota_k \otimes I$. We omit an additional subindex for sake of brevity.

Orthonormality of the Takenaka–Malmquist system implies that ι_k defines an isometric embedding. The orthogonal projector onto $\mathscr{K}_k(\alpha)\otimes\mathbb{C}^p$ is therefore given by

$$P_k = \iota_k \iota_k^* : L^2(0,\infty;\mathbb{C}^p) \to L^2(0,\infty;\mathbb{C}^p). \tag{31}$$

With operators Ψ and $\mathbb{F}$ as in (10) and (11), we define the matrices

$$F_k = \iota_k^* \mathbb{F} \iota_k \in \mathbb{C}^{kp\times km}, \tag{32}$$

$$S_k = \iota_k^* \Psi \in \mathbb{C}^{kp\times n}. \tag{33}$$

We have

$$P_k \Psi = \iota_k S_k, \qquad P_k \mathbb{F} = P_k \mathbb{F} P_k = \iota_k F_k \iota_k^*, \tag{34}$$

where the equality $P_k\mathbb{F} = P_k\mathbb{F}P_k$ follows by taking adjoints in $\mathbb{F}^*P_k = P_k\mathbb{F}^*P_k$ and the latter equality follows from Theorem 2.

Algorithm 1 from [10] provides a recursive method for the determination of S_k and F_k. The determination of S_k is based on the fact that the unnormalized Takenaka–Malmquist system system $(\phi_j)_{j=1}^{\infty}$ (28) fulfills

$$\begin{aligned}
&\Psi^*(\phi_1 v) = (\alpha_1 - A^*)^{-1} C^* v,\\
&\Psi^*(\phi_j v) = \Psi^*(\phi_{j-1} v) - (\alpha_j + \overline{\alpha_{j-1}})(\alpha_j - A^*)^{-1} \Psi^*(\phi_{j-1} v) \qquad \forall v \in \mathbb{C}^p,
\end{aligned}$$

see [10, Corollary 13]. The determination of F_k relies on the following consideration: Let $\Lambda : L^2(0,\infty;\mathbb{C}^n) \to L^2(0,\infty;\mathbb{C}^p)$ be the input-output map of the system (3) with $B = I$ and $D = 0$. Then $\mathbb{F} = \Lambda B + D$, where $B \in \mathbb{C}^{n\times m}$ and $D \in \mathbb{C}^{p\times m}$ are regarded as constant multiplication operators on $L^2(0,\infty;\mathbb{C}^m)$. Then Λ^* satisfies the recursion (here $(\varphi_j)_{j=1}^{\infty}$ is the convolution system from (29))

$$\begin{aligned}
&\Lambda^*(\varphi_1 v) = (\alpha_1 - A^*)^{-1} C^* v \varphi_1,\\
&\Lambda^*(\varphi_j v) = (\alpha_j - A^*)^{-1} C^* v \varphi_j + (\alpha_j - A^*)^{-1} \Lambda^*(\varphi_{j-1} v) \qquad \forall v \in \mathbb{C}^p,
\end{aligned}$$

see [10, Corollary 14]. A transition from the basis $(\varphi_1,\ldots,\varphi_k)$ to the basis $(\psi_1,\ldots,\psi_k)$ then gives rise to the construction of F_k. The precise construction is given in Algorithm 1 (we refer to [10] for further details).

Algorithm 1 ADI iteration for output and input-output maps.

Input: $A \in \mathbb{C}^{n\times n}$ a stable matrix, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{p\times n}$, $D \in \mathbb{C}^{p\times m}$ and shift parameters $\alpha_1,\ldots,\alpha_k \in \mathbb{C}$ with $\mathrm{Re}(\alpha_i) > 0$.
Output: $S_k = \iota_k^* \Psi \in \mathbb{C}^{kp\times n}$, $F_k = \iota_k^* \mathbb{F} \iota_k \in \mathbb{C}^{kp\times km}$

$V_1 = (\alpha_1 - A^*)^{-1} C^*$

$S_1 = \sqrt{2\mathrm{Re}(\alpha_1)} \cdot V_1^*$

$Q_1 = \sqrt{2\mathrm{Re}(\alpha_1)} \cdot V_1^* B$

$L_1 = \frac{1}{\sqrt{2\mathrm{Re}(\alpha_1)}}$

$F_1 = Q_1 L_1 + D$

for $i = 2,3,\ldots,k$ **do**

$\quad V_i = V_{i-1} - (\alpha_i + \overline{\alpha_{i-1}}) \cdot (\alpha_i - A^*)^{-1} V_{i-1}$

$\quad S_i = [S_{i-1}^*, \sqrt{2\mathrm{Re}(\alpha_i)} \cdot V_i]^*$

$\quad Q_i = [Q_{i-1}, \sqrt{2\mathrm{Re}(\alpha_i)} \cdot V_i^* B]$

$\quad \gamma_i = \sqrt{\frac{\mathrm{Re}(\alpha_j)}{\mathrm{Re}(\alpha_{j-1})}}$

$$M_{i,1} = \begin{bmatrix} \frac{1}{\sqrt{2\mathrm{Re}(\alpha_1)}} & & \\ & \ddots & \\ & & \frac{1}{\sqrt{2\mathrm{Re}(\alpha_i)}} \end{bmatrix}, \quad M_{i,2} = \begin{bmatrix} \overline{\alpha_1} + \alpha_i & & & \\ \alpha_1 - \alpha_i & \overline{\alpha_2} + \alpha_i & & \\ & & \ddots & \\ & & \alpha_{i-1} - \alpha_i & \overline{\alpha_i} + \alpha_i \end{bmatrix},$$

$$M_{i,3} = \begin{bmatrix} 1 & \ldots & 1 \\ & \ddots & \vdots \\ & & 1 \end{bmatrix}, \quad M_{i,4} = \begin{bmatrix} 0 & I \\ 1 & 0 \end{bmatrix}, \quad M_{i,5} = \begin{bmatrix} -\sqrt{2\mathrm{Re}(\alpha_1)} & & & \\ & \ddots & & \\ & & -\sqrt{2\mathrm{Re}(\alpha_{i-1})} & \\ & & & 1 \end{bmatrix}$$

$\quad M_i = M_{i,1}^{-1} M_{i,2}^{-1} M_{i,3}^{-1} M_{i,4}^{-1} M_{i,5}^{-1}$

$$L_i = \begin{bmatrix} \gamma_i L_{i-1} & 0 \\ 0 & 0 \end{bmatrix} - M_i \begin{bmatrix} L_{i-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_i(\alpha_i + \overline{\alpha_{i-1}}) I & 0 \\ [0,\gamma_i] & -1 \end{bmatrix}$$

$$F_i = \begin{bmatrix} [F_{i-1}, 0] \\ Q_i(\overline{L_i} \otimes I_m) + [0, D] \end{bmatrix}$$

end for

4 The projected optimal control problem

In this section we consider the optimal control problems (2) & (3) and (9) & (3), and their relations to the corresponding optimal control problems in which the output y is replaced by $P_k y$ with the orthogonal projector P_k as in (31).
We start with the bounded real case.

Theorem 3 *Assume that $A \in \mathbb{C}^{n\times n}$ is stable, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{p\times n}$ and $D \in \mathbb{C}^{p\times m}$. Further assume that the bounded real dissipation inequality*

$$\begin{bmatrix} A^*X + XA + C^*C & XB + C^*D \\ B^*X + D^*C & D^*D - I \end{bmatrix} \leq 0, \quad X = X^* \tag{35}$$

has a solution $X \in \mathbb{C}^{n\times n}$. Define Ψ and $\mathbb{F}$ by (10) *and* (11).
Let $(\alpha_j)_{j=1}^{\infty}$ be a complex sequence with $\mathrm{Re}(\alpha_j) > 0$ for all $j \in \mathbb{N}$, and let $F_k \in \mathbb{C}^{kp\times km}$, $S_k \in \mathbb{C}^{kp\times n}$ be defined as in (32) *and* (33).
Then the matrix $I - F_k^* F_k \in \mathbb{C}^{km\times km}$ is positive semi-definite. In particular, there exists some matrix $F_{\Xi,k} \in \mathbb{C}^{\ell_k \times km}$ with full row rank and

$$I - F_k^* F_k = F_{\Xi,k}^* F_{\Xi,k}. \tag{36}$$

Further, there exists some $S_{\Xi,k} \in \mathbb{C}^{\ell_k\times n}$ such that

$$F_{\Xi,k}^* S_{\Xi,k} = -F_k^* S_k. \tag{37}$$

For the orthogonal projector P_k as in (31)*, the matrix X_k defined by*

$$X_k = S_k^* S_k + S_{\Xi,k}^* S_{\Xi,k}, \tag{38}$$

fulfills

$$x_0^* X_k x_0 = \sup_{u\in L^2(0,\infty;\mathbb{C}^m)} \|P_k \mathbb{F} u + P_k \Psi x_0\|^2 - \|u\|^2. \tag{39}$$

Proof Since the dissipation inequality (35) has a solution and A is stable, we obtain by Remark 3 b) that the bounded real Lur'e equation has a minimal solution $X = X^* \in \mathbb{C}^{n\times n}$. Then Theorem 1 implies that the operator $I - \mathbb{F}^*\mathbb{F}$ is positive semi-definite. Since $P_k \leq I$ we have $\mathbb{F}^* P_k \mathbb{F} \leq \mathbb{F}^*\mathbb{F}$, which implies that $I - \mathbb{F}^* P_k \mathbb{F} \geq I - \mathbb{F}^*\mathbb{F}$. Since $I - \mathbb{F}^*\mathbb{F}$ is positive semi-definite it follows that $I - \mathbb{F}^* P_k \mathbb{F}$ is positive semi-definite. We have

$$I - F_k^* F_k = I - \iota_k^* \mathbb{F}^* \iota_k \iota_k^* \mathbb{F} \iota_k = \iota_k^* (I - \mathbb{F}^* P_k \mathbb{F}) \iota_k \geq 0,$$

so that $I - F_k^* F_k$ is positive semi-definite. Thus, there exists some $F_{\Xi,k} \in \mathbb{C}^{\ell_k\times km}$ with full row rank and satisfying (36).

We prove that $\mathrm{im}(F_k^* S_k) \subset \mathrm{im}(F_{\Xi,k})$. By taking orthogonal complements, this is equivalent to

$$\ker(F_{\Xi,k}) \subset \ker(S_k^* F_k).$$

Let $x_0 \in \mathbb{C}^n$ and $u \in L^2(0,\infty;\mathbb{C}^m)$. Then, by stability of A, the state $x(t)$ of the system (3) tends to zero, if t tends to infinity. Then (20) yields

$$x_0^* X x_0 \geq \|\mathbb{F} u + \Psi x_0\|^2 - \|u\|^2.$$

By further using (34) and (36), we see that

$$\begin{aligned}
x_0^* X x_0 &\geq \|\mathbb{F}u + \Psi x_0\|^2 - \|u\|^2 \\
&\geq \|P_k \mathbb{F}u + P_k \Psi x_0\|^2 - \|P_k u\|^2 - \|(I-P_k)u\|^2 \\
&= \|\iota_k F_k \iota_k^* u + \iota_k S_k x_0\|^2 - \|\iota_k^* u\|^2 - \|(I-P_k)u\|^2 \\
&= \|F_k \iota_k^* u + S_k x_0\|^2 - \|\iota_k^* u\|^2 - \|(I-P_k)u\|^2 \\
&= \langle \iota_k^* u, (F_k^* F_k - I)\iota_k^* u\rangle + 2\mathrm{Re}\langle \iota_k^* u, F_k^* S_k x_0\rangle + \|S_k x_0\|^2 - \|(I-P_k)u\|^2 \\
&= -\langle \iota_k^* u, F_{\Xi,k}^* F_{\Xi,k})\iota_k^* u\rangle + 2\mathrm{Re}\langle \iota_k^* u, F_k^* S_k x_0\rangle + \|S_k x_0\|^2 - \|(I-P_k)u\|^2 \\
&= -\|F_{\Xi,k}\iota_k^* u\|^2 + 2\mathrm{Re}\langle \iota_k^* u, F_k^* S_k x_0\rangle + \|S_k x_0\|^2 - \|(I-P_k)u\|^2.
\end{aligned} \tag{40}$$

Assume that

$$\ker F_{\Xi,k} \not\subset \ker S_k^* F_k.$$

Then there exists some $\hat{u} \in \mathbb{C}^{km}$ with $S_k^* F_k \hat{u} \neq 0$ and $F_{\Xi,k}\hat{u} = 0$, and thus we can choose some $x_0 \in \mathbb{C}^n$ such that $x_0^* S_k^* F_k \hat{u} \neq 0$. Then, for $\lambda \in \mathbb{C}$, substituting x_0 and $u := \iota_k(\lambda \hat{u}) \in L^2(0,\infty;\mathbb{C}^m)$ into (40), we obtain

$$\begin{aligned}
x_0^* X x_0 &\geq -\|F_{\Xi,k}\iota_k^* \iota_k(\lambda \hat{u})\|^2 + 2\mathrm{Re}\langle \iota_k^* \iota_k(\lambda \hat{u}), F_k^* S_k x_0\rangle + \|S_k x_0\|^2 - \|(I-P_k)\iota_k(\lambda \hat{u})\|^2 \\
&= -\|\lambda F_{\Xi,k}\hat{u}\|^2 + 2\mathrm{Re}(\lambda \langle \hat{u}, F_k^* S_k x_0\rangle) + \|S_k x_0\|^2 \\
&= 2\mathrm{Re}(\lambda \langle \hat{u}, F_k^* S_k x_0\rangle) + \|S_k x_0\|^2.
\end{aligned}$$

In particular, by an appropriate choice of $\lambda \in \mathbb{C}$, we can make the expression on the right hand side arbitrarily large, which leads to a contradiction. Hence $\ker(F_{\Xi,k}) \subset \ker(S_k^* F_k)$.

Since $F_{\Xi,k}$ has full row rank, $F_{\Xi,k} F_{\Xi,k}^*$ is invertible and therefore

$$S_{\Xi,k} := (F_{\Xi,k} F_{\Xi,k}^*)^{-1} F_{\Xi,k} F_k S_k \tag{41}$$

is well-defined. We now show that it satisfies (37). Let $x \in \mathbb{C}^n$. By the above established subspace inclusion $\mathrm{im}(F_k^* S_k) \subset \mathrm{im}(F_{\Xi,k}^*)$, there exists a $z \in \mathbb{C}^{km}$ such that $F_k^* S_k x = F_{\Xi,k}^* z$. Then

$$F_{\Xi,k}^* S_{\Xi,k} x = F_{\Xi,k}^* (F_{\Xi,k} F_{\Xi,k}^*)^{-1} F_{\Xi,k} F_k S_k x = F_{\Xi,k}^* (F_{\Xi,k} F_{\Xi,k}^*)^{-1} F_{\Xi,k} F_{\Xi,k}^* z = F_{\Xi,k}^* z = F_k^* S_k x.$$

Since $x \in \mathbb{C}^n$ was arbitrary this proves that $F_{\Xi,k}^* S_{\Xi,k} = F_k^* S_k$, i.e the above defined $S_{\Xi,k}$ satisfies (37).

It remains to prove that X_k as in (38) fulfills (39). Using (36) and (37), we have for all $x_0 \in \mathbb{C}^n$ and $u \in L^2(0,\infty;\mathbb{C}^m)$ that

$$\begin{aligned}
&\|P_k \mathbb{F}u + P_k \Psi x_0\|^2 - \|u\|^2 \\
&= -\langle \iota_k^* u, F_{\Xi,k}^* F_{\Xi,k} \iota_k^* u\rangle + 2\mathrm{Re}\langle \iota_k^* u, F_k^* S_k x_0\rangle + \|S_k x_0\|^2 - \|(I-P_k)u\|^2 \\
&= -\langle \iota_k^* u, F_{\Xi,k}^* F_{\Xi,k} \iota_k^* u\rangle - 2\mathrm{Re}\langle \iota_k^* u, F_{\Xi,k}^* S_{\Xi,k} x_0\rangle + \|S_k x_0\|^2 - \|(I-P_k)u\|^2 \\
&= -\|F_{\Xi,k}\iota_k^* u + S_{\Xi,k} x_0\|^2 + \|S_{\Xi,k} x_0\|^2 + \|S_k x_0\|^2 - \|(I-P_k)u\|^2 \\
&= -\|F_{\Xi,k}\iota_k^* u + S_{\Xi,k} x_0\|^2 - \|(I-P_k)u\|^2 + x_0^* X_k x_0 \\
&\leq x_0^* X_k x_0.
\end{aligned}$$

This gives rise to

$$x_0^* X_k x_0 \geq \sup_{u \in L^2(0,\infty;\mathbb{C}^m)} \|P_k \mathbb{F}u + P_k \Psi x_0\|^2 - \|u\|^2.$$

On the other hand, using the surjectivity of $F_{\Xi,k}$, there exists some $\hat{u} \in \mathbb{C}^{km}$ with $F_{\Xi,k}\hat{u} = -S_{\Xi,k}x_0$. Then, for $u = \iota_k \hat{u}$, we see that equality holds true in the above calculations. This proves (39).

Remark 6 (Bounded real Lur'e equations and projected optimal control problems)

a) Equation (36) can be regarded as a discrete version of the spectral factorization (23). The matrix $S_{\Xi,k}$ takes the role of the operator Ψ_Ξ in Theorem 1.
b) The formula (41) for $S_{\Xi,k}$ shows that X_k equals $S_k^*[I + F_k F_{\Xi,k}^*(F_{\Xi,k}F_{\Xi,k}^*)^{-2}F_{\Xi,k}F_k^*]S_k$. It is easily verified that $F_{\Xi,k}^*(F_{\Xi,k}F_{\Xi,k}^*)^{-2}F_{\Xi,k}$ is the Moore-Penrose pseudo-inverse of $F_{\Xi,k}^*F_{\Xi,k}$. Therefore, $X_k = S_k^*[I + F_k(I - F_k^*F_k)^+F_k^*]S_k$. This formula should be compared to the one given in Remark 4 b) for X.

Next we prove that the sequence (X_k) is monotonically increasing with respect to definiteness. We further present a criterion on the shift parameters such that convergence to the minimal solutions is achieved.

Theorem 4 *Assume that $A \in \mathbb{C}^{n\times n}$ is stable, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{p\times n}$ and $D \in \mathbb{C}^{p\times m}$. Further assume that the bounded real Lur'e equation* (1) *has a minimal solution $X \in \mathbb{C}^{n\times n}$. Define Ψ and $\mathbb{F}$ by* (10) *and* (11).
Let $(\alpha_j)_{j=1}^\infty$ be a complex sequence with $\mathrm{Re}(\alpha_j) > 0$ *for all $j \in \mathbb{N}$, and let $F_k \in \mathbb{C}^{kp\times km}$, $S_k \in \mathbb{C}^{kp\times n}$ be defined as in* (32) *and* (33)*; let X_k be defined as in Theorem 3.*
Then

$$X_k \le X_{k+1}, \quad X_k \le X \qquad \forall k \in \mathbb{N},$$

and the sequence (X_k) converges. If, additionally, $(\alpha_j)_{j=1}^\infty$ satisfies the non-Blaschke condition (7)*, then (X_k) converges to X.*

Proof For $x_0 \in \mathbb{C}^n$ and $u \in L^2(0,\infty;\mathbb{C}^m)$ we have

$$\|P_k\mathbb{F}u + P_k\Psi x_0\|_{L^2}^2 \le \|P_{k+1}\mathbb{F}u + P_{k+1}\Psi x_0\|_{L^2}^2,$$

since $\mathscr{K}_k(\alpha) \subset \mathscr{K}_{k+1}(\alpha)$. It follows that

$$\begin{aligned} x_0^*X_kx_0 &= \sup_{u\in L^2(0,\infty;\mathbb{C}^m)} \|P_k\mathbb{F}u + P_k\Psi x_0\|^2 - \|u\|^2 \\ &\le \sup_{u\in L^2(0,\infty;\mathbb{C}^m)} \|P_{k+1}\mathbb{F}u + P_{k+1}\Psi x_0\|^2 - \|u\|^2 = x_0^*X_{k+1}x_0. \end{aligned}$$

Similarly, using that

$$\|P_k\mathbb{F}u + P_k\Psi x_0\|_{L^2}^2 \le \|\mathbb{F}u + \Psi x_0\|_{L^2}^2,$$

we obtain

$$x_0^*X_kx_0 \le x_0^*Xx_0 \qquad \forall x_0 \in \mathbb{C}^n.$$

Convergence of the sequence (X_k) follows by the fact that it is non-decreasing and bounded from above by X with respect to definiteness.
In the case where the non-Blaschke condition (7) is fulfilled, the union of the spaces $\mathscr{K}_k(\alpha)$ over all $k \in \mathbb{N}$ is dense in $L^2(0,\infty;\mathbb{C}^m)$ [17]. The sequence (P_k) therefore converges to the identity in the strong operator topology, that is

$$\lim_{k\to\infty} P_k y = y \qquad \forall y \in L^2(0,\infty;\mathbb{C}^m). \tag{42}$$

Let $x_0 \in \mathbb{C}^n$ and $\varepsilon > 0$. By (26) there exists some $u \in L^2(0,\infty;\mathbb{C}^m)$ with

$$x_0^* X x_0 < \|\mathbb{F}u + \Psi x_0\|^2 - \|u\|^2 + \tfrac{\varepsilon}{2}.$$

By (42), there exists some $N \in \mathbb{N}$ with $\|(\mathbb{F}u + \Psi x_0) - P_k(\mathbb{F}u + \Psi x_0)\|^2 \leq \frac{\varepsilon}{2}$ for all $k \geq N$. Then we obtain that for all $k \geq N$ there holds

$$\begin{aligned} x_0^* X x_0 &< \|\mathbb{F}u + \Psi x_0\|^2 - \|u\|^2 + \tfrac{\varepsilon}{2} \\ &\leq \|P_k \mathbb{F}u + P_k \Psi x_0\|^2 + \|(\mathbb{F}u + \Psi x_0) - P_k(\mathbb{F}u + \Psi x_0)\|^2 - \|u\|^2 + \tfrac{\varepsilon}{2} \\ &\leq \|P_k \mathbb{F}u + P_k \Psi x_0\|^2 - \|u\|^2 + \varepsilon \leq x_0^* X_k x_0 + \varepsilon. \end{aligned}$$

Using further that $X_k \leq X$, we obtain

$$|x_0^*(X - X_k)x_0| = x_0^* X x_0 - x_0^* X_k x_0 < \varepsilon \qquad \forall k \geq N.$$

It follows that the sequence (X_k) converges to X.

Next we introduce a slighty different, numerically more advantageous, representation for the matrix X_k as in (38).

Theorem 5 *Assume that $A \in \mathbb{C}^{n\times n}$ is stable, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{p\times n}$ and $D \in \mathbb{C}^{p\times m}$. Further assume that the bounded real dissipation inequality* (35) *has a solution $X \in \mathbb{C}^{n\times n}$. Define Ψ and $\mathbb{F}$ by* (10) *and* (11).
Let $(\alpha_j)_{j=1}^{\infty}$ be a complex sequence with $\mathrm{Re}(\alpha_j) > 0$ for all $j \in \mathbb{N}$, and let $F_k \in \mathbb{C}^{kp\times km}$, $S_k \in \mathbb{C}^{kp\times n}$ be defined as in (32) *and* (33).
Then there exists some matrix $G_k \in \mathbb{C}^{\tilde{\ell}_k\times kp}$ with full row rank and

$$I - F_k F_k^* = G_k^* G_k. \tag{43}$$

Further, there exists some $R_k \in \mathbb{C}^{\tilde{\ell}_k\times n}$ such that

$$G_k^* R_k = S_k. \tag{44}$$

The matrix X_k as in (38) *fulfills*

$$X_k = R_k^* R_k. \tag{45}$$

Proof The matrix $I - F_k F_k^* \in \mathbb{C}^{kp\times kp}$ is positive semi-definite by Theorem 3. Therefore, $I - F_k^* F_k \in \mathbb{C}^{km\times km}$ is positive semidefinite as well. This implies the existence of some matrix $G_k \in \mathbb{C}^{\tilde{\ell}_k\times kp}$ with full row rank such that (43) holds.

By (36) we have $\ker(I - F_k^* F_k) = \ker(F_{\Xi,k})$. From (37) we obtain $\ker(F_{\Xi,k}) \subset \ker(S_k^* F_k)$, whence $\ker(I - F_k^* F_k) \subset \ker(S_k^* F_k)$.

We now prove $\mathrm{im}(S_k) \subset \mathrm{im}(I - F_k F_k^*)$. This is equivalent to $\ker(I - F_k F_k^*) \subset \ker(S_k^*)$. Let $y \in \ker(I - F_k F_k^*)$. Then $y = F_k F_k^* y$. Therefore

$$S_k^* y = S_k^* F_k F_k^* y \tag{46}$$

and $F_k^* y = F_k^* F_k F_k^* y$. The latter is equivalent to $(I - F_k^* F_k) F_k^* y = 0$. Thereby we obtain that $F_k^* y \in \ker(I - F_k^* F_k)$, which by the inclusion of nullspaces established in the previous paragraph gives $F_k^* y \in \ker(S_k^* F_k)$. Hence $S_k^* F_k F_k^* y = 0$. From (46) we then obtain $S_k^* y = 0$. We conclude that $\ker(I - F_k F_k^*) \subset \ker(S_k^*)$, as desired.

From (43) we obtain $\ker(I - F_kF_k^*) = \ker(G_k)$, so that $\mathrm{im}(I - F_kF_k^*) = \mathrm{im}(G_k^*)$. Together with the already established subspace inclusion $\mathrm{im}(S_k) \subset \mathrm{im}(I - F_kF_k^*)$, this shows that $\mathrm{im}(S_k) \subset \mathrm{im}(G_k^*)$. Since G_k has full row rank, $G_kG_k^*$ is invertible and therefore

$$R_k := (G_kG_k^*)^{-1}G_kS_k \tag{47}$$

is well-defined. We now show that it satisfies (44). Let $x \in \mathbb{C}^n$. By the above established subspace inclusion $\mathrm{im}(S_k) \subset \mathrm{im}(G_k^*)$, there exists a $z \in \mathbb{C}^{kp}$ such that $S_kx = G_k^*z$. Then

$$G_k^*R_kx = G_k^*(G_kG_k^*)^{-1}G_kS_kx = G_k^*(G_kG_k^*)^{-1}G_kG_k^*z = G_k^*z = S_kx.$$

Since $x \in \mathbb{C}^n$ was arbitrary this proves that $G_k^*R_k = S_k$, i.e the above defined R_k satisfies (44).

By Remark 6 b) we have $X_k = S_k^*[I + F_k(I - F_k^*F_k)^+F_k^*]S_k$. Using the above established subspace inclusion $\mathrm{im}(S_k) \subset \mathrm{im}(I - F_kF_k^*)$ and the fact that $(I - F_kF_k^*)^+(I - F_kF_k^*)$ is the orthogonal projection onto $\mathrm{im}(I - F_kF_k^*)$ we may alternatively write this as

$$X_k = S_k^*[(I - F_kF_k^*)^+(I - F_kF_k^*) + F_k(I - F_k^*F_k)^+F_k^*]S_k.$$

The following identity for Moore-Penrose pseudo-inverses is most easily proven by verifying the Moore-Penrose conditions [6, Sec. 5.5.4]:

$$(I - F_kF_k^*)^+ = (I - F_kF_k^*)^+(I - F_kF_k^*) + F_k(I - F_k^*F_k)^+F_k^*.$$

From this we see that

$$X_k = S_k^*(I - F_kF_k^*)^+S_k. \tag{48}$$

On the other hand we have, using (47),

$$R_k^*R_k = S_k^*G_k^*(G_kG_k^*)^{-2}G_kS_k,$$

and it is easily verified that $G_k^*(G_kG_k^*)^{-2}G_k$ is the Moore-Penrose pseudo-inverse of $G_k^*G_k$. Since $G_k^*G_k = I - F_kF_k^*$ by (43), it follows that $R_k^*R_k = X_k$.

Remark 7 (Bounded real Lur'e equations)

a) It follows from (32) that $\iota_k^*(I - \mathbb{F}_k\mathbb{F}_k^*)\iota_k = I - F_kF_k^*$.
b) The formula (48) for X_k should be compared to the equation for X in Remark 4 b), which in the bounded real case can be re-written as $X = \Psi^*(I - \mathbb{F}\mathbb{F}^*)^{-1}\Psi$.
c) Observing the lower triangular block structure of matrix F_i in Algorithm 1, that is

$$F_i = \begin{bmatrix} [F_{i-1}, 0] \\ Q_i(\overline{L_i} \otimes I_m) + \big[0, D\big] \end{bmatrix}, \tag{49}$$

we can determine the matrices $G_i \in \mathbb{C}^{\tilde{\ell}_i \times ip}$ and $R_i \in \mathbb{C}^{\tilde{\ell}_i \times n}$ recursively as follows: We have

$$I - F_iF_i^* \\ = \begin{bmatrix} I - F_{i-1}F_{i-1}^* & -\big[F_{i-1}\ 0\big]\big(Q_i(\overline{L_i} \otimes I_m)\big)^* \\ -\big(Q_i(\overline{L_i} \otimes I_m)\big)\big[F_{i-1}\ 0\big]^* & I - \big(Q_i(\overline{L_i} \otimes I_m) + \big[0, D\big]\big)\big(Q_i(\overline{L_i} \otimes I_m) + \big[0, D\big]\big)^* \end{bmatrix}.$$

By making the ansatz

$$G_i = \begin{bmatrix} G_{i-1} & G_{12,i} \\ 0 & G_{22,i} \end{bmatrix},$$

we obtain

$$\begin{bmatrix} G_{i-1}^* G_{i-1} & G_{i-1}^* G_{12,i} \\ G_{12,i}^* G_{i-1} & G_{12,i}^* G_{12,i} + G_{22,i}^* G_{22,i} \end{bmatrix}$$
$$= G_i^* G_i = I - F_i F_i^*$$
$$= \begin{bmatrix} I - F_{i-1} F_{i-1}^* & -\begin{bmatrix} F_{i-1} & 0 \end{bmatrix} \left(Q_i(\overline{L_i} \otimes I_m) \right)^* \\ -\left(Q_i(\overline{L_i} \otimes I_m) \right) \begin{bmatrix} F_{i-1} & 0 \end{bmatrix}^* & I - \left(Q_i(\overline{L_i} \otimes I_m) + \begin{bmatrix} 0, D \end{bmatrix} \right) \left(Q_i(\overline{L_i} \otimes I_m) + \begin{bmatrix} 0, D \end{bmatrix} \right)^* \end{bmatrix}.$$

Thus, the matrix $G_{12,i}$ is the unique solution of the linear equation

$$G_{i-1}^* G_{12,i} = -\begin{bmatrix} F_{i-1} & 0 \end{bmatrix} \left(Q_i(\overline{L_i} \otimes I_m) \right)^*.$$

Thereafter, the matrix $G_{22,i}$ can be obtained by a factorization

$$G_{22,i}^* G_{22,i} = I - \left(Q_i(\overline{L_i} \otimes I_m) + \begin{bmatrix} 0, D \end{bmatrix} \right) \left(Q_i(\overline{L_i} \otimes I_m) + \begin{bmatrix} 0, D \end{bmatrix} \right)^* - G_{12,i}^* G_{12,i}.$$

Since, by Algorithm 1, S_i is obtained from S_{i-1} by

$$S_i = \begin{bmatrix} S_{i-1} \\ \sqrt{2\mathrm{Re}(\alpha_i)} \cdot V_i^* \end{bmatrix}, \tag{50}$$

we can, by making the ansatz

$$R_i = \begin{bmatrix} R_{i-1} \\ R_{2,i} \end{bmatrix},$$

rewrite equation (44) as

$$\begin{bmatrix} G_{i-1}^* & 0 \\ G_{12,i}^* & G_{22,i}^* \end{bmatrix} \begin{bmatrix} R_{i-1} \\ R_{2,i} \end{bmatrix} = \begin{bmatrix} S_{i-1} \\ \sqrt{2\mathrm{Re}(\alpha_i)} \cdot V_i^* \end{bmatrix}.$$

Hence, $R_{2,i}$ is the solution of the linear equation

$$G_{22,i}^* R_{2,i} = \sqrt{2\mathrm{Re}(\alpha_i)} \cdot V_i^* - G_{12,i}^* R_{i-1}.$$

By Theorem 5 and Remark 7 c), we can set up the following algorithm for the determination of the minimal solution of bounded real Lur'e equations.

Algorithm 2 ADI iteration for the bounded real Lur'e equation.

Input: a stable matrix $A \in \mathbb{C}^{n\times n}$, and $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{p\times n}$, $D \in \mathbb{C}^{p\times m}$ such that the bounded real Lur'e equation (1) has the minimal solution $X \in \mathbb{C}^{n\times n}$, and shift parameters $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$ with $\mathrm{Re}(\alpha_i) > 0$.
Output: $R_k \in \mathbb{C}^{\tilde{l}_k \times n}$ such that $R_k^* R_k = X_k \approx X$.

1: Perform steps 1–5 in Algorithm 1
2: Determine a matrix G_1 with full row rank and $G_1^* G_1 = I - F_1 F_1^*$
3: Determine a matrix R_1 with $G_1^* R_1 = S_1$
4: **for** $i = 2, 3, \ldots, k$ **do**
5: Perform steps 7–14 in Algorithm 1.
6: Determine a matrix $G_{12,i}$ with $G_{i-1}^* G_{12,i} = -\begin{bmatrix} F_{i-1} & 0 \end{bmatrix} \left(Q_i(\overline{L_i} \otimes I_m) \right)^*$
7: Determine a matrix $G_{22,i}$ with full row rank and
$G_{22,i}^* G_{22,i} = I - \left(Q_i(\overline{L_i} \otimes I_m) + \begin{bmatrix} 0, D \end{bmatrix} \right) \left(Q_i(\overline{L_i} \otimes I_m) + \begin{bmatrix} 0, D \end{bmatrix} \right)^* - G_{12,i}^* G_{12,i}$
8: $G_i = \begin{bmatrix} G_{i-1} & G_{12,i} \\ 0 & G_{22,i} \end{bmatrix}$
9: Determine a matrix $R_{2,i}$ with $G_{22,i}^* R_{2,i} = \sqrt{2\mathrm{Re}(\alpha_i)} \cdot V_i^* - G_{12,i}^* R_{i-1}$
10: $R_i = \begin{bmatrix} R_{i-1} \\ R_{2,i} \end{bmatrix}$
11: **end for**

Remark 8 If $A \in \mathbb{C}^{n\times n}$ is stable, $B = 0 \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{p\times n}$ and $D = 0 \in \mathbb{C}^{p\times m}$, then the bounded real Lur'e equations reduce to the Lyapunov equation

$$A^*X + XA + C^*C = 0.$$

In this case, the matrices in Algorithm 2 read $F_i = 0$, $G_i = I$ and $S_i = R_i$. Then Algorithm 2 reduces to the well-known and established ADI iteration for Lyapunov equations [7, 9, 19].

Now we consider positive real Lur'e equations. First we present a version of Theorem 3 for positive real systems. The proof can be done by adapting the lines of the proof of Theorem 3.

Theorem 6 *Assume that $A \in \mathbb{C}^{n\times n}$ is stable, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{m\times n}$ and $D \in \mathbb{C}^{m\times m}$. Further assume that the positive real dissipation inequality*

$$\begin{bmatrix} A^*X + XA & XB - C^* \\ B^*X - C & -(D^* + D) \end{bmatrix} \leq 0, \quad X = X^* \tag{51}$$

has a solution $X \in \mathbb{C}^{n\times n}$.
Define Ψ and $\mathbb{F}$ by (10) *and* (11). *Let $(\alpha_j)_{j=1}^{\infty}$ be a complex sequence with* $\mathrm{Re}(\alpha_j) > 0$ *for all $j \in \mathbb{N}$, and let $F_k \in \mathbb{C}^{kp\times km}$, $S_k \in \mathbb{C}^{kp\times n}$ be defined as in* (32) *and* (33).
Then the matrix $F_k^* + F_k \in \mathbb{C}^{km\times km}$ is positive semi-definite. In particular, there exists some $F_{\Xi,k} \in \mathbb{C}^{\ell_k\times km}$ with full row rank and

$$F_k^* + F_k = F_{\Xi,k}^* F_{\Xi,k}. \tag{52}$$

Further, there exists some $S_{\Xi,k} \in \mathbb{C}^{\ell_k\times n}$ such that

$$F_{\Xi,k}^* S_{\Xi,k} = S_k. \tag{53}$$

For the orthogonal projector P_k as in (31), *the matrix X_k defined by*

$$X_k = S_{\Xi,k}^* S_{\Xi,k}. \tag{54}$$

fulfills,

$$x_0^* X_k x_0 = \sup_{u\in L^2(0,\infty;\mathbb{C}^m)} -2\mathrm{Re}\langle u, P_k\mathbb{F}u + P_k\Psi x_0\rangle. \tag{55}$$

Remark 9 Using (34) and the self-adjointness of P_k, we obtain from (34) and (55) that

$$\begin{aligned} x_0^* X_k x_0 &= \sup_{u\in L^2(0,\infty;\mathbb{C}^m)} -2\mathrm{Re}\langle u, P_k\mathbb{F}u + P_k\Psi x_0\rangle \\ &= \sup_{u\in L^2(0,\infty;\mathbb{C}^m)} -2\mathrm{Re}\langle u, P_k\mathbb{F}P_k u + P_k\Psi x_0\rangle \\ &= \sup_{u\in L^2(0,\infty;\mathbb{C}^m)} -2\mathrm{Re}\langle P_k u, \mathbb{F}P_k u + \Psi x_0\rangle \\ &= \sup_{u\in \mathscr{K}_k(\alpha)\otimes\mathbb{C}^m} -2\mathrm{Re}\langle u, \mathbb{F}u + \Psi x_0\rangle. \end{aligned}$$

Again, we can formulate a convergence result. The proof is analogous to that of Theorem 4 and therefore omitted.

Theorem 7 *Assume that $A \in \mathbb{C}^{n\times n}$ is stable, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{m\times n}$ and $D \in \mathbb{C}^{m\times m}$. Further assume that the positive real Lur'e equation* (8) *has a minimal solution $X \in \mathbb{C}^{n\times n}$. Define Ψ and $\mathbb{F}$ by* (10) *and* (11).
Let $(\alpha_j)_{j=1}^{\infty}$ be a complex sequence with $\mathrm{Re}(\alpha_j) > 0$ for all $j \in \mathbb{N}$, and let $F_k \in \mathbb{C}^{kp\times km}$, $S_k \in \mathbb{C}^{kp\times n}$ be defined as in (32) *and* (33)*; let X_k be defined as in Theorem 6.*
Then

$$X_k \le X_{k+1}, \quad X_k \le X \qquad \forall k \in \mathbb{N},$$

and the sequence (X_k) converges. If, additionally, $(\alpha_j)_{j=1}^{\infty}$ satisfies the non-Blaschke condition (7)*, then (X_k) converges to X.*

Remark 10 (Positive real Lur'e equations and projected optimal control problems)

a) If the Popov operator $\mathbb{F}^* + \mathbb{F}$ is positive definite and boundedly invertible, then the matrix $F_k^* + F_k$ is positive definite. In this case, the matrix X_k fulfills

$$X_k = S_k^*(F_k^* + F_k)^{-1} S_k,$$

cf. Remark 4 b).

b) In the following we show that, by using the fact that the matrix F_i has the lower triangular block structure as in (49), the matrices $F_{\Xi,i} \in \mathbb{C}^{\ell_i\times im}$ and $S_{\Xi,i} \in \mathbb{C}^{\ell_i\times n}$ can be recursively determined (cf. Remark 7 c):
We have

$$F_i + F_i^* = \begin{bmatrix} F_{i-1} + F_{i-1}^* & \begin{bmatrix} I_{(i-1)m} & 0 \end{bmatrix} (Q_i(\overline{L_i} \otimes I_m))^* \\ (Q_i(\overline{L_i} \otimes I_m)) \begin{bmatrix} I_{(i-1)m} \\ 0 \end{bmatrix} & D + D^* + \begin{bmatrix} 0 & I_m \end{bmatrix} (Q_i(\overline{L_i} \otimes I_m))^* + (Q_i(\overline{L_i} \otimes I_m)) \begin{bmatrix} 0 \\ I_m \end{bmatrix} \end{bmatrix}.$$

By making the ansatz

$$F_{\Xi,i} = \begin{bmatrix} F_{\Xi,i-1} & F_{\Xi 12,i} \\ 0 & F_{\Xi 22,i} \end{bmatrix},$$

we obtain

$$\begin{aligned} &\begin{bmatrix} F_{\Xi,i-1}^* F_{\Xi,i-1} & F_{\Xi,i-1}^* F_{\Xi 12,i} \\ F_{\Xi 12,i}^* F_{\Xi,i-1} & F_{\Xi 12,i}^* F_{\Xi 12,i} + F_{\Xi 22,i}^* F_{\Xi 22,i} \end{bmatrix} \\ &= F_{\Xi,i}^* F_{\Xi,i} = F_i + F_i^* \\ &= \begin{bmatrix} F_{i-1} + F_{i-1}^* & \begin{bmatrix} I_{(i-1)m} & 0 \end{bmatrix} (Q_i(\overline{L_i} \otimes I_m))^* \\ (Q_i(\overline{L_i} \otimes I_m)) \begin{bmatrix} I_{(i-1)m} \\ 0 \end{bmatrix} & D + D^* + \begin{bmatrix} 0 & I_m \end{bmatrix} (Q_i(\overline{L_i} \otimes I_m))^* + (Q_i(\overline{L_i} \otimes I_m)) \begin{bmatrix} 0 \\ I_m \end{bmatrix} \end{bmatrix}. \end{aligned}$$

Thus, the matrix $F_{\Xi 12,i}$ is the unique solution of the linear equation

$$F_{\Xi,i-1}^* F_{\Xi 12,i} = \begin{bmatrix} I_{(i-1)m} & 0 \end{bmatrix} (Q_i(\overline{L_i} \otimes I_m))^*.$$

Thereafter, the matrix $F_{\Xi 22,i}$ can be obtained by a factorization

$$F_{\Xi 22,i}^* F_{\Xi 22,i} = D + D^* + \begin{bmatrix} 0 & I_m \end{bmatrix} (Q_i(\overline{L_i} \otimes I_m))^* + (Q_i(\overline{L_i} \otimes I_m)) \begin{bmatrix} 0 \\ I_m \end{bmatrix} - F_{\Xi 12,i}^* F_{\Xi 12,i}.$$

Since, by Algorithm 1, the matrices S_i and S_{i-1} are related by (50) we see, by making the ansatz

$$S_{\Xi,i} = \begin{bmatrix} S_{\Xi,i-1} \\ S_{\Xi 2,i} \end{bmatrix},$$

that equation (53) now reads

$$\begin{bmatrix} F_{\Xi,i-1}^* & 0 \\ F_{\Xi 12,i}^* & F_{\Xi 22,i}^* \end{bmatrix} \begin{bmatrix} S_{\Xi,i-1} \\ S_{\Xi 2,i} \end{bmatrix} = \begin{bmatrix} S_{i-1} \\ \sqrt{2\mathrm{Re}(\alpha_i)} \cdot V_i^* \end{bmatrix}.$$

Hence, $S_{\Xi 2,i}$ is the solution of the linear equation

$$F_{\Xi 22,i}^* S_{\Xi 2,i} = \sqrt{2\mathrm{Re}(\alpha_i)} \cdot V_i^* - F_{\Xi 12,i}^* S_{\Xi,i-1}.$$

Algorithm 3 ADI iteration for the positive real Lur'e equation.

Input: $A \in \mathbb{C}^{n\times n}$ a stable matrix, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{p\times n}$, $D \in \mathbb{C}^{p\times m}$ such that the positive real Lur'e equation (8) has the minimal solution $X \in \mathbb{C}^{n\times n}$, and shift parameters $\alpha_1,\ldots,\alpha_k \in \mathbb{C}$ with $\mathrm{Re}(\alpha_i) > 0$.
Output: $S_{\Xi,k} \in \mathbb{C}^{\ell_k \times n}$ such that $S_{\Xi,k}^* S_{\Xi,k} = X_k \approx X$.

```
Perform steps 1–5 in Algorithm 1
Determine a matrix F_{Ξ,1} with full row rank and F*_{Ξ,1} F_{Ξ,1} = F_1 + F*_1
Determine a matrix S_{Ξ,1} with F*_{Ξ,1} S_{Ξ,1} = S_1
for i = 2,3,...,k do
    Perform steps 7–14 in Algorithm 1.
    Determine a matrix F_{Ξ12,i} with F*_{Ξ,i-1} F_{Ξ12,i} = [I_{(i-1)m} 0] (Q_i (conj(L_i) ⊗ I_m))*
    Determine a matrix F_{Ξ22,i} with full row rank and
      F*_{Ξ22,i} F_{Ξ22,i} = D + D* + [0 I_m] (Q_i (conj(L_i) ⊗ I_m))* + (Q_i (conj(L_i) ⊗ I_m)) [0; I_m] − F*_{Ξ12,i} F_{Ξ12,i}
    F_{Ξ,i} = [F_{Ξ,i-1} F_{Ξ12,i}; 0 F_{Ξ22,i}]
    Determine a matrix S_{Ξ2,i} with F*_{Ξ22,i} S_{Ξ2,i} = sqrt(2Re(α_i)) · V_i − F*_{Ξ12,i} S_{Ξ,i-1}
    S_{Ξ,i} = [S_{Ξ,i-1}; S_{Ξ2,i}]
end for
```

Remark 11 We note that Algorithm 2 reduces to well-known ADI iteration for Lyapunov equations [7, 9, 19] (cf. Remark 8): If $A \in \mathbb{C}^{n\times n}$ is stable, $B = 0 \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{m\times n}$ and $D = \frac{1}{2} I_m \in \mathbb{C}^{m\times m}$, then the positive real Lur'e equation reduces to the Lyapunov equation

$$A^* X + XA + C^* C = 0.$$

The matrices in Algorithm 3 then read $F_i = 0$, $F_{\Xi,i} = \frac{1}{2} I$ and $S_{\Xi,i} = S_i$, whence Algorithm 2 then again reduces to ADI iteration for Lyapunov equations.

5 Numerical Example

We present a numerical example to show the applicability of our algorithm and to demonstrate the expected performance of the ADI iteration for the positive real Lur'e equation in terms of monotonicity and convergence behavior.

We consider a convection-diffusion equation on the unit square $\Omega := [0,1] \times [0,1]$, namely

$$\frac{\partial x}{\partial t}(\xi,t) = k\Delta x(\xi,t) + b^\top \nabla x(\xi,t), \quad (\xi,t) \in \Omega \times \mathbb{R}_{\geq 0}. \tag{56}$$

The input is a scalar function formed by the Robin boundary condition

$$u(t) = \nu(\xi)^\top \nabla x(\xi,t) + \alpha x(\xi,t), \quad (\xi,t) \in \partial\Omega \times \mathbb{R}_{\geq 0},$$

and the output consists of the integral of Dirichlet boundary values, i.e.

$$y(t) = \int_{\partial\Omega} x(\xi,t)\, d\sigma_\xi,$$

where $\partial\Omega$ denotes the boundary of Ω, σ_ξ denotes the surface measure, and $\nu(\xi)$ denotes the outward unit normal.

We consider $b = \left[\begin{smallmatrix}10\\10\end{smallmatrix}\right]$ and set $k = \alpha = 1$. To discretize the PDE (56), we apply a finite element discretization with uniform triangular elements of fixed size $h = \frac{1}{N-1}$, where $N \in \mathbb{N}$ is the number of points in each coordinate direction. In addition, we define the subspace $V_h \subset H^1(\Omega)$ using piecewise-linear basis functions. As a result, we obtain a finite dimensional dynamical system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) \end{aligned} \tag{57}$$

with state space dimension $n = N^2$. $E \in \mathbb{R}^{n,n}$ is a symmetric positive definite mass matrix, $A \in \mathbb{R}^{n,n}$ is a non-symmetric stiffness matrix, $B \in \mathbb{R}^{n,1}$ is the input matrix, and $C \in \mathbb{R}^{1,n}$ is the output matrix.

The system is asymptotically stable and the matrix $A + A^*$ is negative definite. Furthermore, we have $B = C^*$. A simple calculation then shows that the system is passive. The positive real Popov operator has no bounded inverse, since the positive real Popov function $\Pi(\imath\omega) = G(\imath\omega) + G(\imath\omega)^*$ (with $G(s) = C(sE-A)^{-1}B = C(sI - E^{-1}A)^{-1}E^{-1}B$) vanishes at infinity.

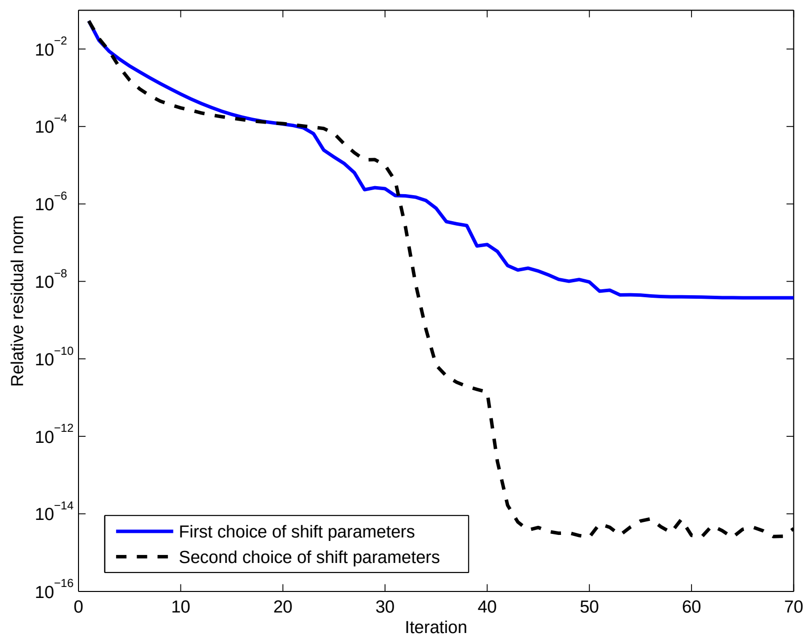

Fig. 1 Comparison of different shift parameters for ADI iteration: convection-diffusion equation with the space dimension $n = 4900$

We find an approximate solution $X \in \mathbb{C}^{n\times n}$ of the positive real Lur'e equation (8) by applying Algorithm 3. Thereby, we use the modifications proposed in [10, Remark 7.1] & [8, Remark 3.3] which allow computations without explicit inversion of E. In addition, in steps 6 and 7 of Algorithm 3, we do not need to compute the expression $Q_i(\overline{L_i} \otimes I_m)$, because we compute it once in step 14 of Algorithm 1. In fact, we need to just access the last p rows of the matrix F_i in order to obtain the value of this expression (cf. Remark 10 b)).

The choice of shift parameters has a major effect on the convergence speed of the ADI algorithm. In our example, we choose the following two different sets of shift parameters.

1. As a first set of shift parameters, we generate 30 parameters using the Wachspress method [19] on the basis of 4900 eigenvalues of the Dirichlet Laplacian given by $\pi^2(i^2 + j^2)$, $i, j = 1, 2, \ldots, 70$. We use the obtained shift parameters in the first 30 iterations. Afterwards, we select a subset of these parameters which provided the highest reduction in the value of residual norm. In our case, we chose 13 shift parameters and re-use them every 13 iterations.
2. The second set of shift parameters is motivated by the statements in Remark 4 e). Specifically, we generate a set of 30 shift parameters using Penzl's heuristic procedure [13] on negatives of the stable eigenvalues of the even matrix pencil

$$\lambda\mathscr{E} - \mathscr{A} = \begin{bmatrix} 0 & -\lambda E + A & B \\ \lambda E + A^* & 0 & -C^* \\ B^* & -C & 0 \end{bmatrix}.$$

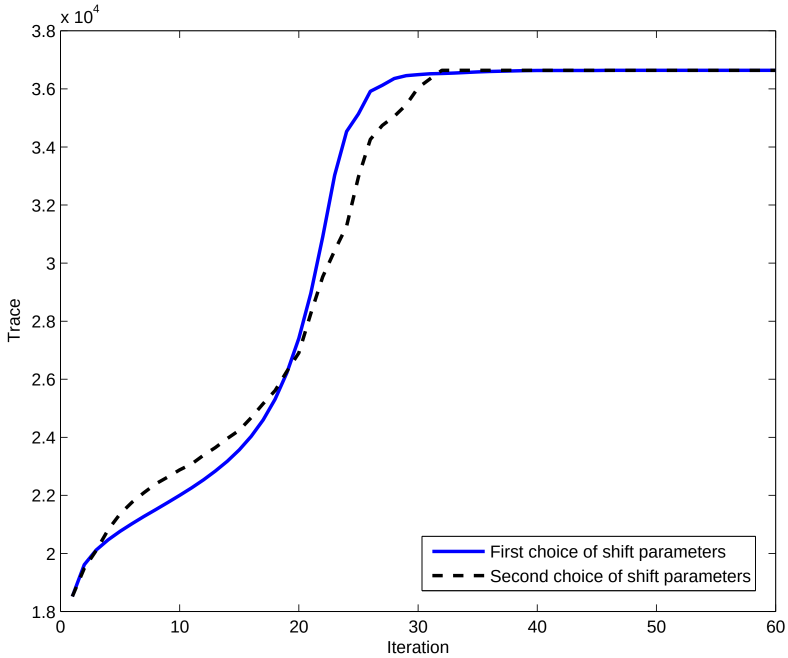

Fig. 2 Monotonicity of ADI iteration: convection-diffusion equation with the space dimension $n = 4900$

We sort the obtained 30 parameters in a decreasing order with respect to the values of their real part in order to obtain a smooth convergence. We perform 30 iterations of Algorithm 3 using these shift parameters. Subsequently, we extract a subset of these parameters which provided the highest reduction in the value of residual norm. From this set of shift parameters, we extract 8 parameters to re-use every 8 iterations.

We add a large real shift parameter of order 10^{12} to the above two sets of shift parameters and consider it to be the first parameter in the set. We use this large shift parameter just in the first iteration of Algorithm 3 and do not repeat it in the further iterations. The reason for adding a very big shift parameter can be explained as follows. Since in the positive real case the Popov function has a zero at infinity, a delta impulse will occur in the optimal control. The Takenaka-Malmquist basis function corresponding to a big shift parameter should suitably approximate the behavior of this delta impulse.

We have performed the calculations with several state space dimensions using MATLAB 7.10.0 (R2010a). At each iteration k, we observe the relative residual norm of the positive real Lur'e equation using the approach proposed in [14, Sec. 6]. Figure 1 shows the relative residual norm with respect to the iteration for the space dimension $n = 4900$ and for the two different choices of shift parameters which we have introduced earlier. We can conclude from this figure that the second set of shift parameters provide a faster convergence behavior to the solution of positive real Lur'e equation corresponding to the system (57). In fact, with a tolerance of 10^{-14} on the relative residual norm for the problem with the space dimension $n = 4900$, the second choice of shift parameters leads to convergence in 41 iterations whereas the first set of parameters requires more than 70 iterations for the desired convergence.

In order to illustrate the monotonicity of the ADI iteration, we observe the trace of X_k, denoted by $\operatorname{trace}(X_k)$, at each iteration of Algorithm 3. The trace of X_k can be computed efficiently as

$$\operatorname{trace}(X_k) = \operatorname{trace}\left(S_{\Xi,k}^{*} S_{\Xi,k}\right) = \|S_{\Xi,k}\|_F^2 ,$$

where $\|\cdot\|_F$ denotes the Frobenius norm. Figure 2 shows the trace of solutions X_k generated by Algorithm 3 with the two sets of shift parameters introduced earlier in this example. From this figure we observe that $tr(X_k) \leq tr(X_{k+1})$, for all $k \in \mathbb{N}$, which is consistent with Theorem 7.

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