Funnel control via funnel pre-compensator for minimum phase systems with relative degree two

Thomas Berger and Timo Reis

***Abstract*—We consider tracking control for linear minimum phase single-input, single-output systems with relative degree two. For a class of sufficiently smooth reference signals we introduce a dynamic controller which achieves that the tracking error evolves within a prespecified performance funnel. This controller is based on the recently developed funnel pre-compensator combined with a proportional-derivative funnel controller. Altogether, this yields a dynamic controller which satisfies the control objective and uses only the output of the system and NOT the derivative of the output. The system parameters do not have to be known for the controller design.**

***Index Terms*—Linear systems, funnel control, funnel pre-compensator, relative degree, minimum phase.**

I. Introduction

In the present paper we consider output tracking for linear minimum phase systems with relative degree two by funnel control. The concept of funnel control has been developed in [1], see also the survey [2] and the references therein. In particular, the funnel controller proved to be the appropriate tool for tracking problems in various applications, such as chemical reactor models [3], industrial servo-systems [4], [5] and rigid, revolute joint robotic manipulators [6], voltage and current control of electrical circuits [7], and control of peak inspiratory pressure [8].

An obstacle for high-gain adaptive controllers are systems of relative degree higher than one [2], [9]. In [10], [11], Ilchmann et al. introduce a funnel controller for higher relative degree systems by implementing a "backstepping" procedure in conjunction with a filter. The controller achieves tracking with prescribed transient behavior for a large class of systems governed by nonlinear (functional) differential equations, however the backstepping procedure is quite complicated and impractical since it involves high powers of a gain function which typically takes large values, cf. [12, Sec. 4.4.3]. Backstepping is also used for an adaptive λ-tracker in an earlier work by Ye [13].

In the case of relative degree two systems, an alternative funnel controller has been proposed in [4] (see also the modification in [14]), where the backstepping procedure is avoided by using a linear combination of the output and its derivative instead. Generalizations of this approach to systems with higher relative degree are the bang-bang funnel controller introduced in [15] and the recent funnel controller developed in [16]. However, the incorporation of output derivatives means in practice that measurements have to be differentiated. The latter is an ill-posed problem in particular in the presence of noise, see e.g. [12, Sec. 1.4.4].

In [17], a "Prescribed Performance Controller" for systems with higher strict relative degree is introduced. Though this controller is applicable to a large class of systems, its drawback is that it requires the full information of the state. In [18], an adaptive λ-tracker is introduced by composing a high-gain observer, a high-gain observer-state feedback and a common adaptation scheme for both high-gain parameters. The controller achieves tracking with prescribed asymptotic accuracy $\lambda > 0$ for a class of systems which are affine in the control, of known relative degree, and with affine linearly bounded drift term. The advantage of this controller is that no derivatives of the output are required due to the high-gain observer, however the transient behavior of the tracking error cannot be influenced.

In the present paper we overcome the drawbacks of [4] by incorporating a funnel pre-compensator, which resembles an adaptive high-gain observer, so that derivatives of the output are not required anymore. The combination of the funnel pre-compensator with the funnel controller from [4] results in a dynamic controller achieving prescribed transient behavior of the tracking error.

A. Nomenclature

$\mathbb{R}_{\geq 0}$	$= [0,\infty)$
$\mathscr{L}^\infty_{\text{loc}}(I\to\mathbb{R}^n)$	the set of locally essentially bounded functions $f: I\to\mathbb{R}^n$, $I\subseteq\mathbb{R}$ an interval
$\mathscr{L}^\infty(I\to\mathbb{R}^n)$	the set of essentially bounded functions $f: I\to\mathbb{R}^n$
$\|f\|_\infty$	= ess $\sup_{t\in I}\|f(t)\|$
$\mathscr{W}^{k,\infty}(I\to\mathbb{R}^n)$	the set of k-times weakly differentiable functions $f: I\to\mathbb{R}^n$ such that $f,\dots,f^{(k)}\in\mathscr{L}^\infty(I\to\mathbb{R}^n)$
$\mathscr{C}(V\to\mathbb{R}^n)$	the set of continuous functions $f: V\to\mathbb{R}^n$, $V\subseteq\mathbb{R}^m$
$f\|_W$	restriction of the function $f: V\to\mathbb{R}^n$ to $W\subseteq V$

B. System class

In the present paper we consider linear single-input, single-output systems given by

$$\begin{aligned} \dot x(t) &= Ax(t)+bu(t), \qquad x(0)=x^0,\\ y(t) &= cx(t), \end{aligned} \tag{1}$$

where $A\in\mathbb{R}^{n\times n}$ and $b,c^\top,x^0\in\mathbb{R}^n$. The functions $u,y:\mathbb{R}_{\geq 0}\to\mathbb{R}$ are called *input* and *output* of the system (1), resp. We assume that (1) has relative degree two, positive high-frequency gain and is minimum phase (equivalently, the zero dynamics are asymptotically stable, cf. [19]), that is

(A1) $cb=0$ and $cAb>0$;

(A2) $\det\begin{bmatrix}\lambda I_n - A & b\\ c & 0\end{bmatrix}\neq 0$ for all $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda\geq 0$.

Adaptive control of minimum phase linear systems (1) is well-studied, see e.g. [20]–[23]. We formulate our control objective in the following.

C. Control objective

The objective is to design a dynamic output feedback

$$\begin{aligned} \dot z(t) &= F\big(t,z(t),y(t),y_{\text{ref}}(t)\big),\\ u(t) &= G\big(t,z(t),y(t),y_{\text{ref}}(t)\big), \end{aligned} \tag{2}$$

where y_{ref} is a sufficiently smooth reference signal, such that in the closed-loop system the tracking error $e(t)=y(t)-y_{\text{ref}}(t)$ evolves within a prescribed performance funnel

$$\mathscr{F}_\varphi := \{\ (t,e)\in\mathbb{R}_{\geq 0}\times\mathbb{R}\ |\ \varphi(t)|e|<1\ \}, \tag{3}$$

which is determined by a function φ belonging to

$$\Phi := \left\{ \varphi\in\mathscr{W}^{1,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R}) \;\middle|\; \begin{array}{l}\varphi(s)>0 \text{ for all } s>0 \text{ and}\\ \text{for all } \varepsilon>0:\\ \varphi^{-1}|_{[\varepsilon,\infty)}\in\mathscr{W}^{1,\infty}([\varepsilon,\infty)\to\mathbb{R})\end{array}\right\}.$$

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Furthermore, all involved signals should remain bounded.

The funnel boundary is given by the reciprocal of φ, see Fig. 1. The case $\varphi(0)=0$ is explicitly allowed and puts no restriction on the initial value since $\varphi(0)|e(0)|<1$; in this case the funnel boundary $1/\varphi$ has a pole at $t=0$.

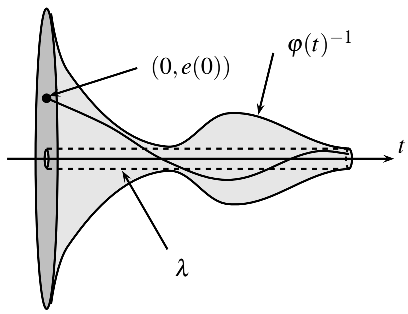

Fig. 1: Error evolution in a funnel $\mathscr{F}_\varphi$ with boundary $\varphi(t)^{-1}$.

An important property is that each performance funnel $\mathscr{F}_\varphi$ with $\varphi \in \Phi$ is bounded away from zero, i.e., due to boundedness of φ there exists $\lambda > 0$ such that $1/\varphi(t) \geq \lambda$ for all $t>0$. The funnel boundary is not necessarily monotonically decreasing, while in most situations it is convenient to choose a monotone funnel. However, there are situations where widening the funnel over some later time interval might be beneficial, e.g., when the reference trajectory changes strongly or the system is perturbed by some calibration so that a large tracking error would enforce a large input action.

In the present paper we show that the control objective can be achieved by the combination of a funnel pre-compensator (see Sec. II) with a proportional-derivative funnel controller for relative degree two systems (see Sec. I-D).

D. Funnel control without pre-compensator

For relative degree two systems of the form (1) a funnel controller has been developed in [4]. However, it is not of type (2), since it uses derivative feedback of the form

$$\boxed{\begin{aligned} u(t)&=-k_0(t)^2e(t)-k_1(t)\dot e(t)+u_d(t), \qquad e(t)=y(t)-y_{\mathrm{ref}}(t),\\ k_i(t)&=\frac{\varphi_i(t)}{1-\varphi_i(t)|e^{(i)}(t)|}, \qquad\qquad i=0,1,\end{aligned}} \tag{4}$$

where $u_d \in \mathscr{L}^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R})$ is an input disturbance and $(\varphi_0,\varphi_1)\in\Phi_2$; the latter class is defined by

$$\Phi_2 := \left\{ (\varphi_0,\varphi_1)\in\Phi\times\Phi \;\middle|\; \begin{array}{l} \exists\,\delta>0 \text{ for a.a. } t>0: \\ (1/\varphi_1)(t)+\frac{\mathrm{d}}{\mathrm{d}t}(1/\varphi_0)(t)\geq\delta \end{array} \right\}.$$

The motivation for the definition of Φ_2 is that the derivative funnel $\mathscr{F}_{\varphi_1}$ must be large enough to allow the error to follow the funnel boundaries; for more details see [4].

The controller (4) even works for a large class of nonlinear systems governed by functional differential equations of the form

$$\begin{aligned} \ddot y(t) &= f\big(d(t),T(y,\dot y)(t)\big)+g\big(d(t),T(y,\dot y)(t)\big)u(t),\\ y|_{[-h,0]} &= y^0\in\mathscr{W}^{1,\infty}([-h,0]\to\mathbb{R}),\end{aligned} \tag{5}$$

where $h>0$ is the "memory" of the system, and

- $d\in\mathscr{L}^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^p)$, $p\in\mathbb{N}$, is a disturbance;
- $f\in\mathscr{C}(\mathbb{R}^p\times\mathbb{R}^q\to\mathbb{R})$, $g\in\mathscr{C}(\mathbb{R}^p\times\mathbb{R}^q\to\mathbb{R})$, $q\in\mathbb{N}$, and $g(v,w)>0$ for all $(v,w)\in\mathbb{R}^p\times\mathbb{R}^q$;
- $T:\mathscr{C}([-h,\infty)\to\mathbb{R})^2\to\mathscr{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}\to\mathbb{R}^q)$ is an operator with the following properties:
  a) there exists $\psi\in\mathscr{C}(\mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0})$ such that for all bounded $(\zeta_1,\zeta_2)\in\mathscr{C}([-h,\infty)\to\mathbb{R})^2$ we have that $T(\zeta_1,\zeta_2)$ is bounded with $\|T(\zeta_1,\zeta_2)\|_\infty\leq\psi(\|\zeta_1\|_\infty,\|\zeta_2\|_\infty)$;
  b) T is causal, i.e., for all $t\geq 0$ and all $\zeta,\xi\in\mathscr{C}([-h,\infty)\to\mathbb{R})^2$:
  $$\zeta|_{[-h,t)}=\xi|_{[-h,t)} \implies T(\zeta)|_{[0,t]}\overset{\text{a.e.}}{=}T(\xi)|_{[0,t]};$$
  c) T is "locally Lipschitz" continuous in the following sense: for all $t\geq 0$ there exist $\tau,\delta,c>0$ such that for all $\zeta,\Delta\zeta\in\mathscr{C}([-h,\infty)\to\mathbb{R})^2$ with $\Delta\zeta|_{[-h,t]}=0$ and $\|\Delta\zeta|_{[t,t+\tau]}\|_\infty<\delta$ we have
  $$\Big\|\big(T(\zeta+\Delta\zeta)-T(\zeta)\big)|_{[t,t+\tau]}\Big\|_\infty\leq c\|\Delta\zeta|_{[t,t+\tau]}\|_\infty.$$

In [4], the existence of global solutions of the closed-loop system (5), (4) is investigated. To this end, $y:[-h,\omega)\to\mathbb{R}$ is called a *solution* of (5), (4) on $[-h,\omega)$, $\omega\in(0,\infty]$, if $y|_{[-h,0]}=y^0$ and $y|_{[0,\omega)}$ is twice weakly differentiable and satisfies (5), (4) for almost all $t\in[0,\omega)$; y is called *maximal*, if it has no right extension that is also a solution. Note that uniqueness of solutions of (5), (4) is not guaranteed in general.

The following result is in [4, Thm. 3.1].

Theorem I.1. *Consider a system* (5) *with initial trajectory* $y^0\in\mathscr{W}^{1,\infty}([-h,0]\to\mathbb{R})$, *a reference signal* $y_{\mathrm{ref}}\in\mathscr{W}^{2,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R})$, *an input disturbance* $u_d\in\mathscr{L}^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R})$ *and a pair of funnels* $(\varphi_0,\varphi_1)\in\Phi_2$ *such that*

$$\varphi_0(0)|y^0(0)-y_{\mathrm{ref}}(0)|<1 \quad \text{and} \quad \varphi_1(0)|\dot y^0(0)-\dot y_{\mathrm{ref}}(0)|<1.$$

Then the controller (4) *applied to* (5) *yields a closed-loop system which has a solution, and every maximal solution* $y:[0,\omega)\to\mathbb{R}$ *has the properties:*

(i) $\omega=\infty$;
(ii) *all involved signals* $y(\cdot),\dot y(\cdot),k_0(\cdot)$ *and* $k_1(\cdot)$ *are bounded;*
(iii) *the tracking error and its derivative evolve uniformly within the respective performance funnels in the sense*

$$\forall\, i\in\{0,1\}\ \exists\,\varepsilon_i>0\ \forall\, t>0:\ |e^{(i)}|\leq\varphi_i(t)^{-1}-\varepsilon_i.$$

E. Contribution of the present paper

The drawback of the funnel controller (4) is that it involves derivative feedback and thus it does not satisfy the control objective. The derivative of the output is usually not available or very hard to compute [12, Sec. 1.4.4]. Therefore, a dynamic error feedback of the form (2) is sought.

In the present paper we resolve this drawback by first applying the funnel pre-compensator developed in [24] to system (1) to obtain an interconnection with certain properties so that the controller (4) may be applied to it. In the end, the combination of the pre-compensator and (4) yields a new funnel control design which achieves the control objective. For the precise controller structure see Sec. III.

II. The funnel pre-compensator

An integral part of the controller that we propose in the present paper is the funnel pre-compensator developed in [24]. We like to point out that in [24] it is called "funnel observer" since it resembles an (adaptive) high-gain observer, however it does not have the corresponding properties. The funnel pre-compensator is of the form

$$\boxed{\begin{aligned} \dot z_1(t)&=z_2(t)+\big(q_1+p_1k_2(t)\big)\big(y(t)-z_1(t)\big),\\ \dot z_2(t)&=\tilde\gamma u(t)+\big(q_2+p_2k_2(t)\big)\big(y(t)-z_1(t)\big),\\ k_2(t)&=\frac{1}{1-\varphi_2(t)^2|y(t)-z_1(t)|^2},\end{aligned}} \tag{6}$$

with initial conditions

$$z_i(0)=z_i^0\in\mathbb{R}, \quad i=1,2, \tag{7}$$

where $\varphi_2 \in \Phi$, $\tilde{\gamma} > 0$ and $q_i > 0$, $p_i > 0$ for all $i = 1,2$. The functions $z_i : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $i = 1,2$, are the pre-compensator states and $k_2 : \mathbb{R}_{\geq 0} \to [1,\infty)$ is the pre-compensator gain. Note that the matrix $Q := \begin{bmatrix} -q_1 & 1 \\ -q_2 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}$ is Hurwitz, i.e., all its eigenvalues have negative real part. The constants p_i depend on the choice of the q_i in the following way: Let

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_4 \end{bmatrix}, \quad P_1, P_2, P_4 \in \mathbb{R}$$

be such that

$$Q^\top P + PQ + I_2 = 0, \quad P > 0.$$

The matrix P depends only on the choice of the constants q_i. The constants p_i are defined by

$$p_1 = 1, \quad p_2 = -\tfrac{P_2}{P_4}. \tag{8}$$

Then

$$\left(1, -\tfrac{P_2}{P_4}\right) P \begin{pmatrix} 1 \\ -\frac{P_2}{P_4} \end{pmatrix} = P_1 - \tfrac{P_2^2}{P_4} > 0 \tag{9}$$

and we will see later that (9) guarantees that P defines a quadratic Lyapunov function for the pre-compensator error dynamics.

The pre-compensator (6) is a nonlinear and time-varying system, nevertheless it is simple in its structure and only two-dimensional. Similar to an observer, the funnel pre-compensator (6) only requires the input signal $u(t)$ and the output signal $y(t)$, see Fig. 2; no further knowledge of system parameters is required.

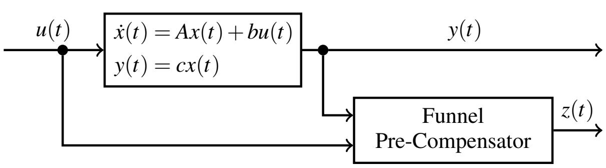

Fig. 2: Interconnection of (1) with funnel pre-compensator (6).

Note that by the design of the pre-compensator (6), the gain $k_2(t)$ increases if the norm of the error $|y(t) - z_1(t)|$ approaches the funnel boundary $1/\varphi_2(t)$, and decreases if a high gain is not necessary. This guarantees prescribed transient behavior of the error $e_1(t) = y(t) - z_1(t)$ as shown in [24, Thm. 4.1].

III. Controller structure

We propose a novel and simple funnel controller for trajectory tracking with prescribed transient behavior for relative degree two systems such that a derivative of the output is not required. The first part of the controller is a funnel pre-compensator (6). Considering the interconnection of system (1) with the funnel pre-compensator (6) we treat the state z_1 as an output of this system and apply the controller (4) to it. We stress that the controller (4) requires the derivative of this artificial output, which however is available since $\dot{z}_1 = z_2 + \big(q_1 + p_1 k_2\big)(y - z_1)$ and k_2 only depends on the available variables y, z_1 and the funnel function $\varphi_2 \in \Phi$. Therefore, we arrive at a controller of the form (2), namely

$$\boxed{\begin{aligned}
\dot{z}_1(t) &= z_2(t) + \big(q_1 + p_1 k_2(t)\big)\big(y(t) - z_1(t)\big), \\
\dot{z}_2(t) &= \tilde{\gamma} u(t) + \big(q_2 + p_2 k_2(t)\big)\big(y(t) - z_1(t)\big), \\
u(t) &= -k_0(t)^2\big(z_1(t) - y_{\text{ref}}(t)\big) \\
&\quad - k_1(t)\big(\dot{z}_1(t) - \dot{y}_{\text{ref}}(t)\big) + u_d(t), \\
k_0(t) &= \tfrac{\varphi_0(t)}{1 - \varphi_0(t)|z_1(t) - y_{\text{ref}}(t)|}, \\
k_1(t) &= \tfrac{\varphi_1(t)}{1 - \varphi_1(t)|\dot{z}_1(t) - \dot{y}_{\text{ref}}(t)|}, \\
k_2(t) &= \tfrac{1}{1 - \varphi_2(t)^2 |y(t) - z_1(t)|^2},
\end{aligned}} \tag{10}$$

where $\tilde{\gamma} > 0$, $y_{\text{ref}} \in \mathscr{W}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ is the reference trajectory, $u_d \in \mathscr{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$ is an input disturbance, $(\varphi_0, \varphi_1) \in \Phi_2$ and $\varphi_2 \in \Phi$ define the funnel boundaries, and $q_1, q_2, p_1, p_2 > 0$ are such that (8) is satisfied for corresponding matrices P and Q. The controller structure is depicted in Fig. 3.

IV. Main result

The intuition for the funnel controller (10) to work for system (1) is that the error dynamics of the funnel pre-compensator act as internal dynamics of the interconnection of system (1) with the funnel pre-compensator (6) when the state z_1 is taken as output. These internal dynamics are bounded-input, bounded-output stable since, as we will show, all signals involved in this interconnection are bounded and can hence be modeled by an operator which maps bounded signals to bounded signals, thus allowing the application of Theorem I.1. Since $\dot{z}_1$ in (6) does not require derivatives of y and $y - z_1$ evolves in $\mathscr{F}_{\varphi_2}$ we obtain prescribed performance of the tracking error $e = y - y_{\text{ref}}$.

Theorem IV.1. *Consider a linear system* (1) *which satisfies (A1) and (A2) with initial value $x^0 \in \mathbb{R}^n$, a reference signal $y_{\text{ref}} \in \mathscr{W}^{2,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$, an input disturbance $u_d \in \mathscr{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$ and a pair of funnels $(\varphi_0, \varphi_1) \in \Phi_2$ such that*

$$\varphi_0(0)|cx^0 - y_{\text{ref}}(0)| < 1 \quad \textit{and} \quad \varphi_1(0)|cAx^0 - \dot{y}_{\text{ref}}(0)| < 1.$$

Further choose initial values (7) *and $\varphi_2 \in \Phi$ such that*

$$\varphi_2(0)|cx^0 - z_1^0| < 1,$$

$\tilde{\gamma} > 0$ and $q_1, q_2, p_1, p_2 > 0$ such that (8) *is satisfied for corresponding matrices P and Q.*
Then the controller (10) *applied to* (1) *yields a closed-loop system which has a unique maximal solution $(x, z_1, z_2) : [0, \omega) \to \mathbb{R}$ with the properties:*

(i) *$\omega = \infty$;*
(ii) *all involved signals $x(\cdot), z_1(\cdot), z_2(\cdot), k_0(\cdot), k_1(\cdot), k_2(\cdot)$ are bounded;*
(iii) *the errors evolve uniformly within the respective performance funnels in the sense*

$$\begin{aligned}
\exists\, \varepsilon_0, \varepsilon_1, \varepsilon_2 > 0\ \forall\, t > 0 :\ & |z_1(t) - y_{\text{ref}}(t)| \leq \varphi_0(t)^{-1} - \varepsilon_0, \\
& |\dot{z}_1(t) - \dot{y}_{\text{ref}}(t)| \leq \varphi_1(t)^{-1} - \varepsilon_1, \\
& |y(t) - z_1(t)| \leq \varphi_2(t)^{-1} - \varepsilon_2.
\end{aligned} \tag{11}$$

In particular, the tracking error satisfies

$$\forall\, t > 0 :\ |y(t) - y_{\text{ref}}(t)| \leq \varphi_0(t)^{-1} + \varphi_2(t)^{-1} - \varepsilon_0 - \varepsilon_2. \tag{12}$$

Proof. Since system (1) has relative degree two by (A1), we may without loss of generality assume that it is in Byrnes-Isidori form:

$$\begin{aligned}
\ddot{y}(t) &= r_1 y(t) + r_2 \dot{y}(t) + s\eta(t) + \gamma u(t), \\
\dot{\eta}(t) &= w y(t) + V\eta(t),
\end{aligned} \tag{13}$$

where $r_1, r_2 \in \mathbb{R}$, $w, s^\top \in \mathbb{R}^{n-2}$, $V \in \mathbb{R}^{(n-2)\times(n-2)}$ and $\gamma = cAb > 0$. See [25] and [11, Lem. 3.5] for an explicit derivation of the transformation which leads to (13). By the minimum phase assumption (A2) we further obtain that all eigenvalues of V have negative real part. We proceed in several steps.

Step 1: We show existence and uniqueness of a local solution of the closed-loop system consisting of the controller (10) applied to (13). Define

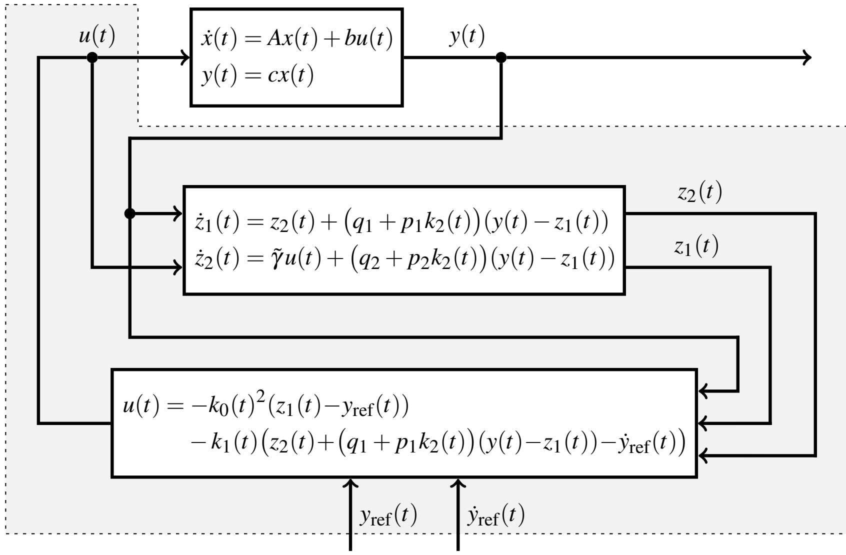

Fig. 3: The funnel controller (10) applied to system (1) consisting, indicated by the grey box, of a funnel pre-compensator (6) and a controller (4).

$$\mathscr{D} := \left\{ (t, y_0, y_1, \eta, z_1, z_2) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{n+2} \;\middle|\; \begin{array}{l} \varphi_0(t)|z_1 - y_{\text{ref}}(t)| < 1 \\ \varphi_1(t)\left| z_2 + \left(q_1 + \frac{p_1}{1 - \varphi_2(t)^2 |y_0 - z_1|^2}\right)(y_0 - z_1) - \dot{y}_{\text{ref}}(t) \right| < 1 \\ \varphi_2(t)|y_0 - z_1| < 1 \end{array} \right\}$$

and $F : \mathscr{D} \to \mathbb{R}^{n+2}$ by

$$F(t, y_0, y_1, \eta, z_1, z_2) = \begin{pmatrix} y_1 \\ r_1 y_0 + r_2 y_1 + s\eta - \gamma \left(\frac{z_1 - y_{\text{ref}}(t)}{1 - \varphi_0(t)^2 |z_1 - y_{\text{ref}}(t)|^2} + \frac{z_2 + \left(q_1 + \frac{p_1}{1 - \varphi_2(t)^2 |y_0 - z_1|^2}\right)(y_0 - z_1) - \dot{y}_{\text{ref}}(t)}{1 - \varphi_1(t)^2 \left| z_2 + \left(q_1 + \frac{p_1}{1 - \varphi_2(t)^2 |y_0 - z_1|^2}\right)(y_0 - z_1) - \dot{y}_{\text{ref}}(t) \right|^2} - u_d(t) \right) \\ w y_0 + V\eta \\ z_2 + \left(q_1 + \frac{p_1}{1 - \varphi_2(t)^2 |y_0 - z_1|^2}\right)(y_0 - z_1) \\ -\tilde{\gamma} \left(\frac{z_1 - y_{\text{ref}}(t)}{1 - \varphi_0(t)^2 |z_1 - y_{\text{ref}}(t)|^2} + \frac{z_2 + \left(q_1 + \frac{p_1}{1 - \varphi_2(t)^2 |y_0 - z_1|^2}\right)(y_0 - z_1) - \dot{y}_{\text{ref}}(t)}{1 - \varphi_1(t)^2 \left| z_2 + \left(q_1 + \frac{p_1}{1 - \varphi_2(t)^2 |y_0 - z_1|^2}\right)(y_0 - z_1) - \dot{y}_{\text{ref}}(t) \right|^2} - u_d(t) \right) + \left(q_2 + \frac{p_2}{1 - \varphi_2(t)^2 |y_0 - z_1|^2}\right)(y_0 - z_1) \end{pmatrix}.$$

Then the closed-loop system (13), (10) is equivalent to

$$\begin{pmatrix} \dot{y}(t) \\ \ddot{y}(t) \\ \dot{\eta}(t) \\ \dot{z}_1(t) \\ \dot{z}_2(t) \end{pmatrix} = F\left(t, \begin{pmatrix} y(t) \\ \dot{y}(t) \\ \eta(t) \\ z_1(t) \\ z_2(t) \end{pmatrix}\right), \quad \begin{pmatrix} y(0) \\ \dot{y}(0) \\ \eta(0) \\ z_1(0) \\ z_2(0) \end{pmatrix} = \begin{pmatrix} cx^0 \\ cAx^0 \\ \eta^0 \\ z_1^0 \\ z_2^0 \end{pmatrix} =: X^0, \quad (14)$$

where $\eta^0 \in \mathbb{R}^{n-2}$ is chosen in terms of x^0 and the transformation to the form (13). Thus, $(0, X^0) \in \mathscr{D}$ and F is measurable in t and locally Lipschitz in $(y_0, y_1, \eta, z_1, z_2)$. Hence, by the theory of ordinary differential equations (see e.g. [26, § 10, Thm. VI]) there exists a unique maximal absolutely continuous solution $(y, \dot{y}, \eta, z_1, z_2) : [0, \omega) \to \mathbb{R}^{n+2}$, $\omega \in (0, \infty]$, of (14) satisfying the initial conditions. Further, the closure of the graph of this solution is not a compact subset of $\mathscr{D}$.

Step 2: We aim to reformulate the interconnection of (13) with the funnel pre-compensator (6) as an initial value problem

$$\begin{aligned} \ddot{z}_1(t) &= f\big(d(t), T(z_1, \dot{z}_1)(t)\big) + \tilde{\gamma} u(t), \\ z_1(0) &= z_1^0, \quad \dot{z}_1(0) = z_2^0 + (q_1 + p_1)(cx^0 - z_1^0) \end{aligned} \quad (15)$$

so that Theorem I.1 can be applied. Define the new variables

$$\begin{aligned} v_1(t) &:= y(t) - z_1(t), \\ v_2(t) &:= \dot{y}(t) - \tfrac{\gamma}{\tilde{\gamma}} z_2(t) - r_2\big(y(t) - z_1(t)\big). \end{aligned}$$

Then we obtain

$$\begin{aligned} \dot{v}_1(t) &= -\tfrac{\gamma}{\tilde{\gamma}}\left(q_1 - \tfrac{\tilde{\gamma}}{\gamma} r_2 + p_1 k_2(t)\right) v_1(t) + v_2(t) + \tfrac{\gamma - \tilde{\gamma}}{\tilde{\gamma}} \dot{z}_1(t), \\ \dot{v}_2(t) &= -\tfrac{\gamma}{\tilde{\gamma}}\left(q_2 - \tfrac{\tilde{\gamma}}{\gamma} r_1 + p_2 k_2(t)\right) v_1(t) + s\eta(t) + r_1 z_1(t) \\ &\quad + r_2 \dot{z}_1(t), \\ \dot{\eta}(t) &= w v_1(t) + V\eta(t) + w z_1(t), \\ k_2(t) &= \tfrac{1}{1 - \varphi_2(t)^2 v_1(t)^2}. \end{aligned} \quad (16)$$

To put the system (13), (6) into an equation of the form (15) we define the operator $T: \mathscr{C}([0,\infty)\to\mathbb{R})^2 \to \mathscr{L}^\infty_{\rm loc}(\mathbb{R}_{\ge 0}\to\mathbb{R}^{n+2})$ (essentially) as the solution operator of (16), i.e., for $\zeta_1,\zeta_2\in\mathscr{C}([0,\infty)\to\mathbb{R})$ let $(v_1,v_2,\eta):[0,\beta)\to\mathbb{R}^n$, $\beta\in(0,\infty]$, be the unique maximal solution of (16) for $z_1=\zeta_1$, $\dot z_1=\zeta_2$ corresponding to the initial values $v_1(0)=cx^0-z_1^0$, $v_2(0)=cAx^0-\frac{\gamma}{\tilde\gamma}z_2^0-r_2v_1(0)$, $\eta(0)=\eta^0$, and define

$$T(\zeta_1,\zeta_2)(t) := \big(\zeta_2(t), v_1(t), v_2(t), \eta(t)^\top, k_2(t)\big)^\top,\quad t\in[0,\beta).$$

We now show that T is well-defined, i.e., $\beta=\infty$, and has the properties a)–c) as defined in Sec. I-D. Note that $(t,v_1(t),v_2(t),\eta(t))\in\tilde{\mathscr{D}}$ for all $t\in[0,\beta)$, where

$$\tilde{\mathscr{D}} := \{\ (t,v_1,v_2,\eta)\in\mathbb{R}_{\ge0}\times\mathbb{R}^n\ \mid\ \varphi_2(t)|v_1|<1\ \}$$

and the closure of the graph of the solution (v_1,v_2,η) is not a compact subset of $\tilde{\mathscr{D}}$.

Step 2a: Assume that ζ_1 and ζ_2 are bounded on $[0,\beta)$. We show that v_1,v_2 and η are bounded as well. As $\varphi_2(t)|v_1(t)|<1$ for all $t\in[0,\beta)$ it is clear that v_1 is bounded and thus, as all eigenvalues of V have negative real part, η is bounded as well. Let $v(t):=(v_1(t),v_2(t))^\top$ and observe that

$$\begin{aligned}\dot v(t) &= Qv(t) - k_2(t)\tfrac{\gamma}{\tilde\gamma}\begin{pmatrix}p_1\\p_2\end{pmatrix}v_1(t)\\ &\quad+\begin{pmatrix}\frac{\gamma-\tilde\gamma}{\tilde\gamma}(\zeta_2(t)-q_1v_1(t))+r_2v_1(t)\\ q_2\frac{\tilde\gamma-\gamma}{\tilde\gamma}v_1(t)+s\eta(t)+r_1\zeta_1(t)+r_2\zeta_2(t)+r_1v_1(t)\end{pmatrix}\end{aligned}$$

for almost all $t\in[0,\beta)$. Boundedness of v_1,ζ_1,ζ_2 and η gives that, for some $M_1>0$ and for all $t\in[0,\beta)$,

$$\left\|\begin{pmatrix}\frac{\gamma-\tilde\gamma}{\tilde\gamma}(\zeta_2(t)-q_1v_1(t))+r_2v_1(t)\\ q_2\frac{\tilde\gamma-\gamma}{\tilde\gamma}v_1(t)+s\eta(t)+r_1\zeta_1(t)+r_2\zeta_2(t)+r_1v_1(t)\end{pmatrix}\right\|\le M_1.$$

We now find that, for almost all $t\in[0,\beta)$,

$$\begin{aligned}&\tfrac{\rm d}{{\rm d}t}v(t)^\top Pv(t)\\ &= v(t)^\top\Big(Q^\top P+PQ\Big)v(t) - 2k_2(t)\tfrac{\gamma}{\tilde\gamma}v(t)^\top P\begin{pmatrix}p_1\\p_2\end{pmatrix}v_1(t)\\ &+2v(t)^\top P\begin{pmatrix}\frac{\gamma-\tilde\gamma}{\tilde\gamma}(\zeta_2(t)-q_1v_1(t))+r_2v_1(t)\\ q_2\frac{\tilde\gamma-\gamma}{\tilde\gamma}v_1(t)+s\eta(t)+r_1\zeta_1(t)+r_2\zeta_2(t)+r_1v_1(t)\end{pmatrix}\\ &\le -v(t)^\top v(t) - 2k_2(t)\tfrac{\gamma}{\tilde\gamma}(P_1-P_2^2/P_4)v_1(t)^2\\ &\quad+2M_1\|P\|\|v(t)\|.\end{aligned}$$

With $M_2:=2M_1\|P\|>0$ and $\mu:=\|P\|^{-1}$ we have, using (9),

$$\tfrac{\rm d}{{\rm d}t}v(t)^\top Pv(t)\le -\mu v(t)^\top Pv(t)+M_2\|v(t)\|.$$

Let $\lambda_{\min}(P)$ denote the smallest eigenvalue of P and $\delta:=\frac12\mu\lambda_{\min}(P)$. Then, using that $ab\le\frac12(a^2+b^2)$ for all $a,b\ge0$, it follows that

$$\begin{aligned}\tfrac{\rm d}{{\rm d}t}v(t)^\top Pv(t) &\le -\mu v(t)^\top Pv(t) + \big(\sqrt{2\delta}\|v(t)\|\big)\left(\frac{M_2}{\sqrt{2\delta}}\right)\\ &\le -\mu v(t)^\top Pv(t)+\delta\|v(t)\|^2+\frac{M_2^2}{4\delta}\\ &\le -\frac{\mu}{2}v(t)^\top Pv(t)+\frac{M_2^2}{2\mu\lambda_{\min}(P)}\end{aligned}$$

for almost all $t\in[0,\beta)$. Gronwall's lemma now implies that

$$v(t)^\top Pv(t)\le v(0)^\top Pv(0)e^{-\frac{\mu}{2}t}+\tfrac{M_2^2}{\mu^2\lambda_{\min}(P)},$$

and hence

$$\forall\,t\in[0,\beta):\ \|v(t)\|^2\le\tfrac{\lambda_{\max}(P)}{\lambda_{\min}(P)}e^{-\frac{\mu}{2}t}\|v(0)\|^2+\tfrac{M_2^2}{\mu^2\lambda_{\min}(P)}.$$

In particular, we obtain $v\in\mathscr{L}^\infty\big([0,\beta)\to\mathbb{R}^2\big)$.

Step 2b: We show that $k_2\in\mathscr{L}^\infty([0,\beta)\to\mathbb{R})$, still provided that ζ_1 and ζ_2 are bounded on $[0,\beta)$. Let $\kappa\in(0,\beta)$ be arbitrary but fixed and $\lambda:=\inf_{t\in(0,\beta)}\varphi_2(t)^{-1}>0$. Since $\varphi_2\in\Phi$ we find that $\dot\varphi_2$ and $\varphi_2|_{[\kappa,\infty)}(\cdot)^{-1}$ are bounded and hence $\frac{\rm d}{{\rm d}t}\varphi_2|_{[\kappa,\infty)}(\cdot)^{-1}$ is bounded, thus there exists a Lipschitz bound $L>0$ of $\varphi_2|_{[\kappa,\infty)}(\cdot)^{-1}$. By Step 2a, v_2 is bounded and we may choose $\varepsilon>0$ small enough so that

$$\varepsilon\le\min\left\{\frac{\lambda}{2},\inf_{t\in(0,\kappa]}(\varphi_2(t)^{-1}-|v_1(t)|)\right\}$$

and

$$L\le -S+\frac{\gamma}{\tilde\gamma}\left(\frac{\tilde q_1\lambda}{2}+\frac{\lambda^2}{4\varepsilon}\right),\tag{17}$$

where

$$S=\sup_{t\in[0,\beta)}|v_2(t)|+\tfrac{|\gamma-\tilde\gamma|}{\tilde\gamma}\sup_{t\in[0,\beta)}|\zeta_2(t)|,\qquad \tilde q_1=q_1-\tfrac{\tilde\gamma}{\gamma}r_2.\tag{18}$$

We show that

$$\forall\,t\in(0,\beta):\ \varphi_2(t)^{-1}-|v_1(t)|\ge\varepsilon.\tag{19}$$

By definition of ε, (19) holds on $(0,\kappa]$. Seeking a contradiction suppose that

$$\exists\,t_1\in[\kappa,\beta):\ \varphi_2(t_1)^{-1}-|v_1(t_1)|<\varepsilon.$$

Then for $t_0:=\max\{\ t\in[\kappa,t_1)\ \mid\ \varphi_2(t)^{-1}-|v_1(t)|=\varepsilon\ \}$, we have for all $t\in[t_0,t_1]$ that

$$\begin{aligned}\varphi_2(t)^{-1}-|v_1(t)|&\le\varepsilon,\\ |v_1(t)|\ge\varphi_2(t)^{-1}-\varepsilon&\ge\lambda-\varepsilon\ge\tfrac{\lambda}{2},\\ k_2(t)=\tfrac{1}{1-\varphi_2(t)^2v_1(t)^2}&\ge\tfrac{1}{2\varepsilon\varphi_2(t)}\ge\tfrac{\lambda}{2\varepsilon}.\end{aligned}\tag{20}$$

Now we calculate, for all $t\in[t_0,t_1]$,

$$\begin{aligned}\tfrac12\tfrac{\rm d}{{\rm d}t}v_1(t)^2 &\overset{(16)}{=} v_1(t)\left(v_2(t)-\tfrac{\gamma}{\tilde\gamma}(\tilde q_1+k_2(t))v_1(t)+\tfrac{\gamma-\tilde\gamma}{\tilde\gamma}\zeta_2(t)\right)\\ &\overset{(18)}{\le} -\tfrac{\gamma}{\tilde\gamma}(\tilde q_1+k_2(t))v_1(t)^2+S|v_1(t)|\\ &\overset{(20)}{\le} -\tfrac{\gamma}{\tilde\gamma}\left(\tfrac{\tilde q_1\lambda}{2}+\tfrac{\lambda^2}{4\varepsilon}\right)|v_1(t)|+S|v_1(t)|\overset{(17)}{\le}-L|v_1(t)|.\end{aligned}$$

Therefore, using $\frac12\frac{\rm d}{{\rm d}t}v_1(t)^2=|v_1(t)|\frac{\rm d}{{\rm d}t}|v_1(t)|$, and that $|v_1(t)|>0$ for all $t\in[t_0,t_1]$, we find that

$$\begin{aligned}|v_1(t_1)|-|v_1(t_0)| &= \int_{t_0}^{t_1}\tfrac12|v_1(t)|^{-1}\tfrac{\rm d}{{\rm d}t}v_1(t)^2\,{\rm d}t\\ &\le -L(t_1-t_0)\le -|\varphi_2(t_1)^{-1}-\varphi_2(t_0)^{-1}|\\ &\le\varphi_2(t_1)^{-1}-\varphi_2(t_0)^{-1},\end{aligned}$$

and hence

$$\varepsilon=\varphi_2(t_0)^{-1}-|v_1(t_0)|\le\varphi_2(t_1)^{-1}-|v_1(t_1)|<\varepsilon,$$

a contradiction. As a consequence, (19) holds and this implies boundedness of k_2.

Step 2c: We show $\beta=\infty$ (not requiring boundedness of ζ_1,ζ_2). Seeking a contradiction, assume that $\beta<\infty$. Then ζ_1 and ζ_2 are bounded on $[0,\beta)$ and hence v_1, v_2, η and k_2 are bounded by Steps 2a and 2b. Therefore, it follows that the closure of the graph of the solution (v_1,v_2,η) is a compact subset of $\tilde{\mathscr{D}}$, a contradiction, thus $\beta=\infty$.

Step 2d: We show that the interconnection of (13) with (6) can be reformulated in the form (15). By Steps 2a–2c the operator $T:\mathscr{C}([0,\infty)\to\mathbb{R})^2\to\mathscr{L}^\infty_{\rm loc}(\mathbb{R}_{\ge0}\to\mathbb{R}^{n+1})$ is well-defined. Furthermore, it clearly has properties b) and c) as defined in Sec. I-D and property a) is an immediate consequence of Step 2a. Differentiating the first equation in (6), we may write the interconnection (13), (6) in the form

$$\begin{aligned}\ddot{z}_1(t) = \tilde{\gamma}u(t) + \big(q_2 + p_2 k_2(t)\big)v_1(t) + 2p_1 k_2(t)^2 \varphi_2(t)\dot{\varphi}_2(t)v_1(t)^3 \\ + \Big(2p_1 k_2(t)^2 \varphi_2(t)^2 v_1(t)^2 + q_1 + p_1 k_2(t)\Big) \times \\ \times \Big(v_2(t) - \tfrac{\gamma}{\tilde{\gamma}}\Big(q_1 - \tfrac{\tilde{\gamma}}{\gamma} r_2 + p_1 k_2(t)\Big) v_1(t) + \tfrac{\gamma-\tilde{\gamma}}{\tilde{\gamma}} \dot{z}_1(t)\Big),\end{aligned}$$

and hence it takes the form (15) for some appropriate function $f \in \mathscr{C}(\mathbb{R}^2 \times \mathbb{R}^{n+2} \to \mathbb{R})$ and

$$d := \begin{pmatrix} \varphi_2 \\ \dot{\varphi}_2 \end{pmatrix} \in \mathscr{L}^\infty\Big(\mathbb{R}_{\geq 0} \to \mathbb{R}^2\Big).$$

Step 3: We show (i)–(iii). From Theorem I.1 we may conclude that the application of the control

$$\begin{aligned} u(t) &= -k_0(t)^2\big(z_1(t) - y_{\text{ref}}(t)\big) - k_1(t)\big(\dot{z}_1(t) - \dot{y}_{\text{ref}}(t)\big) + u_d(t), \\ k_0(t) &= \tfrac{\varphi_0(t)}{1-\varphi_0(t)|z_1(t)-y_{\text{ref}}(t)|}, \\ k_1(t) &= \tfrac{\varphi_1(t)}{1-\varphi_1(t)|\dot{z}_1(t)-\dot{y}_{\text{ref}}(t)|} \end{aligned} \tag{21}$$

to the system (15) yields a closed-loop system where every solution z_1 can be extended to a global solution, the signals $z_1, \dot{z}_1, k_0$ and k_1 are bounded and the first two conditions in (11) are satisfied.

In particular, the unique maximal solution $(y, \dot{y}, \eta, z_1, z_2)$ of (14) obtained in Step 1 constitutes a maximal solution of (15), (21) by observing that

$$T(z_1, \dot{z}_1) = \big(\dot{z}_1, y - z_1, \dot{y} - r_2(y - z_1) - z_2, \eta^\top, k_2\big)^\top.$$

Therefore, $\omega = \infty$ and $z_1, \dot{z}_1, k_0$ and k_1 are bounded, and by invoking Steps 2a and 2b it follows that v_1, v_2 and k_2 are bounded. This implies boundedness of z_2 and hence of $\dot{y}$. We have thus shown (i) and (ii), and (iii) follows from the boundedness of k_0, k_1 and k_2 which completes the proof of the theorem. □

Remark IV.2. We stress that the original control objective as stated in Sec. I-C was prescribed transient behavior of the tracking error $e(t) = y(t) - y_{\text{ref}}(t)$. The funnel controller (10) is indeed able to achieve this: Given $\varphi \in \Phi$ with the aim that $(t, e(t)) \in \mathscr{F}_\varphi$ for all $t \geq 0$, we may set $\varphi_0 = \varphi_2 = 2\varphi$ and choose $\varphi_1 \in \Phi$ such that $(\varphi_0, \varphi_1) \in \Phi_2$. By Theorem IV.1 an application of the funnel controller (10) yields the error evolution (12) and we calculate

$$|e(t)| \leq \varphi_0(t)^{-1} + \varphi_2(t)^{-1} - \varepsilon_0 - \varepsilon_2 = \varphi(t)^{-1} - \varepsilon_0 - \varepsilon_2,$$

thus $e(t)$ evolves uniformly within the funnel $\mathscr{F}_\varphi$.

Remark IV.3. We discuss some extensions of Theorem IV.1.

(i) It is a straightforward modification of the proof of Theorem IV.1 to show that its statement remains valid when a nonlinear perturbation affects system (1). More precise, we may consider the nonlinearly perturbed system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + bu(t) + \Delta\big(t, x(t)\big), \qquad x(0) = x^0, \\ y(t) &= cx(t) \end{aligned} \tag{22}$$

where, additionally to (A1) and (A2), we assume that the perturbation Δ satisfies

(A3) $\Delta \in \mathscr{C}(\mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n)$ is locally Lipschitz continuous w.r.t. x and there exists $\vartheta \in \mathscr{C}(\mathbb{R} \to \mathbb{R}_{\geq 0})$ such that

$$\forall (t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n : \ \|\Delta(t, x)\| \leq \vartheta(cx).$$

Tracking in the presence of perturbations has been studied in [27] for relative degree one systems and in [10], [13], [28] for systems of arbitrary relative degree. As discussed before, in the latter works the control law requires derivatives of the output and/or a complicated backstepping procedure.

(ii) As shown in [12], [14] the equation for $u(t)$ in the controller (4) can be modified such that

$$u(t) = -k_0(t)^2 e(t) - k_0(t)k_1(t)\dot{e}(t) + u_d(t)$$

and Theorem I.1 is still true; in [12], [14] this is shown for a certain class of linear systems, but the extension to nonlinear systems (5) is straightforward. As a consequence, a careful inspection of the proof of Theorem IV.1 reveals that it is still true when we modify $u(t)$ in (10) to

$$\begin{aligned} u(t) = &-k_0(t)^2\big(z_1(t) - y_{\text{ref}}(t)\big) \\ &- k_0(t)k_1(t)\big(\dot{z}_1(t) - \dot{y}_{\text{ref}}(t)\big) + u_d(t). \end{aligned} \tag{23}$$

The modification (23) is advantageous compared to (4), since the latter yields a badly damped closed-loop system response and may lead to admissibility problems in applications since speed measurement is usually very noisy; for more details see [12], [14].

V. Simulations

We illustrate Theorem IV.1 and compare our controller to the funnel controller proposed in [10]. To this end, we consider the same situation as in [10]: The controller is applied to a controlled pendulum modelled by the nonlinearly perturbed relative degree two system

$$\ddot{y}(t) + a \sin y(t) = bu(t), \tag{24}$$

with parameters $a, b \in \mathbb{R}$, $b \neq 0$. For the simulation, the parameters are chosen as $a = 1/2$, $b = 1$, the initial values as $y(0) = 0, \dot{y}(0) = 0$ and the reference trajectory is $y_{\text{ref}}(t) = (1/2)\cos t$. Obviously, the system can be reformulated in the form (22), cf. also [10], and satisfies the assumptions (A1)–(A3). We use the controller (10) with the modification (23) as discussed in Remark IV.3, and choose the funnel functions

$$\begin{aligned} \varphi_0(t) = \varphi_2(t) &= \begin{cases} 20(1 - (0.1t - 1)^2), & 0 \leq t < 10, \\ 20, & t \geq 10, \end{cases} \\ \varphi_1(t) &= \big(e^{-t} + 1\big)^{-1}. \end{aligned}$$

This guarantees that the tracking error remains in the same funnel as suggested in [10]; in particular, a tracking accuracy of $|e(t)| < 0.1$ is guaranteed for $t \geq 10$. Furthermore, we choose $\tilde{\gamma} = 2 (\neq \gamma = b)$ and $q_1 = q_2 = p_1 = 1$, $p_2 = 1/3$ which satisfy (8). Remark IV.3 together with Theorem IV.1 yields that the application of the controller (10) with the modification (23) to the system (24) is feasible. We compare the simulation to that of the controller in [10].

The simulation of the controller (10) with the modification (23) applied to (24) over the time interval $[0, 20]$ has been performed in MATLAB (solver: `ode45`, rel. tol.: 10^{-14}, abs. tol.: 10^{-10}) and is depicted in Fig. 4. Fig. 4a shows the tracking error, while Fig. 4b shows the input function generated by the controller. The corresponding gain functions are depicted in Fig. 4c. It can be seen that our proposed funnel controller requires much less input action than the controller in [10] when compared to [10, Fig. 3 & 4] and provides an excellent performance.

VI. Conclusion

In the present paper we have proposed a new dynamic funnel controller for tracking of linear minimum phase single-input, single-output systems (1) with relative degree two. Our controller is based on the funnel pre-compensator from [24] combined with a proportional-derivative funnel controller from [4] or [14]; we stress that in the resulting dynamic controller (10) no derivative feedback is involved,

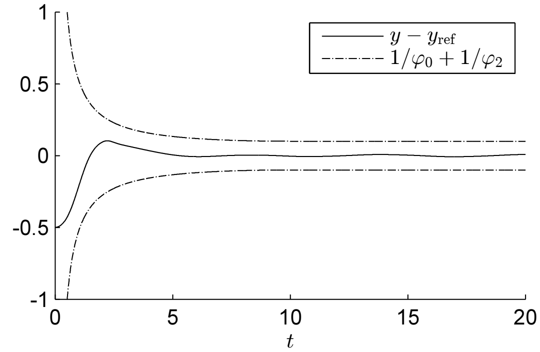

Fig. 4a: Funnel and tracking error

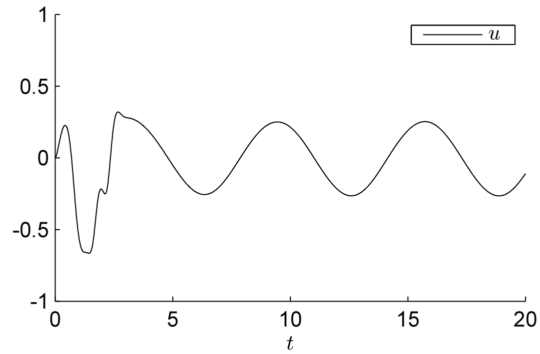

Fig. 4b: Input function

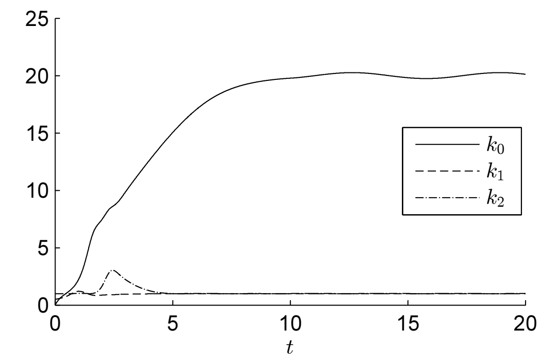

Fig. 4c: Gain functions

Fig. 4: Simulation of the controller (10) with the modification (23) for the system (24).

i.e., derivatives of the output of (1) are not used by (10). The controller (10) achieves, for a given sufficiently smooth reference signal, that the tracking error evolves within a prespecified performance funnel. Furthermore, no knowledge of the system parameters is required for the controller design.

We have shown that feasibility of the funnel controller (10) is not limited to linear systems; a straightforward extension to certain nonlinearly perturbed systems is possible. The extension of the proposed controller methodology to more general classes of nonlinear systems and systems with higher relative degree, based on the recent results obtained in [16], is the topic of future research.

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