Additive Combinatorics 2 – 1st problem set
Summer 2021
Christian Reiher

---

1. Prove that $G(4) \geqslant 16$, i.e., that there are arbitrarily large natural numbers which cannot be represented as a sum of 15 biquadrates.

2. Let $k \geqslant 2$. Prove $G(k) \geqslant k+1$, i.e., that there exist infinitely many natural numbers not belonging to the set
$$A_k = \{x_1^k + \dots + x_k^k \colon x_1, \dots, x_k \in \mathbb{N}_0\}\,.$$

3. Suppose that $t \geqslant 2$ is an integer and that n is odd. Prove that there exists an integer i such that $0 \leqslant i < 2^{t-2}$ and either $n \equiv 5^i \pmod{2^t}$ or $n \equiv -5^i \pmod{2^t}$.

4. Hardy and Littlewood conjectured that for every $k \geqslant 2$ and every $\varepsilon > 0$ there exists some $n_0 = n_0(k, \varepsilon)$ such that for every $n \geqslant n_0$ there are at most n^ε solutions $(x_1, \dots, x_k) \in \mathbb{N}_0^k$ of the equation $n = x_1^k + \dots + x_k^k$. Prove that this is false for $k = 3$.

   Hint: $(1 - 9x^3)^3 + (3x - 9x^4)^3 + (9x^4)^3 = 1$

Discussion on Tuesday, April 20

Hints

1. Prove $x^4 \equiv 0, 1 \pmod{16}$ for every integer x.
2. In fact, there is $\varepsilon > 0$ such that $|A_k \cap [n]| < (1 - \varepsilon)n$ holds for every sufficiently large n.
3. Determine the largest power of 2 dividing $5^{2^m} - 1$.