Passive Infinite-Dimensional Descriptor Systems

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1 Introduction

We consider infinite-dimensional descriptor systems

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where $E : X \to Z$, $A : D(A) \subset X \to Z$, $B : U \to Z$, $C : X \to Y$, $D : U \to Y$ are linear operators acting on complex Hilbert spaces U, X, Y and Z. This type for instance arises when input-output systems are considered whose internal behavior is modeled by coupled partial differential and differential-algebraic equations. Although infinite-dimensional descriptor systems appear in many applications, research in this field is a very young area, especially inside linear systems theory [15–18]. Practical examples for infinite-dimensional descriptor systems are electrical circuits with spatially distributed components [15] or heat exchanger models [12].
In this work, the concept of input-output-passivity (io-passivity) is considered for systems of type (1). Io-passivity means that $U = Y$ and that the real part of the Lebesgue inner product of input and output of the trivially initialized system is always non-negative, i.e. for all $T > 0$ and all $u(\cdot) \in L_2([0, T], U)$, such that (1) with $x_0 = 0$ has a solution $x(\cdot) : [0, T] \to X$, the output satisfies

$$\operatorname{Re} \int_0^T \langle y(\tau), u(\tau) \rangle_U \, d\tau \geq 0. \tag{2}$$

The property of io-passivity has been studied in various works (see e.g. [22,23]) and is for instance very important in the stability analysis of switched systems [14] and in the field of synthesis of electrical circuits [2]. The expression on the left hand side usually has the physical interpretation of energy that is lost by the system. Io-passivity therefore means that the system cannot generate energy.
For finite-dimensional state-space systems (i.e., $E = I$, $A \in \mathbb{C}^{n,n}$, $B, C^* \in \mathbb{C}^{n,p}$, $D \in \mathbb{C}^{p,p}$), io-passivity is equivalently characterized via the positivity of its transfer function. That is, the rational $\mathbb{C}^{p,p}$-valued function $\boldsymbol{G}(s) = D + C(sI - A)^{-1}B$ has no pole on the right complex half-plane $\mathbb{C}^+ := \{s \in \mathbb{C}^+ : \operatorname{Re}(s) > 0\}$ and additionally, $\boldsymbol{G}(s) + \boldsymbol{G}^*(s)$ is positive semi-definite ($\boldsymbol{G}(s) + \boldsymbol{G}^*(s) \geq 0$) for all $s \in \mathbb{C}^+$.

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A further criterion for io-passivity of finite-dimensional state-space systems is the existence of $H = H^* \in \mathbb{C}^{n,n}$ with $H \geq 0$ and

$$\begin{bmatrix} -A^*H - HA & C^* - HB \\ C - B^*H & D + D^* \end{bmatrix} \geq 0. \tag{3}$$

The linear matrix inequality (3) is called Kalman-Yakubovich-Popov (KYP) inequality. The solvability of the KYP inequality is sufficient for io-passivity. If the system is controllable and observable, then io-passivity implies the solvability of the KYP inequality for some $H \in \mathbb{C}^{n,n}$ with $H \geq 0$. The real version of this criterion is known as the *positive real lemma* [1,2].
In this work we review known generalizations for finite-dimensional descriptor systems and infinite-dimensional state-space systems. We will furthermore consider infinite-dimensional descriptor systems and present approaches for the characterization of io-passivity. The presented results are illustrated by means of an example from electrical circuit theory.

2 Infinite-Dimensional State-Space Systems

Characterizations of io-passivity for infinite-dimensional state-space systems are considered in [7,8]. The most general results are in the works [3, 19, 20] by Staffans et al. where *linear system nodes* are considered. For sake of simplicity, we review the results of [3] for the slightly smaller class of *well-posed linear systems*, which are of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \\ x(0) &= x_0 \end{aligned} \tag{4}$$

and satisfy the following assumptions:

- $A : D(A) \subset X \to X$ is a generator of a strongly continuous semigroup on X.
- For $D(A^*)'$ being the dual of $D(A^*)$ equipped with the graph norm of A^* (the dual of X is identified with X itself), $B : U \to D(A^*)'$ is linear and bounded.
- $C : D(A) \to U$ and $D : U \to U$ are linear and bounded.
- For all $T > 0$ there exists some $c_T > 0$ such that the solutions of (4) satisfy

$$\|x(t)\| + \|y(\cdot)\|_{L_2([0,T],U)} \leq c_t \cdot (\|x_0\| + \|u(\cdot)\|_{L_2([0,T],U)}). \tag{5}$$

First we consider a frequency domain characterization for io-passivity of (4), i.e. we consider the transfer function $\boldsymbol{G}(s) = D + C(sI - A)^{-1}B$. Note that the well-posedness of the system implies that $\boldsymbol{G}$ is bounded and analytic on some half-plane $\mathbb{C}_\omega^+ := \{s \in \mathbb{C} :\ \mathrm{Re}(s) > \omega\}$. The following result can be obtained from [3, 19, 20].

Theorem 1. *We assume that system (4) is wellposed. Then the following statements are equivalent:*

1. *For all $T > 0$ and $u(\cdot) \in L_2([0,T], U)$, the output $y(\cdot)$ of (4) with $x_0 = 0$ satisfies*

$$\mathrm{Re}\langle u(\cdot), y(\cdot)\rangle_{L_2([0,T],U)} \geq 0.$$

2. *The transfer function $\boldsymbol{G}(s)$ can be analytically extended to $\mathbb{C}^+$ and satisfies $\boldsymbol{G}(s) + \boldsymbol{G}^*(s) \geq 0$ for all $s \in \mathbb{C}^+$.*

Another characterization is the generalization of the Kalman-Yakubovich-Popov inequality (3) to the infinite-dimensional case. The following criterion is formulated in [3]. Note that the operator A has an extension to a bounded operator mapping from X to $D(A^*)' \supset X$. Therefore, for a given $u_0 \in U$, the set of $x_0 \in X$ with $Ax_0 + Bu_0 \in X$ is an affine linear space. We use the following notation: $H^k([0,T],U)$ is the Sobolev space of measurable functions $f(\cdot)$ whose first k distributional derivatives are square integrable. Furthermore, the dual of a space $\widetilde{X}$ is denoted by $\widetilde{X}'$. The evaluation of $x' \in \widetilde{X}'$ at $x \in \widetilde{X}$ is denoted by $\langle x', x\rangle_{\widetilde{X}',\widetilde{X}}$.

Criterion 1 (Passive infinite-dimensional state-space systems).
There exists some Hilbert space $\widetilde{X} \subset X$ and a bounded operator $H : \widetilde{X} \to \widetilde{X}'$ with the following properties:

1. *$\langle Hx, x\rangle_{\widetilde{X}',\widetilde{X}}$ is real and non-negative for all $x \in \widetilde{X}$.*

2. *For all $T > 0$ and $u(\cdot) \in H^2([0,T])$ with $u(0) = 0$, the solution $x(\cdot) : [0,T] \to X$ of (4) with $x_0 = 0$ is continuously differentiable with $x(t), \dot{x}(t) \in D(\widetilde{X})$ for all $t \in [0,T]$.*

3. *For all $u_0 \in U$, $x_0 \in X$ such that $Ax_0 + Bu_0 \in D(\widetilde{X})$ holds*

$$-\operatorname{Re}\langle Hx_0, Ax_0 + Bu_0\rangle_{\widetilde{X}',\widetilde{X}} + \operatorname{Re}\langle Cx_0, u_0\rangle_U + \operatorname{Re}\langle u_0, Du_0\rangle_U \geq 0. \tag{6}$$

Note that the above criterion slightly differs from that formulated in [3]. However, Criterion 1 is equivalent to the formulation in [3].
The sufficiency of Criterion 1 for io-passivity follows from the fact that for a solution of (4) described in Part 2, Part 1 and 3 imply that

$$0 \leq \langle Hx(T), x(T)\rangle_{\widetilde{X}',\widetilde{X}} \leq \operatorname{Re}\langle u(\cdot), y(\cdot)\rangle_{L_2([0,T],U)}. \tag{7}$$

It is shown in [3] that for systems which are both approximately controllable and approximately observable, the validity of Criterion 1 is also necessary for io-passivity.

3 Finite-Dimensional Descriptor Systems

In this section we review some results for descriptor systems (1) with $B, C^* \in \mathbb{C}^{n,p}$ and $E, A \in \mathbb{C}^{n,n}$ such that $\det(sE - A)$ does not vanish identically. The characterization of io-passivity for finite-dimensional descriptor systems is considered in from [4,5,10,21]. Note that these works treat real finite-dimensional descriptor systems. However, the extension of these results to the complex case is straightforward.
The output of descriptor systems may contain derivatives of the input. In frequency domain, this corresponds to the fact that the transfer function $\boldsymbol{G}(s) = D + C(sE - A)^{-1}B$ may have a pole at infinity. The finite-dimensionality implies that the transfer function is rational. Therefore, there exists a representation $\boldsymbol{G}(s) = \boldsymbol{G}_p(s) + \sum_{k=1}^{\nu} s^k M_k$ for some $M_1, \ldots, M_\nu \in \mathbb{C}^{p,p}$ and a rational function G_p which is bounded in some right half plane $\mathbb{C}^+_\omega$. As in the case of state-space systems, io-passivity can be characterized via the transfer function. The following result is presented in [2,4,5,10,21].

Theorem 2. *Let a descriptor system (1) be given with $B, C^* \in \mathbb{C}^{n,p}$ and $E, A \in \mathbb{C}^{n,n}$ such that $\det(sE - A)$ does not vanish identically. Then the following statements are equivalent:*

1. *(1) is io-passive, i.e. for all $T > 0$ and $u(\cdot) : [0,T] \to \mathbb{R}^p$ such that (1) with $x_0 = 0$ has a solution, the output of the system with $x_0 = 0$ satisfies (2).*

2. *The transfer function $\boldsymbol{G}(s) = D + C(sE - A)^{-1}B$ has no poles in $\mathbb{C}^+$. Moreover, for all $s \in \mathbb{C}^+$ we have $\boldsymbol{G}(s) + \boldsymbol{G}^*(s) \geq 0$.*

3. *The transfer function* $\boldsymbol{G}(s) = D + C(sE - A)^{-1}B$ *has a representation* $\boldsymbol{G}(s) = \boldsymbol{G}_p(s) + sM_1$, *where* $M_1 = M_1^* \geq 0$ *and* $\boldsymbol{G}_p$ *is bounded on* $\mathbb{C}_\omega^+$ *for some* $\omega > 0$. *Moreover, for all* $s \in \mathbb{C}^+$ *we have* $\boldsymbol{G}(s) + \boldsymbol{G}^*(s) \geq 0$.

In particular, Part 3 implies that $\boldsymbol{G}(s)$ has at most linear growth in $\mathbb{C}^+$. In time domain, this consequences that for all $T > 0$, there exists some constant $c_T > 0$ such that for all $u(\cdot) \in H^1([0,T],\mathbb{R}^p)$ with $u(0) = 0$ we have

$$\|y(\cdot)\|_{L_2([0,T],\mathbb{R}^p)} \leq \|u(\cdot)\|_{H^1([0,T],\mathbb{R}^p)}. \tag{8}$$

In the following we present a criterion from [5] for io-passivity that generalizes the KYP inequality (3).

Criterion 2 (Passive finite-dimensional descriptor systems).
There exist matrices $H \in \mathbb{C}^{n,n}$, $K \in \mathbb{C}^{n,p}$ *with the following properties:*

(i) $E^*X = X^*E \geq 0$ *and* $E^*K = 0$.

(ii) $\begin{bmatrix} -A^*X - X^*A & C^* - A^*K - X^*B \\ C - K^*A - B^*X & -K^*B - B^*K + D + D^* \end{bmatrix} \geq 0.$

The claim (ii) implies that for all $u_0 \in \mathbb{R}^p$, $x_0 \in \mathbb{R}^n$ with $Ax_0 + Bu_0 \in \operatorname{im} E$ we have

$$\begin{bmatrix} x_0^* & u_0^* \end{bmatrix} \begin{bmatrix} -A^*X - X^*A & C^* - X^*B \\ C - B^*X & D + D^* \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \geq 0. \tag{9}$$

By using the fact that for all $t \in [0,T]$ the solutions of the descriptor system satisfy $Ax_0 + Bu_0 \in \operatorname{im} E$ (and thus $K^*Ax(t) + K^*Bu(t) = 0$), the sufficiency of Criterion 2 for passivity can be shown analogous to the corresponding result for infinite-dimensional state-space systems.
Furthermore it is shown in [5] that Criterion 2 is necessary for io-passivity, if the descriptor system is controllable and observable.

4 Infinite-Dimensional Descriptor Systems

The aim is to find a characterization for passivity that generalizes the so far presented criteria for the case of infinite-dimensional state-space systems as well as for finite-dimensional descriptor systems. The general assumptions on the descriptor system are

a) U, X, Y and Z are Hilbert spaces and $E : X \to Z$, $B : U \to Z$, $C : X \to Y$ and $D : U \to Y$ are linear and bounded;

b) $A : D(A) \subset X \to Z$ is closed and there exists some complex half-plane $\mathbb{C}_\omega^+$ such that $(sE - A)^{-1} : Z \to X$ is defined for all $s \in \mathbb{C}_\omega^+$. Moreover, there exists $M > 0$, $\nu \in \mathbb{N}$ such that for all $s \in \mathbb{C}_\omega^+$ we have $\|(sE - A)^{-1}\| \leq M \cdot (1 + |s|)^\nu$.

The above assumptions imply that the transfer function $\boldsymbol{G}(s) = D + C(sE - A)^{-1}B$ of the descriptor system is well-defined on $\mathbb{C}_\omega^+$.
Note that for infinite-dimensional descriptor systems, it is no loss of generality to consider bounded B and C, that is, B does not map to a larger space than Z, and C is defined on whole X. This is due to the fact that unbounded operators B and C can be avoided by an artificial enlargement of the spaces X and Z [17].
A natural question is whether the io-passivity is equivalent to the fact that the transfer function $\boldsymbol{G}$ is positive, i.e. $\boldsymbol{G}$ can be extended analytically to $\mathbb{C}^+$ satisfying $\boldsymbol{G}(s) + \boldsymbol{G}^*(s) \geq 0$ for all $s \in \mathbb{C}^+$. The sufficiency can be shown by Parseval's identity. The necessity is more complicated and will

be treated in a forthcoming paper [13].
The work [11] treats the class of scalar holomorphic and positive functions, i.e. $\boldsymbol{G}:\mathbb{C}^+\to\mathbb{C}$ with $\operatorname{Re}(\boldsymbol{G}(s))\geq 0$ for all $s\in\mathbb{C}^+$. It is in particular shown that for all $\omega>0$, $\boldsymbol{G}$ has at most quadratic growth on $\mathbb{C}^+_\omega$. i.e. there exists some M_ω such that $|\boldsymbol{G}(s)|\leq M_\omega(1+|s|)^2$ for all $s\in\mathbb{C}^+_\omega$. It is further shown in [11] that $\boldsymbol{G}$ has at most linear growth on the real axis and $\lim_{s\to\infty,s\in\mathbb{R}}\frac{1}{s}\boldsymbol{G}(s)=M$ for some real $M>0$.
Indeed, the quadratic growth of the transfer function causes that an inequality (8) is in general not possible, but only an estimate of the L_2-norm of the output $y(\cdot)$ by the Sobolev-norm H^2 of the input $u(\cdot)$.
We will now generalize Criterion 1 and Criterion 2 to infinite-dimensional descriptor systems. For a system (1) and a subspace $\widetilde{Z}\subset Z$, we introduce the space

$$S_{\widetilde{Z}}:=\left\{\begin{bmatrix}u_0\\x_0\end{bmatrix}\in U\times D(A):\ Ax_0+Bu_0\in\widetilde{Z}\cap\operatorname{im}E\right\}.$$

Criterion 3 (Passive infinite-dimensional descriptor systems).
There exists some Hilbert spaces $\widetilde{X}\subset X$, $\widetilde{Z}\subset Z$ and some bounded operator $H:\widetilde{X}\to\widetilde{Z}'$ with the following properties:

(i) $E\widetilde{X}\subset\widetilde{Z}$ *and* $\langle Hx,Ex\rangle_{\widetilde{Z}',\widetilde{Z}}$ *is real and positive for all* $x\in\widetilde{X}$.

(ii) There exists some $k\geq\nu$ *such that for all* $T>0$ *and* $u(\cdot)\in H^k([0,T],U)$ *with* $u(0)=\dot{u}(0)=\cdots=u^{(k-1)}(0)=0$, *the solution* $x(\cdot):[0,T]\to X$ *of system (1) with* $x_0=0$ *is continuously differentiable with* $x(t),\dot{x}(t)\in\widetilde{X}$ *for all* $t\in[0,T]$.

(iii) For all $u_0\in U$, $x_0\in X$ *such that* $[\,u_0\,,\,x_0\,]\in S_{\widetilde{Z}}$ *holds*

$$-\operatorname{Re}\langle Hx_0,Ax_0+Bu_0\rangle_{\widetilde{Z}',\widetilde{Z}}+\operatorname{Re}\langle Cx_0,u_0\rangle_U+\operatorname{Re}\langle u_0,Du_0\rangle_U\geq 0. \tag{10}$$

Theorem 3. *Let a descriptor system (1) satisfying assumptions a) and b) with moreover $U=Y$ be given. Further, assume that Criterion 3 is fulfilled. Then (1) is io-passive.*

Proof. The fact that $\langle Hx,Ex\rangle_{\widetilde{Z}',\widetilde{Z}}$ is real and positive for all $x\in\widetilde{X}$, implies that for all $x_1,x_2\in\widetilde{X}$ holds $\langle Hx_1,Ex_2\rangle_{\widetilde{Z}',\widetilde{Z}}=\overline{\langle Hx_2,Ex_1\rangle_{\widetilde{Z}',\widetilde{Z}}}$.
Now let $u(\cdot):[0,T]\to U$ and $x(\cdot):[0,T]\to X$ be solutions of system (1) described in (ii) of Criterion 3. Then we have that

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\langle Hx(t),Ex(t)\rangle_{\widetilde{Z}',\widetilde{Z}}=\operatorname{Re}\langle Hx(t),E\dot{x}(t)\rangle_{\widetilde{Z}',\widetilde{Z}}.$$

Moreover, due to $E\dot{x}(t)=Ax(t)+Bu(t)$ and $E\dot{x}(t)\in\widetilde{Z}$, we have $[\,u(t)\,,\,x(t)\,]^T\in S_{\widetilde{Z}}$ for all $t\in[0,T]$. Then we compute

$$\begin{aligned}0\leq\tfrac{1}{2}\cdot\langle Hx(T),Ex(T)\rangle_{\widetilde{Z}',\widetilde{Z}}&=\operatorname{Re}\int_0^T\langle Hx(t),E\dot{x}(t)\rangle_{\widetilde{Z}',\widetilde{Z}}dt\\&=\operatorname{Re}\int_0^T\langle Hx(t),Ax(t)+Bu(t)\rangle_{\widetilde{Z}',\widetilde{Z}}dt\\&\leq\operatorname{Re}\int_0^T\langle u(t),Cx(t)+Du(t)\rangle_U dt=\operatorname{Re}\langle u(\cdot),y(\cdot)\rangle_{L_2([0,T],U)}.\end{aligned}$$

Therefore, the system is passive. □

An open question is whether Criterion 3 is also sufficient for passivity. This will be subject of future research.

5 Example: Electrical Circuits with Transmission Lines

Consider an electrical circuit with voltage sources, current sources, capacitances, inductances, resistances and transmission lines. The transmission lines (which are assumed to have normalized length) fulfill the so-called *telegraph equations*

$$\begin{aligned} L_T(\xi)\dot{I}_T(\xi,t) &= -R_T(\xi)I_T(\xi,t) - \tfrac{\partial}{\partial\xi}V_T(\xi,t),\\ C_T(\xi)\dot{V}_T(\xi,t) &= -G_T(\xi)I_T(\xi,t) - \tfrac{\partial}{\partial\xi}V_T(\xi,t), \qquad t>0,\ \ \xi\in[0,1], \end{aligned}$$

where $L_T(\cdot)$, $R_T(\cdot)$, $C_T(\cdot)$, $G_T(\cdot) : [0,1] \to \mathbb{R}^{n_T,n_T}$ are measurable matrix-valued functions which are symmetric and positive semi-definite almost everywhere. We further claim that there exists $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that for almost every $\xi \in [0,1]$ holds $\varepsilon_1 I_{n_T} > C_T(\xi) > \varepsilon_2 I_{n_T}$ and $\varepsilon_1 I_{n_T} > L_T(\xi) > \varepsilon_2 I_{n_T}$.
The *modified nodal analysis* [6] leads to the following equations

$$\begin{aligned} A_C C A_C \dot{e}(t) =& -A_R R^{-1} A_R^T e(t) - A_L i_L(t) - A_V i_V - A_{T0} I_T(0,t) - A_{T1} I_T(1,t) - A_I i_I(t),\\ L\dot{i}_L(t) =& A_L^T e(t),\\ 0 =& A_V^T e(t) - u_V(t), \end{aligned} \tag{11}$$

where R, C and L are the resistance, capacitance and inductance matrix, which are positive definite by assumption. The matrices A_C, A_R, A_L, A_V, A_I, A_{T0}, A_{T1} denote the *element-related incidence matrices* of capacitances, resistances, inductances, voltage and current sources, initial and terminal ports of transmission lines. We assume that the input of the system is given by the voltages of voltage sources and currents of current sources, i.e., $u(t) = [\,u_V(t)\,,\, i_I(t)\,]$, whereas the output consists of the negative of currents of voltage sources and voltages of current sources, i.e. $y(t) = [-i_V(t)\,,\,-u_I(t)\,]$. Defining $C_0, C_1 : H^1([0,1],\mathbb{R}^k) \to \mathbb{R}^k$ by $C_0 f(\cdot) = f(0)$, $C_1 f(\cdot) = f(1)$, we then obtain a descriptor system (1) with state $x(t) = [\,e(t)\,,\, i_L(t)\,,\, i_V(t)\,,\, I_T(t,\cdot)\,,\, V_T(t,\cdot)\,]$ and spaces

$$\begin{aligned} X &= \mathbb{R}^{n_e} \times \mathbb{R}^{n_L} \times \mathbb{R}^{n_V} \times L_2([0,1],\mathbb{R}^{n_T}) \times L_2([0,1],\mathbb{R}^{n_T}),\\ D(A) &= \mathbb{R}^{n_e} \times \mathbb{R}^{n_L} \times \mathbb{R}^{n_V} \times H^1([0,1],\mathbb{R}^{n_T}) \times H^1([0,1],\mathbb{R}^{n_T}),\\ Z &= \mathbb{R}^{n_e} \times \mathbb{R}^{n_L} \times \mathbb{R}^{n_V} \times L_2([0,1],\mathbb{R}^{n_T}) \times L_2([0,1],\mathbb{R}^{n_T}) \times \mathbb{R}^{n_T} \times \mathbb{R}^{n_T} \end{aligned} \tag{12}$$

and operators

$$E = \begin{bmatrix} A_C C A_C^T & 0 & 0 & 0 & 0\\ 0 & L & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & C_T & 0\\ 0 & 0 & 0 & 0 & L_T\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -A_R R^{-1} A_R^T & -A_L & -A_V & 0 & \begin{matrix}-A_{T0}C_0\\ +A_{T1}C_1\end{matrix}\\ A_L^T & 0 & 0 & 0 & 0\\ A_V^T & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -G_T & -\frac{\partial}{\partial\xi}\\ 0 & 0 & 0 & -\frac{\partial}{\partial\xi} & -R_T\\ A_{T0}^T & 0 & 0 & -C_0 & 0\\ A_{T1}^T & 0 & 0 & -C_1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} -A_I & 0\\ 0 & 0\\ 0 & -I\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} -A_I^T & 0 & 0 & 0 & 0\\ 0 & 0 & -I & 0 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0\\ 0 & 0 \end{bmatrix}. \tag{13}$$

In [15], it is shown that if A_V and $[A_C\ A_R\ A_L\ A_V\ A_{T0}\ A_{T1}]^T$ have full column rank, then there exists some $M > 0$, $\omega \in \mathbb{R}$ such that for all $s \in \mathbb{C}_\omega^*$ we have

$$\|(sE - A)^{-1}\| < M(|s| + 1).$$

The conditions on the incidence matrices correspond to the absence of loops of voltage sources and cutsets of currents sources [9].
For

$$X = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{14}$$

we have that E^*X is self-adjoint and positive semidefinite. Therefore, the claim (i) in Criterion 3 is fulfilled for $\widetilde{X} = X$ and $\widetilde{Z} = Z$. Defining $Q_C \in \mathbb{R}^{n_e, n_e}$ to be a projector onto $\ker A_C^T$, the space $S_{\widetilde{X}}$ reads

$$\left\{ \begin{bmatrix} e \\ i_L \\ i_V \\ V_T \\ I_T \\ i_I \\ u_V \end{bmatrix} \in \mathbb{R}^{n_e+n_L+n_V} \times H^1([0,1], \mathbb{R}^{2n_T}) \times \mathbb{R}^{2n_T} \;\middle|\; \begin{array}{r} Q_C^T A_R R^{-1} A_R^T e + Q_C^T A_L i_L + Q_C^T A_V i_V \\ + Q_C^T A_{T0} I_T(0) - Q_C^T A_{T1} I_T(1) + Q_C^T A_I i_I = 0 \\ A_V^T e - u_V = 0 \\ A_{T0}^T e - V_T(0) = 0 \\ A_{T1}^T e - V_T(1) = 0 \end{array} \right\}. \tag{15}$$

Then for $[\, x_0 \,,\, u_0 \,] \in S_{\widetilde{X}}$ holds

$$\begin{aligned} & - \mathrm{Re}\langle Hx_0, Ax_0 + Bu_0 \rangle_{\widetilde{Z}', \widetilde{Z}} + \mathrm{Re}\langle Cx_0, u_0 \rangle_U + \mathrm{Re}\langle u_0, Du_0 \rangle_U \\ = & - 2e^T A_R R^{-1} A_R^T e - 2e^T A_L i_L - 2e^T A_{T0} I_T(0) + 2e^T A_{T1} I_T(1) + 2i_L^T A_L^T e \\ & - 2\int_0^1 V_T^T(\xi) G_T V_T(\xi) d\xi - 2\int_0^1 V_T^T(\xi) \tfrac{\partial}{\partial \xi} I_T(\xi) d\xi \\ & - 2\int_0^1 \tfrac{\partial}{\partial \xi} V_T^T(\xi) I_T(\xi) d\xi - 2\int_0^1 V_T^T(\xi) G_T V_T(\xi) d\xi - e^T A_I i_I - e^T A_I i_I. \end{aligned}$$

Using integration by parts and the equalities in (15), we obtain that this equals to

$$-2e^T A_R R^{-1} A_R^T e - 2\int_0^1 V_T^T(\xi) G_T V_T(\xi) d\xi - 2\int_0^1 V_T^T(\xi) G_T V_T(\xi) d\xi,$$

which is negative by assumptions on R, C, L and $G_T(\cdot)$. Therefore, the system is passive.

6 Conclusion

We have considered the class of input-output-passive infinite-dimensional descriptor systems. Criteria for input-output passivity have been given in terms of the transfer function. Another sufficient condition generalizing the Kalman-Yakubovich-Popov inequality has been presented. The criteria have been compared to the existing results for infinite-dimensional state-space and finite-dimensional descriptor systems.

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