Diplomarbeit

On Moduli for Toric Sheaves on Weighted Projective Spaces

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1 Introduction

1.1 Motivation

Starting with a toric variety X the almost trivial observation that multiplication by a torus element $t \in T$ provides an isomorphism $t\colon X \to X$ gives rise to a natural and interesting question for sheaves on X. Given a coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on X we may ask whether the pull back via the morphism t induces an isomorphism $t^*\mathcal{F} \to \mathcal{F}$. A sheaf having this and some additional, technical properties is called a toric sheaf.

The topic of this thesis is to describe some toric sheaves and the vector bundles among them on weighted projective spaces $\mathbb{P}(Q)$. Our main tool to do so is the theorem of Beilinson which gives us - first only on $\mathbb{P}^n$ - bounded resolutions of the desired sheaves, more precisely we obtain the sheaf as the cohomology of the complex, and substantially reduce moduli problems to problems of linear algebra. We will see that the building blocks of these resolutions inherit a toric structure and we only have to care about the correct choice of morphisms between them. So, knowing when a matrix is toric as a morphism, actually means knowing how to describe toric sheaves via the Beilinson monads. Since the problems of linear algebra, which are known to be pretty hard, do not reduce as much as one might hope in the case of the much smaller subclass of toric sheaves, we can unfortunately only consider some special cases. We show that the moduli space of semistable rank two sheaves on the projective plane with vanishing first Chern class contains only 6 toric bundles. Moreover we explicitly describe the toric sheaves occurring in the boundary of the compactified moduli space in the case where the second Chern class is not greater than three.

To extend the approach described so far to weighted projective spaces we proceed as follows. We recall that there is a finite toric morphism $\pi\colon \mathbb{P}^n \to \mathbb{P}(Q)$ for all weighted projective spaces, fix discrete invariants of a coherent sheaf $\mathcal{F}$ on $\mathbb{P}(Q)$, such as Chern classes and rank and pull the sheaf back. We use the theorem of Beilinson to obtain a finite resolution, try to simplify things using the Hirzebruch-Riemann-Roch theorem and Serre duality and finally push the whole resolution down to $\mathbb{P}(Q)$ again. Of course there are some technical problems involved, for example we have to show how the invariants behave under pull back, that the morphism π does not destroy the structure of the sequence and that this procedure really gives us a resolution of $\mathcal{F}$. Having shown all this we can go on as in the case of $\mathbb{P}^n$ with the difference that the morphisms can in general not be described that easy anymore and it is combinatorially much harder to classify the sheaves, since we get different possible resolutions by pushing the classical Beilinson sequence down and the complexity depends on how complicated the weight Q is.

1.2 Overview

The thesis consists of four parts. The first chapter develops the basics needed for the other chapters. We begin by giving a short introduction to the theory of toric geometry, in particular the for our purposes very important construction of Cox which enables us to work with coordinates. Then we define the category of toric sheaves and state first properties. After having done this we review certain topics of intersection theory, such as the description of Chow rings via Minkowski weights. Since most of the papers about weighted projective spaces do not even mention the very useful toric structure, we will develop the theory in chapter two from a more toric point of view, including properties of sheaves and morphisms we need to talk about moduli spaces later on.

In chapter three we will reprove the theorem of Beilinson on $\mathbb{P}^n$ as well as a version on $\mathbb{P}(Q)$ and give examples of concrete resolutions. In order to do so we recall parts of the theory of derived categories and functors which are essential to state the theorems.

The last chapter splits up in two parts. In the first part we give characterizations for a morphism between structure and differential sheaves to be toric. In the second part we review basic properties from the theory of moduli space and finally apply the developed theory and give examples of moduli for toric sheaves on $\mathbb{P}^2$ and $\mathbb{P}(1,1,2)$.

1.3 Conventions

We assume familiarity with basic algebraic geometry and commutative algebra as treated in [Har77] and [Eis96]. By k we will denote an algebraically closed field, $k^* = k \setminus \{0\}$ is the group of units and rings are commutative with 1. By a scheme we will always mean a separated algebraic scheme over k, i.e. irreducible and of finite type over k. Neglecting the additional structure of a scheme, the reader might instead also think of those objects as classical varieties without loosing much. Whenever needed, an algebraic variety or simply variety will then just be a reduced scheme in our sense.

2 Basic Tools

This chapter should be thought of as help for the reader. The topics were chosen having two criteria in mind: First of all they are too important for the understanding of the thesis to be left out and second, they are not contained or not sufficiently explained in the standard textbooks of Hartshorne [Har77] and Eisenbud [Eis96]. So this basics chapter should enable the reader, having those books at hand, to read through the thesis without any problems.
We start by introducing the theory of toric varieties.

2.1 Toric Varieties

A toric variety can be defined as an irreducible variety X over some algebraically closed ground field k, which contains an algebraic torus T as a Zariski open and therefore dense subset acting on X, with the additional property that this action is an extension of the action of T on itself. Let's make this precise:

Definition 2.1. Let X be an irreducible n-dimensional normal variety over k with a torus $T \cong (k^*)^n$ acting on it via σ: $T \times X \to X$ extending the torus multiplication μ: $T \times T \to T$. We call X a **toric variety** if it contains an open, dense and torus invariant subset $X_0 \cong T$, such that the following diagram commutes

$$\begin{tikzcd}
T \times X \arrow[r, "\sigma"] \arrow[d] & X \arrow[d] \\
T \times X_0 \arrow[r, "\sigma|_{X_0}"] \arrow[d, "\cong"'] & X_0 \arrow[d, "\cong"] \\
T \times T \arrow[r, "\mu"] & T.
\end{tikzcd}$$

This way of introducing toric varieties might be the fastest, but has at the same time the disadvantage that it hides the fruitful relations to convex geometry which allows us to compute several properties of the varieties explicitly. Besides this toric geometry works best when we assume our varieties to be normal and we will see in the next section why this is true. Therefore we give a rather detailed introduction which is intended to motivate the definitions and give insight to the structure and behavior of toric varieties.
For further discussions we refer to the books of Ewald [Ewa96], Fulton [Ful93] and Oda [Oda88], as well as to the papers of Danilov [Dan78]and Cox [Cox].

Tori and Lattices

We start with basic notions from the the theory of linear algebraic groups to explain where the name torus comes from.

Definition 2.2. An **algebraic group** G is a variety which carries the additional structure of a group, such that the two maps

$$\begin{aligned} \sigma\colon G \times G &\longrightarrow G \\ (g,h) &\longmapsto \sigma(g,h) =: g.h \end{aligned} \qquad\qquad \begin{aligned} \iota\colon G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned}$$

are morphisms of varieties.
If the variety is affine, we call G a **linear algebraic group**.

Remark. The name linear algebraic group stems from the fact that each such group is isomorphic to a closed subgroup of the general linear group $GL_n(k)$ for some $n \in \mathbb{N}$. See for example [Spr98] or [Bor91] .

Definition 2.3. We call a linear algebraic group G **diagonizable** if it is isomorphic to a closed subgroup of the group D_n of diagonal $n \times n$ matrices over k with non-vanishing determinant for some $n \in \mathbb{N}$.
If G is isomorphic to D_n itself, we call it an n-dimensional **algebraic torus**, or simply a **torus**.

Remark. The name torus stems from the fact that the role those tori play here is similar to that of topological tori, i.e. finite products of the group S^1, in the theory of Lie groups.

Definition 2.4. A **morphism of algebraic groups** is a group homomorphism and a morphism of algebraic varieties. For any two algebraic groups G and H we denote by

$$\mathrm{Hom}(G,H) := \mathrm{Hom}_{alg.gr.}(G,H)$$

the set of morphism from G to H, which naturally has a group structure.

There are two special classes of morphisms associated to a single algebraic group, that are very important for toric varieties:

Definition 2.5. Let G be an algebraic group. We call

$$\chi(G) := \mathrm{Hom}(G,k^*)$$

the group of rational characters or simply the **character group** of G and

$$\Lambda(G) := \mathrm{Hom}(k^*,G)$$

the **one-parameter subgroup** of G.

Let T be an n-dimensional torus and fix an isomorphism $T \cong (k^*)^n$. This induces isomorphisms

$$\begin{aligned} \chi(T) &\cong \mathrm{Hom}(T,k^*) \cong \mathrm{Hom}((k^*)^n,k^*) \cong \mathbb{Z}^n \\ \text{and } \Lambda(T) &\cong \mathrm{Hom}(k^*,T) \cong \mathrm{Hom}(k^*,(k^*)^n) \cong \mathbb{Z}^n. \end{aligned}$$

With such a basis it is easy to see that every $m \in \chi(T)$, isomorphic to $(m_1,\dots,m_n) \in \mathbb{Z}^n$, sends an element $t \in T$, isomorphic to $(t_1,\dots,t_n) \in (k^*)^n$, to $t^m := t_1^{m_1} \dots t_n^{m_n}$ which we call a **Laurent monomial**. Clearly those monomials lie in the ring $k[t_1,t_1^{-1}\dots,t_n,t_n^{-1}]$. On the other hand if $u \in \Lambda(T)$ is an element isomorphic to $u_1,\dots,u_n$ then the corresponding one parameter map is given by sending $t \in k^*$ to $(t^{u_1},\dots,t^{u_n})$. Moreover we see that choosing a basis of T is in fact equivalent to choosing a basis for $\chi(T)$ or $\Lambda(T)$.

Definition 2.6. A free abelian group of finite rank is called a **lattice**.

We will see in the next section that the whole business of toric geometry is based on the fact that we can perfectly relate algebraic tori T, which act "properly" on a scheme, to the combinatorics of the corresponding lattices $\chi(T)$ and $\Lambda(T)$.
Before we go on, we should say a few words about group actions on schemes.

Definition 2.7. Let X be a scheme and G be an algebraic group. We say that G acts on X via α or there is a **group action** α of G on X if there is a morphism

$$\begin{aligned} \alpha\colon\ & G \times X \to X \\ & (g,x) \mapsto g.x := \alpha(g,x) \end{aligned}$$

such that for the neutral element e of G and for all x in X we have $e.x = x$ and for all g and h in G the equality $g.(h.x) = (g.h).x$ holds.

Let now X and X' be schemes on which G, respectively G', act on via α, respectively α', then we can introduce another useful notion:

Definition 2.8. A pair of morphisms

$$(\phi,\psi)\colon\ G \times X \to G' \times X',$$

where ϕ additionally is a group homomorphism, i.e. $\phi \in \mathrm{Hom}(G,G')$, is called an **equivariant morphism** with respect to G and G' if it is compatible with the actions of α and α', i.e.

$$\alpha' \circ (\phi \times \psi) = \psi \circ \alpha.$$

Example 2.9. Let $T = G$ be a torus acting on varieties X and X' via α and α'. Then a pair (ϕ, ψ): $G \times X \to G' \times X'$, is equivariant if for every element $t \in T$ and all $x \in X$ the following equation holds

$$(\alpha' \circ (\phi \times \psi))(t, x) = \phi(t).\psi(x) = \psi(t.x) = (\psi \circ \alpha)(t, x).$$

Note in particular that if we choose ψ to be multiplication by some $t \in T$, i.e. $\psi = \alpha(t, \_)$, we get an equivariant automorphism (id, ψ).

Affine Toric Varieties: Cones

The following notation is more or less canonically used in books on toric varieties and we will keep this notation throughout this chapter.

Notation. For every torus T of dimension n we denote its character group by M and its one-parameter subgroup by N. Both groups are lattices and, after fixing a basis, isomorphic to $\mathbb{Z}^n$. Recall that for every isomorphism $T \cong (k^*)^n$ we have induced isomorphisms for the lattices $M \cong \mathbb{Z}^n \cong N$.

Note that M and N are dual to each other, i.e. $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ and $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. Consequently, there exists a natural bilinear pairing

$$M \times N \to \mathbb{Z} \qquad (m, n) \mapsto m(n) =: \langle m, n \rangle,$$

which reduces to the usual dot product after choosing a basis, respectively a dual basis, for N and M, whence the notation. Thus N and M not only determine each other, but also provide an additional structure allowing us to introduce the concept of convexity, since for $N_{\mathbb{R}} := N \otimes \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes \mathbb{R}$ this pairing extends to an $\mathbb{R}$-bilinear pairing

$$M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}.$$

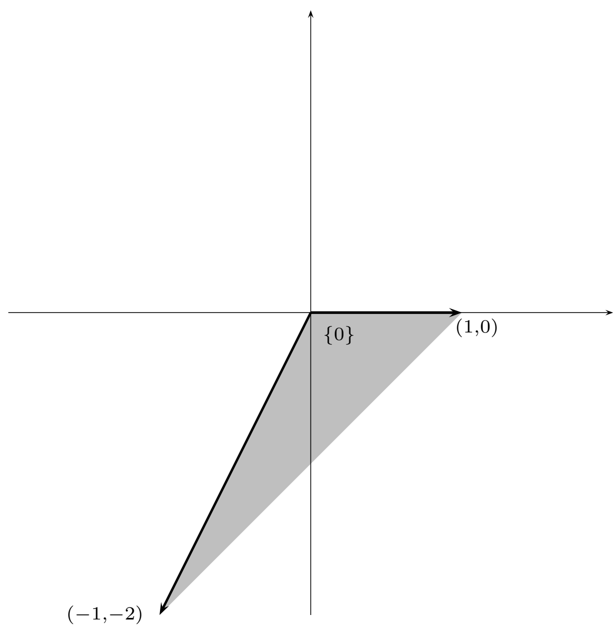
The next definitions and lemmata can be formulated in a more general setting (see for example [Ewa96] or [Oda88] for a general treatment of convex geometry), but we will restrict ourselves to the properties of the lattices N and M.

Definition 2.10. A subset $\sigma \subset N_{\mathbb{R}}$ is called a **convex rational polyhedral cone** if there is a finite number of lattice points $s_1, \ldots, s_r \in N \cong \mathbb{Z}^n$ such that

$$\sigma = \mathrm{Cone}(s_1, \ldots, s_r) := \left\{ \sum_{i=1}^{r} \lambda_i s_i | \lambda_i \in \mathbb{R}_{\geq 0} \right\}.$$

Conversely we say that σ is **generated** by $s_1, \ldots, s_r$ and we put $\mathrm{Cone}(\varnothing) := \{0\}$. Such a cone σ is called **strongly convex** if it moreover contains no positive dimensional subvectorspace of $N_{\mathbb{R}}$, that is iff $\sigma \cap (-\sigma) = \{0\}$. We will call a convex rational polyhedral cone by abuse of notation a **cone**.
We define the dimension of σ to be the dimension of the smallest subspace containing it, i.e. $\dim_{\mathbb{R}}(\sigma \otimes \mathbb{R})$.

Example 2.11. The cone spanned by the vectors $(1, 0)$ and $(-1, -2)$ can be pictured as follows



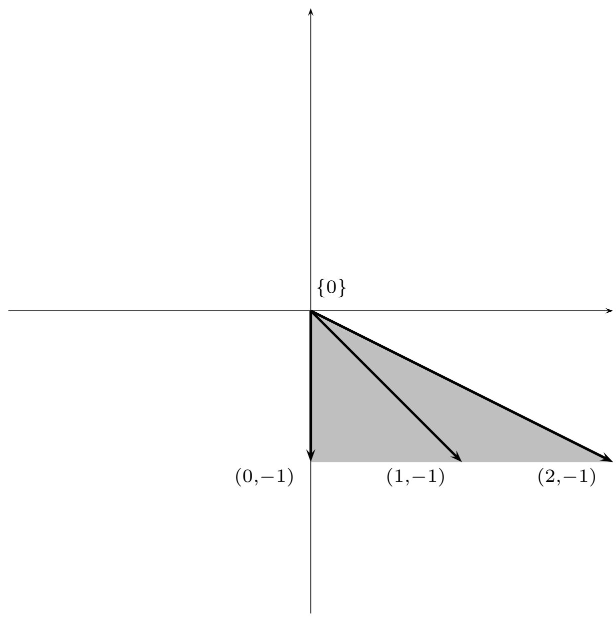
Remark. The expression "rational" used for cones stems from an alternative definition of the vectors s_i. One could also define the s_i to be points in $N_{\mathbb{R}}$ with the extra condition that all coefficients in those vectors are rational, i.e. elements of $\mathbb{Q}$. But then there are multiples s_i' of s_i with integer coefficients, i.e. $s_i' \in N$ such that $\mathrm{Cone}(s_1, \ldots, s_r) = \mathrm{Cone}(s_1', \ldots, s_r')$.Thus the two definitions are equivalent and we use the easier one. Note moreover that the cones we defined are in fact convex in the sense that for each $x, y \in \sigma$ and $\lambda \in [0,1]$ the vector $\lambda x + (1-\lambda)y$ is also an element of σ.

Definition 2.12. Let σ be a cone in N, then we define the **dual cone of** σ to be the set

$$\sigma^\vee := \{u \in M_{\mathbb{R}} | \langle u, v\rangle \geq 0, \forall v \in \sigma\}.$$

This set is, as the name suggests, again a convex rational polyhedral cone, see e.g. [Ful93, p. 11].

Example 2.13. The dual of the cone from example 2.11 has the three generators $(0,-1)$, $(1,-1)$ and $(2,-1)$.



Notation. For every cone $\sigma \in N_{\mathbb{R}}$ we denote the set $\sigma^\vee \cap M$ by S_σ.

Note that the zero cone, which is by definition contained in every cone, has as dual the whole space $M_{\mathbb{R}}$ and the integral lattice points are just the points of M corresponding to the torus (as character group), i.e $S_{\{0\}} = M$.

Lemma 2.14 (Gordan's lemma)**.** *Let σ be a cone. Then S_σ is a finitely generated subsemigroup of M.*

Proof. Let $s_1, \dots, s_r$ be the generators in $\sigma^\vee$, which we by the remark can choose to be in $\sigma^\vee \cap M$. Consider the, with respect to the classical topology in $\mathbb{R}^n$, compact set $K := \{\sum_{i=1}^r \lambda_i s_i | 0 \leq \lambda_i \leq 1\}$. We claim that the finite set $K \cap M$ generates $\sigma^\vee \cap M$. Let therefore $u = \sum_{i=1}^r \mu_i s_i \in \sigma^\vee \cap M$ be arbitrary, i.e $\mu_i \geq 0$. Since we can write $\mu_i = n_i + \lambda_i$ for $n_i \in \mathbb{N}$ and $0 \leq t_i \leq 1$ it is clear that

$$u = \sum_{i=1}^r (n_i + \lambda_i) \cdot s_i = \sum_{i=1}^r n_i s_i + \sum_{i=1}^r \lambda_i s_i \in \langle K \cap M \rangle_{\mathbb{Z}_{\geq 0}},$$

which finishes the proof. □

Definition 2.15. The **semigroup algebra $\mathbf{k}[\mathbf{S}_\sigma]$ associated to $\mathbf{S}_\sigma$** is defined as the the k-vector space generated by elements of S_σ.

Note that by Gordan's lemma we find finitely many elements $m_1, \dots, m_r$ that form a basis of this finite-dimensional vector space. Since the lattice M can be identified with the character group of the torus we usually say that $m \in M$ corresponds to the character $\chi^m \in \mathrm{Hom}(T, k^*)$. Furthermore since M has an additive structure and we think of the χ^m as monomials we will write

$$\chi^m \cdot \chi^{m'} = \chi^{m+m'}$$

to imitate the exponential rule. Note that by definition every toric variety contains a torus as a dense open subset, thus each χ^m can be seen as a rational function on this variety.

Definition 2.16. Let $\sigma \subset N_{\mathbb{R}}$ be a strongly convex cone. Then we call

$$X_\sigma := \text{m-Spec}(k[S_\sigma])$$

the **affine toric variety associated to** σ.

Example 2.17.

1. Let $\sigma = \{0\}$ then by what we have seen before choosing a basis $(t_1, \dots, t_n)$ of M, which induces an isomorphism $M \cong \mathbb{Z}^n$, gives

$$\begin{aligned} X_\sigma = \text{m-Spec}(k[S_\sigma]) &= \text{m-Spec}(k[M]) \\ &= \text{m-Spec}(k[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]) \cong (k^*)^n. \end{aligned}$$

2. The dual of the cone σ generated by the vectors $(1, 0)$ and $(0, 1)$ is σ itself. Thus the corresponding affine toric variety

$$X_\sigma = \text{m-Spec}(k[S_\sigma]) = \text{m-Spec}(k[t_1, t_2]) \cong k^2$$

is the 2-dimensional affine space over k.

3. The cone from example 2.11 generated by $(1, 0)$ and $(-1, -2)$ needs some more calculation. The three generators $(1, -1)$, $(0, -1)$ and $(2, -1)$ of the the dual cone correspond to the Laurent monomials

$$t_1 \cdot t_2^{-1} \qquad t_2^{-1} \qquad t_1^2 \cdot t_2^{-1}.$$

Introducing new variables $x := t_1 \cdot t_2^{-1}$, $y := t_2^{-1}$ and $z := t_1^2 \cdot t_2^{-1}$ we see that there the relation $x^2 - yz$ and we obtain that

$$X_\sigma = V(x^2 - yz)$$

is just a cone in k^3.

4. Let σ be generated by the vectors $(0,1)$ and $(-1,-2)$. Then $\sigma^\vee$ is spanned by $(-1,0)$ and $(-2,1)$ corresponding to the linearly independent Laurent monomials t_1^{-1} and $t_1^{-2} \cdot t_2$. Hence we see that
$$X(\sigma) = \text{m-Spec}(k[S_\sigma]) \cong k^2.$$

The following theorem is very important, it particularly makes our notation consistent.

Theorem 2.18. *The affine toric variety associated to a cone is a toric variety in the sense of definition 2.1, in particular it is normal. Conversely any toric variety which is also affine is isomorphic to an affine toric variety associated to a cone.*

Proof. [Cox, Theorem 1.13] $\square$

Thus we see that subsemigroups of N respectively M and affine toric varieties correspond to each other. This fact carries over to arbitrary toric varieties and suitable collections of cones. Before we can define these collections we need the notions of half spaces and faces, which all come from convex geometry.

Definition 2.19. For each $\varnothing \neq u \in M$ we define

$$H_u := \{\nu \in N_\mathbb{R} | \langle u, \nu \rangle \geq 0\}$$

to be the **half space** associated to u. Moreover by ∂H_u we denote the boundary of this hyperplane, i.e.
$$\partial H_u := \{\nu \in N_\mathbb{R} | \langle u, \nu \rangle = 0\}$$

Lemma 2.20. *Let $\sigma = Cone(s_1, \ldots, s_r) \subset N_\mathbb{R}$ be a cone. Then we have $\sigma^{\vee\vee} = \sigma$ and $\sigma^\vee = \cap_{i=1}^r H_{s_i}$. In particular we see that every cone is the intersection of finitely many half spaces.*

Proof. See [Ful93, p. 9 and p.11] for a proof. $\square$

Definition 2.21. Let σ be a cone in $N_\mathbb{R}$ and $u \in M \setminus \{0\}$ such that $\sigma \subset H_u$. Then we call $\tau := \sigma \cap \partial H_u$ a **face of** σ. If $\tau \neq \sigma$ we call τ a **proper face** and in order to have a partial order relation, we regard every σ as a face of itself and we write $\tau \prec \sigma$ whenever τ is a face of σ. If $\dim(\tau) = \dim(\sigma) - 1$ we call τ a **facet**.

Lemma 2.22. *Let σ and τ be cones. Then*

- *all faces of σ are cones.*
- *If σ and τ have non-empty intersection, then $S_{\sigma\cap\tau} = S_\sigma + S_\tau$.*
- *If τ is a face of σ, then there is an $m \in S_\sigma$ such that $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-m)$ as semigroup algebras.*

Proof. [Ful93, p.10, p.14 Proposition 3 and Proposition 2 on p.13] $\square$

Note that $\tau \prec \sigma$ gives an inclusion in reversed order for the semigroup rings, i.e. $k[S_\sigma] \subset k[S_\tau]$. The last lemma now implies that this can be expressed more precisely by localisation, namely

$$k[S_\sigma] \subset k[S_\sigma]_{\chi^m} = k[S_\tau].$$

Considering the maximal spectra of both algebras then gives $X_\tau \subset X_\sigma$ for the varieties.

Remark. The last fact might clarify why we prefer the geometry of the lattice N to that of M and therefore the maybe confusing habit to start with cones in N, although we primarily use the dual cones in M to construct toric varieties. It would suffice to take a cone in M, but then we would loose the quite intuitive relation

$$\tau \prec \sigma \Rightarrow X_\tau \subset X_\sigma.$$

Definition 2.23. A cone is called **smooth** if its minimal generators form part of a $\mathbb{Z}$-basis of N. We call a cone **simpicial** if its set of minimal generators are linearly independent.

Example 2.24. The cone spanned by the rays $(1,0)$ and $(0,1)$ clearly is smooth and the corresponding affine toric variety k^2 is smooth. On the other hand the variety otained from the rays $(1,0)$ and $(1,-2)$ is just a cone in k^3, singular in the zero point and we see that this cone is not smooth, since the vector $(0,-2)$ is not primitive. Both cones are simplicial, since the generators are linearly independent.

This example can be generalized and we will do so in the next section.

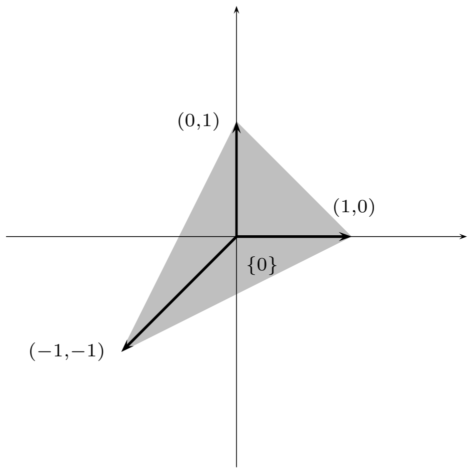
General Toric Varieties: Fans

Definition 2.25. Let Δ be a finite collection of convex rational polyhedral cones. We call Δ a **fan** if for any $\sigma \in \Delta$ every face $\tau \subset \sigma$ also belongs to Δ and for two cones $\sigma, \tau \in \Delta$ the common intersection $\sigma \cap \tau$ is a face of each cone.

The last condition in fact means that the fan is closed with respect to taking intersections, since common faces are by the first condition again elements of the fan. As we have seen before we get for any two cones $\sigma, \tau \in \Delta$ two Zariski open embeddings $X_{\sigma\cap\tau} \to X_\sigma$ and $X_{\sigma\cap\tau} \to X_\tau$ with isomorphic images, which leads to the following

Definition 2.26. Let Δ be a fan of cones in $N_{\mathbb{R}}$. We define the **toric variety associated to Δ** to be the variety obtained by glueing the affine toric varieties associated to the cones in Δ along their common intersection, as for example described in [Har77, p.75].

Example 2.27. The fan of $\mathbb{P}^2$ looks as follows.



It is easily checked that all cones σ_i give the variety k^2 and glue together in the usual way to $\mathbb{P}^2$.

The next two theorems assert, analogously to the affine case, that in fact we get neither something new nor are there more toric varieties than those obtained from fans.

Theorem 2.28. *Let $X(\Delta)$ be the toric variety associated to the fan Δ. Then $X(\Delta)$ is a separated toric variety in the sense of definition 2.1.*

Proof. We give a proof of this theorem in order to give a slight impression how things work for the affine case, whose proof we skipped.
By definition every fan contains the cone $\{0\}$. The dual of the zero cone is the whole lattice M and thus $X_{\{0\}} = T$. Hence the torus is densely contained in $X(\Delta)$, which is therefore irreducible. Since for each $\sigma \in \Delta$ the same torus T acts on the affine parts X_σ by theorem 2.18 and the gluing obviously respects this action (the gluing is defined on the intersections, which are T-invariant). Moreover U_σ is normal and normality is a local property, so $X(\Delta)$ is a toric variety. Separated means that the diagonal morphism of $X(\Delta)$ has a closed image. Because of the gluing, this can be

checked locally by showing that $X_{\sigma\cap\tau} \to X_\sigma \times X_\tau$ is closed for all $\sigma, \tau \in \Delta$ or equivalently that on the level of the semigroup algebras $k[S_\sigma] \otimes_k k[S_\tau] \to k[S_{\sigma\cap\tau}]$ is surjective, but this stems for the fact that $S_{\sigma\cap\tau} = S_\sigma + S_\tau$ by lemma 2.22. □

Theorem 2.29. *All toric varieties X arise from a fan.*

Proof. [GK73, Chapter I.2, Theorem 5] □

Therefore definition 2.1 and definition 2.26 are equivalent and in the following we will write $X = X(\Delta)$ for a toric variety X, since there always has to be a fan Δ that X corresponds to.

Definition 2.30. Let Δ be a fan. We call $|\Delta| := \bigcup_{\sigma\in\Delta} \sigma$ its **support** and say that Δ is **complete** if the support is the whole space, i.e. $|\Delta| = N_\mathbb{R}$. Moreover we define

$$\Delta(r) := \{\sigma \in \Delta | \sigma \text{ is } r\text{-dimensional}\}$$

to be the set of cones of dimension r and call the elements of $\Delta(1)$ the **rays** of Δ.

Theorem 2.31. *Let $X = X(\Delta)$ be a toric variety. Then*

- *X is complete iff Δ is complete.*
- *X is smooth iff all cones in Δ are smooth, i.e. their minimal generators form part of a $\mathbb{Z}$-basis.*
- *X is Cohen-Macauley.*

Proof. [Cox, Theorem 2.3] □

Remark. We needed a lot of theory to handle the normality hypothesis in the definition of a toric variety, which we also could have dropped. Then we would arrive at a bigger class of varieties, eventually not arising from a fan. For example the cusp $V(x^2 - y^3)$, which is not normal, would be a toric, since $T := (k^*)$ acts on this variety by $t \mapsto (t^2, t^3)$.
Moreover it is possible to allow a fan to consist of infinitely many cones, but then we would also obtain schemes, which are not of finite type over k. Besides this varieties arising from infinite fans loose the first property of the theorem. i.e. they are in general not complete, if the fan is.

Morphisms

We have seen in the last section what a T-equivariant morphism between two toric varieties is. Now that we have the description by the fans it would be nice to read of this property from the lattices.

Definition 2.32. Let Δ_1 be a fan in $N^1 \cong \mathbb{Z}^{n_1}$ and Δ_2 a fan in $N^2 \cong \mathbb{Z}^{n_2}$. A $\mathbb{Z}$-linear map $\pi\colon N^1 \to N^2$ is called a **map of fans** if for the induced map $\overline{\pi}\colon N^1_\mathbb{R} \to N^2_\mathbb{R}$ there is a cone $\sigma_2 \in \Delta_2$ for every $\sigma_1 \in \Delta_1$ such that

$$\overline{\pi}(\sigma_1) \subset \sigma_2,$$

i.e. every cone is mapped into a cone.

Keeping the notation from this definition we have the following

Theorem 2.33. *Every map of fans $\pi\colon N^1 \to N^2$ induces a T-equivariant morphism $\pi_*\colon X(\Delta_1) \to X(\Delta_2)$. Conversely every toric morphism f from $X(\Delta_1)$ to $X(\Delta_2)$ gives rise to a map π of fans such that $\pi_* = f$.*

Proof. See [Oda88, Theorem 1.13] □

2.2 Cox coordinates

In this section we will recall from [Cox] that a toric variety $X = X(\Delta)$ has coordinates corresponding to the one dimensional cones of Δ and that X can be represented as quotient of another toric variety. In order to do so we need to state some basic notions from

Geometric Invariant Theory

The definitions made in this section can be found in a more general setting in [Mum65], but since we only apply them in the case of toric varieties it suffices to use weaker versions.

Definition 2.34. Let X be a variety and G an affine algebraic group acting on X via the morphism α. A pair (Y, φ), where $\varphi\colon X \to Y$ is a morphism of varieties, is called a **categorial quotient** of X by G, if

1. the diagram

$$\begin{tikzcd} G \times X \arrow[r, "\alpha"] \arrow[d, "p_2"'] & X \arrow[d, "\varphi"] \\ X \arrow[r, "\varphi"'] & Y, \end{tikzcd}$$

   where p_2 denotes the second projection, commutes and

2. any other pair (Y', φ') making up such a diagram induces a unique morphism $\psi\colon Y \to Y'$ such that φ' factors through ψ, i.e. $\varphi' = \psi \circ \varphi$ and so the following diagram commutes

$$\begin{tikzcd} G \times X \arrow[r, "\alpha"] \arrow[d, "p_2"'] & X \arrow[d, "\varphi"] \arrow[ddr, bend left, "\varphi'"] & \\ X \arrow[r, "\varphi"'] \arrow[drr, bend right, "\varphi'"'] & Y \arrow[dr, dotted, "\psi"] & \\ & & Y'. \end{tikzcd}$$

Keeping the notation, we define three more types of quotients:

Definition 2.35. The pair (Y, φ) is called a **good quotient**, if the first condition from the last definition is satisfied, i.e. $\varphi \circ \alpha = \varphi \circ p_2$ and moreover the following are true

1. φ is surjective and affine,
2. $U \subset Y$ is open if and only if $\varphi^{-1}(U) \subset X$ is open,
3. $\mathcal{O}_Y \cong (\varphi_* \mathcal{O}_X)^G$ and
4. if W is an invariant closed subset of X, then $\varphi(W)$ is a closed subset of Y. Moreover, if W_1 and W_2 are disjoint invariant closed subsets of X, then $\varphi(W_1) \cap \varphi(W_2) = \varnothing$.

(Y, φ) is called a **geometric quotient** if the geometric fibres of φ are the orbits of the geometric points of X.

Definition 2.36. We call a categorial, respectively a geometric quotient (Y, φ) **universal**, if for any morphism $\sigma\colon Y' \to Y$, the pair $(X \times_Y Y', \varphi')$, where

$$\varphi\colon X \times_Y Y' \to Y'$$

denotes the second projection, is also a categorial, respectively a geometric quotient (note that G acts naturally on the fibre product).

Quotients in toric geometry

Note that given a toric variety $X = X(\Delta)$ every $\rho \in \Delta(1)$ corresponds to an irreducible T-invariant Weil-divisor D_ρ, see [Cox95]. In [Ful93] it was shown that those divisors generate the group of all T-invariant Weil divisors and we therefore have

$$\mathrm{WDiv}(X(\Delta))^T = \left\{ \bigoplus_{\rho\in\Delta(1)} a_\rho \cdot D_\rho | a_\rho \in \mathbb{Z} \right\} \cong \mathbb{Z}^{\Delta(1)}$$

and one might ask what the vanishing order of rational functions along those divisors are.

Lemma 2.37. *1. The vanishing order of an element $m \in M$ along D_ρ, considered as rational function χ^m, is given by:*

$$ord_{D_\rho}(\chi^m) = \langle m, n_\rho \rangle$$

where n_ρ denotes the unique generator of ρ in N.

2. The divisor associated to χ^m is given by

$$div(\chi^m) = \sum_{\rho\in\Delta(1)} \langle m, n_\rho \rangle D_\rho$$

Proof. [Ful93, Section 3.4] □

If we denote by $A_{n-1}(X(\Delta))$ the Chow group of Weil divisors modulo rational equivalence we get a complex by composing the map which takes a divisor to its class with the map from the last lemma. Moreover, denoting the invariant Cartier divisors of X by $\mathrm{CDiv}(X)^T$ and its divisor class group by $\mathrm{Pic}(X)$, we have the following

Theorem 2.38. *There is a commutative diagram with exact rows*

$$\begin{tikzcd}
M \arrow[r] \arrow[d, equal] & CDiv(X)^T \arrow[r] \arrow[d, hook] & Pic(X) \arrow[r] \arrow[d, hook] & 0 \\
M \arrow[r] & \bigoplus_{\rho\in\Delta(1)} \mathbb{Z}\cdot D_\rho \arrow[r, "deg_X"'] & A_{n-1}(X) \arrow[r] & 0
\end{tikzcd}$$

where deg_X takes a divisor to its class and the maps on the left are given by the last lemma. Moreover those maps are injective iff $\Delta(1)$ spans $N_\mathbb{R}$.

Proof. [Ful93, Section 3.4] □

Definition 2.39. The **homogeneous coordinate ring** $S := S(X(\Delta))$ of a toric variety $X = X(\Delta)$ is defined by

$$S := k[x_\rho | \rho \in \Delta(1)].$$

Note that a monomial $\prod_\rho x_\rho^{a_\rho}$ in S corresponds to the divisor $D := \sum_\rho a_\rho D_\rho$, so there is a natural grading for S given by

$$\deg(\prod_\rho x_\rho^{a_\rho}) := \deg_X(D) \in A_{n-1}(X(\Delta)).$$

Definition 2.40. We call the grading of

$$S = \bigoplus_{\alpha\in A_{n-1}(X(\Delta))} S_\alpha = \bigoplus_{\alpha\in A_{n-1}(X)} \bigoplus_{\substack{a\in\mathbb{Z}^{\Delta(1)} \\ \deg_X(a)=\alpha}} S_a$$

given by $\deg_X$ the **A-grading**. There is another grading defined as follows: We give very monomial $\prod_\rho x_\rho^{a_\rho}$ the $\mathbb{Z}^{\Delta(1)}$-degree $(a_\rho)_\rho \in \mathbb{Z}^{\Delta(1)}$. We call this the **fine grading** of

$$S = \bigoplus_{a\in\mathbb{Z}^{\Delta(1)}} S_a.$$

Example 2.41. Let $X = \mathbb{P}^n$. We know that $A_{n-1}(X)$ is isomorphic to $\mathbb{Z}$, so a polynomial $\prod_\rho x_\rho^{a_\rho} \in S = S(X)$ has $\mathbb{Z}$-degree $a_{\rho_0} + \ldots + a_{\rho_n}$ which is fine with the observation $\mathbb{P}^n \cong \mathbf{Proj}k[x_0, \ldots, x_n]$ with $\deg(x_i) = 1$ from classical algebraic geometry.

Definition 2.42. Let Δ be a fan and σ a cone in Δ. Define

$$x(\sigma) := \prod_{\Delta(1) \ni \rho \not\subseteq \sigma} x_\rho$$

and call the ideal generated by all $x(\sigma)$,

$$B := \langle x(\sigma) | \sigma \in \Delta \rangle$$

the **irrelevant ideal** of the coordinate ring S.

We apply the contravariant functor $\mathrm{Hom}_\mathbb{Z}(\_, k^*)$ to the sequence

$$M \to \mathbb{Z}^{\Delta(1)} \to A_{n-1}(X) \to 0$$

and obtain

$$1 \to G := \mathrm{Hom}_\mathbb{Z}(A_{n-1}(X), k^*) \to (k^*)^{\Delta(1)} \to T,$$

where we write 1 on the left to emphasize the multiplicative structure. Thus the group G acts on the coordinate space $(k^*)^{\Delta(1)}$ and gives us the torus of X as quotient. We will see soon that this is also possible for the whole variety, but we need some more preparation.
Define $\hat{X}_\sigma := k^{\Delta(1)} \setminus V(x(\sigma))$, then it is clear that

$$\hat{X} := k^{\Delta(1)} \setminus V(B) = \bigcup_{\sigma \in \Delta} \hat{X}_\sigma.$$

This is completely analogous to the fact for the variety X

$$X = \bigcup_{\sigma \in \Delta} X_\sigma$$

which is just the union of all affine cones (as set).

Lemma 2.43. *Let $X = X(\Delta)$ be a toric variety. Then $\hat{X}$ is a toric variety too and there is a natural toric morphism $\hat{X} \to X$.*

Proof. We will construct a fan $\hat{\Delta}$ for $\hat{X}$ in the lattice $\hat{N} := \mathbb{Z}^{\Delta(1)}$. Denote by $e_\rho \in \mathbb{Z}^{\Delta(1)}$ the standard basis vectors and construct for each $\sigma \in \Delta$ the cone

$$\hat{\sigma} = \mathrm{Cone}\{e_\rho | \rho \in \sigma(1)\}.$$

It is easy to verify that the collection $\{\hat{\sigma} | \sigma \in \Delta\}$ is already a fan if and only if X is simplicial, but we can always define $\hat{\Delta}$ to be the fan generated by all such cones $\hat{\sigma}$ and their faces. For the toric morphism note that the map of the lattices $\hat{N} \to N$ given by $e_\rho \mapsto n_\rho$ maps cones of $\hat{\Delta}$ to cones of Δ by construction and therefore we obtain

$$\begin{tikzcd} \hat{X} \arrow[r, equal] \arrow[d] & \bigcup_{\sigma \in \Delta} \hat{X}_\sigma \arrow[d] \\ X \arrow[r, equal] & \bigcup_{\sigma \in \Delta} X_\sigma. \end{tikzcd}$$

□

The main result of this section now asserts the following:

Theorem 2.44. *Let $X = X(\Delta)$ be a toric variety and let $\Delta(1)$ span $N_\mathbb{R}$. Then*

1. *$X \cong (k^{\Delta(1)} \setminus V(B))/G$ is a universal categorial quotient.*
2. *X is a geometric quotient if and only if Δ is simplicial.*

Proof. [Cox95, Theorem 2.1] □

Graded modules

Starting with a toric variety $X = X(\Delta)$ it is clear that the coordinate ring of each affine part $\hat{X}_\sigma$ is just the localization of the Cox coordinate ring S by the monomial $x(\sigma)$, i.e.

$$k[\hat{X}_\sigma] = S_{x(\sigma)} = \left\{ \frac{f}{x(\sigma)^l} \middle| f \in S, l \in \mathbb{N}\} \right\}$$

which is $A_{n-1}(X)$-graded. Thus we can consider the degree 0 part of this module

$$(S_{x(\sigma)})_0 := \left\{ \frac{f}{x(\sigma)^l} \middle| f \text{ is } A_{n-1}(X) \text{ homogeneous }, l \in \mathbb{N} \text{ and } \deg(f) = l \cdot \deg(x_\sigma) \right\}$$

and have the following result.

Lemma 2.45. *Let $\sigma \in \Delta$ be a cone and $\sigma^\perp$ its dual. Then there is an isomorphism*

$$k[X_\sigma] = k[\sigma^\vee \cap M] \cong (S_{x(\sigma)})_0.$$

Proof. See [Cox95, Lemma 2.2]. □

Let now F be an A_{n-1}-graded S-module, i.e.

$$F = \bigoplus_{\alpha \in A_{n-1}(X)} F_\alpha$$

such that $S_\alpha \cdot F_\beta \subset F_{\alpha+\beta}$. Then $F_\sigma := F \otimes_S S_{x(\sigma)}$ is a graded $S_{x(\sigma)}$-module and so it makes sense to talk about the module $(F_\sigma)_0$. Applying the $\sim$-functor, as defined in [Har77, II section 1], gives us a quasi-coherent sheaf $\mathcal{F}_\sigma := (\tilde{F_\sigma})_0$ on X_σ since by the previous we know that $k[X_\sigma] = (S_{x(\sigma)})_0$.

Proposition 2.46. *The sheaves $\mathcal{F}_\sigma$ for all $\sigma \in \Delta$ patch together to give a quasi-coherent sheaf on $X = X(\Delta)$. Moreover*

- *the functor sending F to $\mathcal{F} = \tilde{F}$ is an exact functor from A_{n-1}-graded S-modules to quasi-coherent $\mathcal{O}_X$-modules.*
- *If X is simplicial then every quasi-coherent sheaf is of the form $\mathcal{F} = \tilde{F}$ for some graded F.*
- *If F is finitely generated, then $\mathcal{F}$ is coherent.*
- *If X is simplicial, then every coherent sheaf is of the form $\mathcal{F} = \tilde{F}$.*

Proof. [Cox95, Proposition 3.1, Theorem 3.2 and Proposition 3.3] □

Let $X = X(\Delta)$ be a toric variety and D a torus invariant Weil divisor, i.e. a Weil divisor whose components consist only of the T-invariant codimension 1 subvarieties D_ρ for $\rho \in \Delta(1)$. If we denote by X_{reg} the set of regular points i: $X_{reg} \to X$ the inclusion into X we can define the sheaf

$$\mathcal{O}_X(D) := i_* \mathcal{O}_{X_{reg}}(D|_{X_{reg}}).$$

where $K(X)$ is the function field of X, see [Per00].

Proposition 2.47. *Let $X = X(\Delta)$ be a toric variety and let $D = \sum_{\rho \in \Delta(1)} a_\rho D_\rho$ and $E = \sum_{\rho \in \Delta(1)} b_\rho D_\rho$ be toric divisors.*

- *There is an isomorphism*

  $$\Phi_D : \ S_\alpha \to H^0(X, \mathcal{O}_X(D)),$$

  where $\alpha = deg_X(D) \in A_{n-1}(X)$ denotes the class of D.

- *Let moreover* $\beta = deg_X(E)$ *denote the class of* E*, then there is a commutative diagram*

$$\begin{tikzcd} S_\alpha \otimes S_\beta \arrow[r] \arrow[d, "\Phi_D"'] & S_{\alpha+\beta} \arrow[d, "\Phi_{D+E}"] \\ H^0(X, \mathcal{O}_X(D)) \otimes H^0(X, \mathcal{O}_X(E)) \arrow[r] & H^0(X, \mathcal{O}_X(D+E)) \end{tikzcd}$$

  where the top morphism is multiplication and the one on the bottom tensor product.

- $\mathcal{O}_X(D) \cong \mathcal{O}_X(\alpha)$*, so the above defined sheaves can be identified with twisted structure sheaves.*

Proof. See [Cox95, Proposition 1.2. and Proposition 3.1] for the first two statements and [Per00] for the third statement. □

2.3 Toric Sheaves

Let X be a scheme of finite type over k and let G be a linear algebraic group over k acting on X via

$$\sigma\colon G \times X \to X.$$

Moreover denote by $\alpha\colon G \times G \to G$ the group multiplication an by

$$G \times X \xrightarrow{p_2} X$$
$$G \times (G \times X) \xrightarrow{p_{23}} G \times X$$

the natural projections. With these notations at hand we can define what we mean by equivariance for sheaves.

Definition 2.48. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is called **G-equivariant** or **G-linearized**, if there is an isomorphism of sheaves

$$\sigma^*\mathcal{F} \xrightarrow[\approx]{\Phi} p_2^*\mathcal{F}$$

such that the following cocycle condition holds:

The following diagram is a commutative diagram of isomorphisms

$$\begin{tikzcd}
(\sigma \circ (\mu \times \mathrm{id}))^*\mathcal{F} \arrow[r, "\approx"'] \arrow[d, "(\mu\times\mathrm{id})^*\Phi"', "\approx"] & (\sigma \circ (\mathrm{id} \times \sigma))^*\mathcal{F} \arrow[r, "(\mu\times\mathrm{id})^*\Phi", "\approx"'] & (p_2 \circ (\mathrm{id} \times \sigma))^*\mathcal{F} \arrow[d, "\approx"'] \\
(p_2 \circ (\mu \times \mathrm{id}))^*\mathcal{F} \arrow[r, "\approx"] & (p_2 \circ p_{23})^*\mathcal{F} & (\sigma \circ p_{23})^*\mathcal{F} \arrow[l, "\approx"', "p_{23}^*\Phi"]
\end{tikzcd}$$

Notation. In some cases it is reasonable to call the pair $(\mathcal{F}, \Phi)$ a G-equivariant sheaf to emphasize on the given isomorphism, but in cases the linearization is obvious or simply not important in the context, we will usually abbreviate this notation by $\mathcal{F}$.

The idea behind the definition lies in the observation that any closed point $g \in G$ induces an isomorphism $g\colon X \to X$ given by multiplication and it is an interesting question whether there is an induced isomorphism on the level of sheaves $\Phi_g\colon g^*\mathcal{F} \to \mathcal{F}$ or not. To be more precise the question is, denoting the embedding $X \hookrightarrow G \times X$ given by $x \mapsto (g, x)$ by i_g, if there is a Φ_g making the following diagram commutative

$$\begin{tikzcd}
i_g^*\sigma^*\mathcal{F} \arrow[r, "i_g^*\Phi"] \arrow[d, "\approx"'] & i_g^*p_2^*\mathcal{F} \arrow[d, "\approx"] \\
g^*\mathcal{F} \arrow[r, "\Phi_g"'] & \mathcal{F}
\end{tikzcd}.$$

If $\mathcal{F}$ is a G-equivariant sheaf, such a family $\{\Phi_g\}_{g\in G}$ always exists, but it is in general too much to expect that a given family in turn glues together to a well defined isomorphism Φ with the mentioned cocycle condition. However, given a global Φ, one sees that this cocycle condition reduces on the level of the Φ_g to the commutativity of the diagram

$$\begin{tikzcd}
g_2^*g_1^*\mathcal{F} \arrow[r, "g_2^*\Phi_{g_2}"] \arrow[d, "\approx"'] & g_2^*\mathcal{F} \arrow[d, "\Phi_{g_2}"] \\
(g_1g_2)^*\mathcal{F} \arrow[r, "\Phi_{g_1g_2}"'] & \mathcal{F}
\end{tikzcd}$$

for all closed points $g_1, g_2 \in G$. Besides this our definition of Φ fits with the general definitions from geometric invariant theory from the book [Mum65].

We want to define the category of equivariant sheaves, so we have to say what an equivariant morphism is:

Definition 2.49. Let $(\mathcal{E}, \Psi)$ and $(\mathcal{F}, \Phi)$ be two G-equivariant sheaves on a scheme X. We call a morphism $f\colon \mathcal{E} \to \mathcal{F}$ **G-equivariant**, if it commutes with the given linearizations, i.e. if the square

$$\begin{tikzcd} \sigma^*\mathcal{E} \arrow[r, "\sigma^* f"] \arrow[d, "\Psi"'] & \sigma^*\mathcal{F} \arrow[d, "\Phi"] \\ p_2^*\mathcal{E} \arrow[r, "p_2^* f"'] & p_2^*\mathcal{F} \end{tikzcd}$$

commutes.

Again, we see that for any $g \in G$ there is an induced commutative diagram given by

$$\begin{tikzcd} g^*\mathcal{E} \arrow[r, "g^* f"] \arrow[d, "\Psi"'] & g^*\mathcal{F} \arrow[d, "\Phi"] \\ \mathcal{E} \arrow[r, "p_2^* f"'] & \mathcal{F} \end{tikzcd}.$$

Definition 2.50. Let G be a linear algebraic group acting on a scheme X, then we define G-equ(X) to be the category of **G**-equivariant sheaves, i.e. the category whose objects consist of coherent, G-equivariant sheaves of $\mathcal{O}_X$-modules on X and G-equivariant morphisms as morphisms.

Notation. In the special case where $G = T$ is the torus of a toric variety $X = X(\Delta)$, we will call T-equ(X) the category of **toric sheaves** and accordingly speak of toric sheaves and morphisms.

Remark. In [Per00] it was shown that G-equ(X) is an abelian category, by showing that it admits kernels and cokernels in a natural way.

With the above notation we have an identification

$$\sigma^* D = \sum_{Y \subset X} n_Y [T \times Y] = p_2^* D.$$

Thus the associated reflexive sheaf $\mathcal{O}_X(D)$ inherits a canonical linearization Ψ given by

$$\begin{tikzcd} \sigma^*\mathcal{O}_X(D) \arrow[r, "\Psi"] \arrow[d, "\cong"'] & p_2^*\mathcal{O}_X(D) \arrow[d, "\cong"] \\ \mathcal{O}_{T\times X}(\sigma^*(D)) \arrow[r, equal] & \mathcal{O}_{T\times X}(p_2^* D). \end{tikzcd}$$

Toric Sheaves and Graded Modules

Recall from the proposition 2.46 that the exact $\sim$-functor send every $A_{n-1}(X)$-graded module F to a quasi-coherent sheaf $\mathcal{F}$. So far we did not discuss the fine grading the modules F might have. The surprising result is the following:

Proposition 2.51. *The sheaf $\mathcal{F} = \tilde{F}$ for an arbitrary graded S-module F is T-equivariant if F was already fine graded. The equivariance is then canonically given.*

Proof. We denote the action of the torus on X by θ and by $p_2\colon X \times X \to X$ the second projection. A fine grading of F induces a fine grading on each of the F_σ, i.e. for each $a \in Z^{\Delta(1)}$ the a-th graded part is given by

$$F_\sigma^a = \left\{ \frac{f}{x(\sigma)^l} \middle| f_b \in F_b \text{ for } b \in \mathbb{Z}^{\Delta(1)} \text{ such that } b - l \cdot \exp(\sigma) = a \right\}$$

where $\exp(\sigma)$ denotes the exponent of the monomial $x(\sigma)$. Note that the $\mathbb{Z}^{\Delta(1)}$-grading of F_σ gives us an action

$$\begin{aligned} \psi\colon\ F_\sigma &\to F_\sigma \otimes_k k[M] \\ s &\mapsto s \otimes \chi(a) \end{aligned}$$

for every $\mathbb{Z}^{\Delta(1)}$-homogeneous element s of degree a. Using this we can locally define a morphism

$$\phi_\sigma\colon\ F_\sigma \otimes^{\theta}_{S_\sigma} S_\sigma \otimes_k k[M] \to F_\sigma \otimes^{p_2}_{S_\sigma} S_\sigma \otimes_k k[M] \cong F_\sigma \otimes_k k[M]$$

by the formula

$$\begin{aligned} \phi_\sigma(s \otimes \chi(a') \otimes \chi(a'')) :&= \psi(s) \cdot (\chi(a') \otimes \chi(a'')) \\ &= (s \otimes 1 \otimes \chi(a)) \cdot (\chi(a') \otimes \chi(a'')) \\ &= s \otimes \chi(a') \otimes \chi(a + a''), \end{aligned}$$

for each element s of degree a. Here the symbols $\otimes^\theta$ and $\otimes^{p_2}$ indicate the S_σ-module structures on $S_\sigma \otimes_k k[M]$ induced by the maps θ and p_2, i.e. the dual actions given by $\theta^*(\chi(a)) = \chi(a) \otimes \chi(a)$ and $p_2^*(\chi(a)) = 1 \otimes \chi(a)$. The map ϕ_σ is in fact an isomorphism and so induces

$$\theta^* \widetilde{F_\sigma} \xrightarrow{\Phi_\sigma} p_2^* \widetilde{F_\sigma}.$$

These local T-equivariances patch together and give rise to a canonical toric structure on $\mathcal{F} = \tilde{F}$. See [MP] and [Per00] for the details. □

Proposition 2.52. *Let $\mathcal{F}$ be a quasi-coherent sheaf on a toric variety X. Then $\mathcal{F}$ is of the form $\tilde{F}$ for a fine graded S-module F.*

Proof. [Per00, Proposition 6.5] □

Toric resolutions

We know from the Hilbert Syzygy theorem that every coherent sheaf of $\mathcal{O}_X$-modules on a scheme X admits a finite free resolution. The next theorem asserts that the resolution for a toric sheaf on a variety $X = X(\Delta)$ can always be chosen toric, i.e. all morphisms respect the T-linearizations of the occurring terms.

Theorem 2.53. *Let $\mathcal{F}$ be a toric sheaf on X. Then there exists a finite toric resolution of $\mathcal{F}$, i.e. there are toric divisors $D_{i_j} \in \mathbb{Z}^{\Delta(1)}$ such that*

$$0 \to \bigoplus_{i_l=0}^{m_l} \mathcal{O}_X(D_{i_l}) \to \ldots \to \bigoplus_{i_1=0}^{m_1} \mathcal{O}_X(D_{i_1}) \to \mathcal{F} \to 0$$

is exact and $l \leq \#\Delta(1)$.

Proof. See [MB97, Theorem 1.1]. □

This means that our good knowledge about the sheaves $\mathcal{O}_X(D)$ for a toric divisor D is theoretically sufficient to know about the T-equivariances of arbitrary toric sheaves.

2.4 Intersection Theory

We recall in this section some notions and theorems from the famous book of Fulton [Ful84a], to be abled to use the full strength of intersection theory on toric varieties. For a shorter and probably more readable introduction we refer to [Ful84b] or [Tra]. Recall that the Chow group

$$A_*(X) = \bigoplus_{i=0}^{n} A_i(X)$$

of an n-dimensional scheme was defined as the direct sum of the groups $A_i(X)$ of i-dimensional cycles modulo rational equivalence. On a smooth scheme it is possible to put $A^i(X) := A_{n-i}(X)$ to define the Chow ring $A^*(X)$ of intersection classes, with multiplication given by the cup product. Moreover, $A_*(X)$ becomes an $A^*(X)$-module via the cap product. Since we are mainly interested in the intersection theory of weighted projective spaces and other possibly singular toric varieties, we need to extend these notions.

Chern classes

Before doing so we repeat the construction and some facts about Chern classes. Recall that for an n-dimensional variety X and an invertible sheaf $\mathcal{L}$ on X there is always a Cartier divisor D such that $\mathcal{L} \cong \mathcal{O}_X(D)$. Moreover it is true that

$$D_1 \sim D_2 \Leftrightarrow \mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2),$$

for any two Cartier divisors D_1 and D_2, hence we can establish a one-to-one correspondence between Cartier divisor classes and classes of invertible sheaves.

With that in mind we can make the following

Definition 2.54. We call $c_1(\mathcal{L}) := [D]$ the **first Chern class** of $\mathcal{L}$.

It is clear by the proposition that for any i-dimensional subvariety V of X we find a Cartier divisor C such that $\mathcal{L}_V \cong \mathcal{O}_X(C)$ for the restricted line bundle. This simply means that there is a map

$$\begin{aligned} Z_i(X) &\to A_{i-1}(V) \hookrightarrow A_{i-1}(X) \\ V &\mapsto [C] \mapsto [C], \end{aligned}$$

determined by linear extension for all $i = 0, \ldots, n$. By [Ful84a, propostion 2.5] the operator $c_1(\mathcal{L}) \cap \_\colon Z_i(X) \to A_{i-1}(X)$ obtained from composition of the above maps extends to a well defined operator

$$c_1(\mathcal{L}) \cap \_\colon A_i(X) \to A_{i-1}(X)$$

which we call the **first Chern class operator**. We will often write $C \cdot \alpha := c_1(\mathcal{L}) \cap \alpha$ for a cycle $\alpha \in Z_i(X)$ and $\mathcal{L} \cong \mathcal{O}_X(C)$.

Lemma 2.55. *We have the following properties:*

1. *For any two line bundles $\mathcal{L}$ and $\mathcal{L}'$ the corresponding Chern class operators commute, that is*

$$c_1(\mathcal{L}) \cap (c_1(\mathcal{L}') \cap \alpha) = c_1(\mathcal{L}') \cap (c_1(\mathcal{L}) \cap \alpha) \in A_{i-2}(X)$$

   for all $\alpha \in A_i(X)$.

2. *If $f\colon X' \to X$ is a proper morphism and $\mathcal{L}$ again a line bundle,then we have the* ***projection formula***

$$c_1(f^*\mathcal{L}) \cap \alpha = c_1(\mathcal{L}) \cap f^*\alpha$$

   for all $\alpha \in A_i(X)$.

3. *For two line bundles $\mathcal{L}$ and $\mathcal{L}'$ and a cycle $\alpha \in A_i(X)$ the following equation holds*

$$c_1(\mathcal{L} \otimes \mathcal{L}') \cap \alpha = c_1(\mathcal{L}) \cap \alpha + c_1(\mathcal{L}') \cap \alpha.$$

Proof. [Ful84a, propostion 2.5] □

Note that part two of the lemma can be extended to finitely many line bundles, which means that we can form a polynomial algebra of Chern operators, i.e for line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_r$ and any homogeneous polynomial $f \in \mathbb{Z}[X_0, \ldots, X_r]$ of degree d we obtain a morphism

$$f(c_1(\mathcal{L}_1), \ldots, c_1(\mathcal{L}_r))\colon A_i(X) \to A_{i-d}(X)$$

for all $i \geq d$.

The next step is to define Chern classes for bundles of higher rank, so let $\mathcal{E}$ be a rank $e+1$ bundle on X (we denote the corresponding locally sheaf by the same letter) and consider its **projective bundle**

$$p\colon \mathbb{P}(\mathcal{E}) \to X.$$

For each $x \in X$ we define

$$\mathbb{P}(\mathcal{E}_x) := \mathbb{P}(\mathcal{E}_x/m_x\mathcal{E})$$

for each $x \in X$, where $\mathcal{E}_x$ respectively m_x denote the stalk of $\mathcal{E}$ in x respectively its maximal ideal. By $p_x\colon \mathbb{P}(\mathcal{E}_x) \to \{x\}$ we denote the from p induced projection.

Lemma 2.56. *There is an exact sequence*

$$0 \to \Omega^1_{\mathbb{P}(\mathcal{E})/X}(1) \to p^*\mathcal{E}^* \to \mathcal{O}_{\mathcal{E}}(1) \to 0,$$

where $\mathcal{E}^*$ denotes the dual of $\mathcal{E}$, and on each fiber we get an Euler sequence

$$0 \to \Omega^1_{\mathbb{P}(\mathcal{E}_x)}(1) \to \mathcal{E}_x^* \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_x)} \to \mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}(1) \to 0.$$

Moreover, the sheaf $\mathcal{O}_{\mathcal{E}}(1)$, which is usually called the ***canonical line bundle****, is invertible on $\mathbb{P}(\mathcal{E})$.*

Proof. See [Ful84a, Appendix B.5] for more on such sequences, [Har77, Chapter 2.7] or better [Gro71, II.5.5.3] for a proof. □

Since $\mathcal{O}_{\mathcal{E}}(1)$ is invertible, we can form $c_1(\mathcal{O}_{\mathcal{E}}(1))$ as before. Now for every $\alpha \in A_i(X)$ we get $p^*\alpha \in A_{i+e}(X)$, since the fiber dimension of $\mathbb{P}(\mathcal{E})$ is e. So by pulling back, applying $c_1(\mathcal{O}_{\mathcal{E}}(1)) \cap \_$ in an appropriate power and pushing down again, we obtain the following intersection morphism:

$$\begin{tikzcd} A_i(X) \arrow[r, dotted, "s_j(\mathcal{E})\cap\_"] \arrow[d, "p^*"'] & A_{i-j}(X) \\ A_{i+e}(\mathbb{P}(\mathcal{E})) \arrow[r, "c_1(\mathcal{O}_{\mathcal{E}}(1))^{e+j}\cap\_"'] & A_{i-j}(\mathbb{P}(\mathcal{E})), \arrow[u, "p_*"'] \end{tikzcd}$$

thus explicitly we get the operator

$$s_j(\mathcal{E}) \cap \alpha := p_*(c_1(\mathcal{O}_{\mathcal{E}}(1))^{e+j} \cap p^*\alpha)$$

for any $\alpha \in A_i(X)$.

By setting $s_0 := 1$ we can formally consider those canonically defined classes as an invertible power series in a variable t, i.e. $\sum_{i=0}^{\infty} s_i t^i$ and define the coefficients $c_i(\mathcal{E})$ in the power series

$$(\sum_{i=0}^{\infty} s_i t^i)^{-1} = 1 + c_1(\mathcal{E})t + c_2(\mathcal{E})t^2 + \ldots$$

to be the **i-th Chern class operators** of $\mathcal{E}$. It is known that for any vector bundle $\mathcal{E}$ of rank e we have $c_i(\mathcal{E}) = 0$ for all $i > e$, thus

$$\sum_{i=0}^{\infty} c_i t^i = \sum_{i=0}^{e} c_i t^i,$$

is in fact a polynomial, which we call the **Chern polynomial** of $\mathcal{E}$. Moreover this definition agrees with the definition made before, the operators are commutative and the projection formula from 2.55 is valid for all i, that is for any proper morphism $f\colon X' \to X$ and any cycle α on X we have the equation

$$f_*(c_i(f^*\mathcal{E}) \cap \alpha) = c_i(\mathcal{E}) \cap f_*(\alpha).$$

Let X be a variety of dimension n, then by definition $[X] \in A_n(X) \cong \mathbb{Z}$ is the generating class, so by putting

$$c_i(\mathcal{E}) := c_i(\mathcal{E}) \cap [X] \in A_{n-i}(X)$$

we can think of Chern classes as elements of $A_{n-i}(X)$ and in the case when X is smooth it is true that $c_i(\mathcal{E}) \cap [X]$ determines the operator $c_i(\mathcal{E}) \cap \_$ and one can show that $A^i(X) \cong A_{n-i}(X)$ for all i, so $A^*(X) \cong A_*(X)$ and the i-th Chern class lives in $A^i(X)$. In the general case we have to deal with bivariant classes to define the groups $A^k(X)$.

Definition 2.57. Let X be a scheme and $i \in \mathbb{N}$, then we define the **i-th Chow cohomology group** $A^i(X)$ to be the collection of homomorphisms

$$c_g^{(j)}\colon A_j(X') \to A_{j-i}(X'),$$

where $g\colon X' \to X$ is a morphism of schemes and $j \in \mathbb{N}$, such that $c_g^{(j)}$ is compatible with proper push-forward, flat pull-back and intersections (see [Ful84a, Def. 17.1, p. 320 (C_1 - C_3)] for a precise definition).

The most important thing is that "the usual properties" carry over to the generalized cohomology classes, i.e. there is a cup product

$$A^i(X) \otimes A^j(X) \to A^{i+j}(X),$$

which is given from the theory of bivariant classes (see [Ful84a, Chapter 17] and a cap product

$$\begin{aligned} A^i(X) \otimes A_j(X) &\to A_{j-i}(X) \\ c \otimes \alpha &\mapsto c(\alpha) = c \cap \alpha. \end{aligned}$$

Thus $A_*(X)$ is a graded $A^*(X)$-module. Moreover the projection formula extends to this setting and for any vector bundle $\mathcal{E}$ on X there is a Chern class operator $c_i(\mathcal{E}) \in A^i(X)$ for each i, which is explicitly given by

$$c_i(\mathcal{E})(\alpha) := c_i(g^*\mathcal{E}) \cap \alpha,$$

where $g\colon X' \to X$, $\alpha \in A_j(X')$ and the operator $c_i(g^*\mathcal{E}) \cap \_$ on the right hand side is the Chern class we defined before. Thus we can describe Chow homology and cohomology groups of arbitrary schemes.
We close this section with the following

Lemma 2.58. *Let X be a complete variety, then for each $i \in \mathbb{N}$ there is a homomorphism*

$$\begin{aligned} \mathcal{D}_X\colon A^k(X) &\to \mathit{Hom}(A_k(X), \mathbb{Z}) \\ c &\mapsto (a \mapsto \mathit{deg}(c \cap a)), \end{aligned}$$

called the ***Kronecker duality homomorphism****.*

Remark. Recall that the **degree of a cycle** is a function we first only defined on zero-cycles, i.e. if X is a variety over k and $\alpha = \sum_p n_p[P] \in A_0(X)$ then we put $\deg(\alpha) = \sum_p n_p[R(P) : k]$, and then extended by zero to all other cycles to give a map deg: $A_*(X) \to \mathbb{Z}$.

Intersection Theory on Toric Varieties

In this section we describe explicitly Chow groups and rings for complete toric varieties, following [WF97], which is unfortunately full of typos. In general, for a variety X we only know that the groups $A_k(X)$ are generated by k-dimensional subvarieties and have relations given by divisors of rational functions on $(k+1)$-dimensional subvarieties. However, on toric varieties, we can describe these groups easily using the data from the fan.

Proposition 2.59. *Let $X = X(\Delta)$ be a toric variety, then*

- *$A_k(X)$ is generated by all $V(\sigma)$, i.e. by the orbit closures of all cones σ with $codim(\sigma) = k$.*
- *The relations are given by all $\tau \in \Delta(k+1)$, i.e. by*

$$[div(\chi^u)] = \sum_{\substack{\tau\subset\sigma \\ dim(\sigma)=dim(\tau)+1}} \langle u, n_{\sigma,\tau}\rangle \cdot [V(\sigma)],$$

for all $u \in M(\tau) = \tau^{\perp} \cap M$, where as usual $n_{\sigma,\tau} \in N$ is the point whose image generates N_σ/N_τ.

Proof. See [WF97, Proposition 2.1] □

Remark. It is known that for nonsingular toric varieties X the groups $A_k(X)$ are free abelian for each k. Moreover, they are never trivial, if X is additionally assumed to be complete. In the case of a nonsingular X, the Chow groups might fail to have these two properties, see for example 2.3 in [WF97]. In the following we show that for complete (singular) toric varieties the Chow ring $A^*(X)$ behaves very well and can be calculated very quickly using the fan of X.

The next theorem is somehow the crucial part of the whole section, since it simplifies the in general very complicated, relative notion of Chow rings to the computation of functionals.

Theorem 2.60. *If X is a complete toric variety, then the Kronecker duality homomorphism*

$$\mathcal{D}_X\colon\ A^k(X) \to Hom(A_k(X), \mathbb{Z})$$

is in fact an isomorphism for each $k \in \mathbb{N}$. Moreover, for each T-invariant subvariety Y, $\mathcal{D}_Y$ is an isomorphism, too.

Proof. [WF95, Theorem 3] □

Minkowski Weights

Notation. Let Δ be a fan, then we denote by $\Delta^{(k)}$ the set of cones of codimension k in Δ. Note that we write $\Delta(k)$ for the cones of dimension k.

This notation allows us to give the next definition in a more compact way.

Definition 2.61. For any integer $k \in \mathbb{N}$ and any fan Δ in a lattice N we call a function

$$c\colon\ \Delta^{(k)} \to \mathbb{Z}$$

a weight function or simply a **weight**. If a weight c moreover satisfies the following relations

$$\sum_{\substack{\sigma\in\Delta^{(k)},\tau\in\Delta^{(k+1)} \\ \tau\subset\sigma}} \langle u, n_{\sigma,\tau}\rangle \cdot c(\sigma) = 0 \quad \forall u \in M(\tau),$$

then we call it a **Minkowski weight**. We denote the set of Minkowski weights of codimension k, which has the structure of a group, by $MW^k(X)$.

Corollary 2.62. *Let $X = X(\Delta)$ be a complete toric variety. Then there is a group isomorphism*

$$A^k(X) \cong MW^k(X)$$

for all $k \in \mathbb{N}$.

Proof. From theorem 2.60 we know that every element of $A^k(X)$ can be represented by a function c on cones of codimension k, extending linearly to an element of $\mathrm{Hom}(A_k(X), \mathbb{Z})$. Such a function clearly has to vanish on all the relations given by $A_k(X)$ in order to be well-defined. So by definition c is a Minkowski weight. □

Remark. By the second part of theorem 2.60, it is clear that the statement of corollary 2.62 extends to torus invariant subvarieties of X.

Recall that for any scheme X the group $A^k(X) = A^k(X \overset{\text{id}}{\to} X)$ is given by a collection of morphisms $A_i(Y) \to A_{i-k}(Y)$ for all $i \geq k$ and all morphisms $f\colon Y \to X$. Hence we obtain a canonical homomorphism using the first Chern class operator

$$\begin{aligned} \operatorname{Pic}(X) &\to A^1(X) \\ L &\mapsto (\alpha \mapsto c_1(f^*L) \cap \alpha), \end{aligned}$$

where α is an element of $A_*(Y)$. This map is in general not an isomorphism, but we have the following result due to Brion.

Theorem 2.63. *Let $X = X(\Delta)$ be a complete toric variety, then the map*

$$Pic(X) \to A^1(X)$$

is an isomorphism.

Proof. See [Bri89] for a more general result for spherical varieties. ☐

Now that we know how to compute Chow rings, let us check how morphisms between them look like. For a given toric morphism $f\colon X(\Delta') \to X(\Delta)$, coming from a map of fans $\Psi\colon N' \to N$ which maps every cone in Δ' into a cone in Δ. We know that applying A^* is a contravariant functor, so induces a map

$$f^*\colon A^*(X(\Delta)) \to A^*(X(\Delta')).$$

It would be nice to compute this map in the case where both varieties are complete in terms of Minkowski weights. In fact, if $c \in MW^k(X(\Delta))$, then $f^*(c)$ is also a weight (maybe not Minkowski). Fortunately, if we assume f to be dominant, there is a formula to compute the pull back.

Proposition 2.64. *Let $f\colon X(\Delta') \to X(\Delta)$ be a toric morphism of complete toric varieties induced by a map $\Psi\colon N' \to N$ of fans. Furthermore let $\tau' \in \Delta'^{(k)}$ and $\Psi(\tau') \subset \tau$ be the smallest cone containing it's image. Then the pull back of a Minkowski weight c of codimension k is explicitly given by*

$$(f^*(c)(\tau') = \begin{cases} c(\tau) \cdot [N : (\Psi(N') + N_\tau)] & , codim(\tau) = k \\ 0 & , codim(\tau) \leq k \end{cases},$$

where the brackets on the right stand for the index of a subgroup in a group.

Proof. By the projection formula 2.55 we obtain

$$f_*(f^*(c) \cap [V(\tau')]) = c \cap f_*([V(\tau')]) \in A_0(X(\Delta)).$$

Now that by definition of push-forward of cycles we have the well known formula using field extensions of rational function fields $[R(V(\tau')) : R(V(\tau))]$ (which in our case can be thought of as the cardinality of a fiber of f)

$$\begin{aligned} f_*([V(\tau')]) &= \begin{cases} [R(V(\tau')) : R(V(\tau))] \cdot [V(\tau)] & , \operatorname{codim}(\tau) = k \\ 0 & , \operatorname{codim}(\tau) \leq k \end{cases} \\ &= \begin{cases} [N/N_\tau : \Psi_*(N'/N'_{\tau'})] \cdot [V(\tau)] & , \operatorname{codim}(\tau) = k \\ 0 & , \operatorname{codim}(\tau) \leq k \end{cases} \\ &= \begin{cases} [N : (\Psi(N') + N_\tau)] \cdot [V(\tau)] & , \operatorname{codim}(\tau) = k \\ 0 & , \operatorname{codim}(\tau) \leq k \end{cases}, \end{aligned}$$

where $\Psi_*\colon N'/N'_{\tau'} \to N/N_\tau$ denotes the induced map on quotients. Combining these two facts and taking degrees we obtain

$$\deg(c \cap f_*([V(\tau')])) = \begin{cases} [N : (\Psi(N') + N_\tau)] \cdot c(\tau) & , \operatorname{codim}(\tau) = k \\ 0 & , \operatorname{codim}(\tau) \leq k \end{cases}.$$

□

Now that we know how to compute the ring homomorphisms between Chow rings in some important cases, we want to compute intersection products in a comparably easy combinatorial way. To do this we need the following lemma, which turns out to be very powerful for our means.

Lemma 2.65. *Let X and Y be toric varieties. Then the Künneth map*

$$\begin{gathered} A_*(X) \otimes A_*(Y) \to A_*(X \times Y) \\ [V] \otimes [W] \mapsto [V \times W] \end{gathered}$$

is an isomorphisms.

Proof. From [Ful93, Section 1.4] we know that $V(\sigma) \times V(\tau) \cong V(\sigma \times \tau)$ for all $\sigma, \tau \in \Delta$, therefore the proof is clear by the above description of the Chow ring of a toric variety. □

Starting with a complete toric variety $X = X(\Delta)$, the trick now is to use the diagonal morphism $\delta\colon X \to X \times X$, which is usually denoted by Δ, but already used in our context. Let $\gamma \in \Delta^{(k)}$, then we apply the lemma and get the following presentation for $V(\gamma)$:

$$\delta[V(\gamma)] = [\delta V(\gamma)] = \sum_{\substack{\sigma \in \Delta^{(i)}, \tau \in \Delta^{(k-i)} \\ \gamma \subset \sigma, \gamma \subset \tau}} m^\gamma_{\sigma,\tau} \cdot [V_\sigma \times V_\tau] \in A_k(V(\gamma) \times V(\gamma)).$$

Definition 2.66. Let $N \cong \mathbb{Z}^n$ be a lattice and $\Delta \subset N$ be a fan defining an n-dimensional toric variety. Then for any saturated d-dimensional sublattice $L \subset N$ and any $v \in N$ we define

$$\Delta(v) := \{\sigma \in \Delta | L_{\mathbb{R}} + v \text{ meets } \sigma \text{ in exactly one point}\}.$$

Clearly, we have $\dim(\sigma) \leq n - d$ for all σ in $\Delta(v)$ and we say that v is a **generic lattice point** if equality holds, i.e. if $\dim(\sigma) = n - d$ for all $\sigma \in \Delta(v)$.

With those notations we can formulate the following theorem, which is called the fan displacement rule:

Theorem 2.67 (fan displacement rule)**.** *As before let $X = X(\Delta)$ be complete, $\gamma \in \Delta$ and v be a generic lattice point of N, then the coefficients in the formula*

$$\delta[V(\gamma)] = [\delta V(\gamma)] = \sum_{\substack{\sigma \in \Delta^{(i)}, \tau \in \Delta^{(k-i)} \\ \gamma \subset \sigma, \gamma \subset \tau}} m^\gamma_{\sigma,\tau} \cdot [V_\sigma \times V_\tau] \in A_k(V(\gamma) \times V(\gamma))$$

are given by

$$m^\gamma_{\sigma,\tau} = \begin{cases} [N : (N_\sigma + N_\tau)] & , \textit{if } \sigma \textit{ meets } \tau + v \\ 0 & \textit{else} \end{cases}.$$

Proof. [WF97, theorem 4.2] □

The next proposition shows that knowing the coefficients $m^\gamma_{\sigma,\tau}$, which are in general not unique, is sufficient for the calculation of cap and cup products. Thus the fan displacement rule is a strong tool for the computation of intersection products and the whole intersection theory of a complete toric variety is determined by the $m^\gamma_{\sigma,\tau}$.

Proposition 2.68. *Let $X = X(\Delta)$ be a complete toric variety and identify the Chow ring of X with the set of Minkowski weights. Then the following equations hold true:*

- *If $c \in MW^p(X)$ and $\gamma \in \Delta^{(k)}$, then the cap product is*

$$c \cap [V(\gamma)] = \sum_{\substack{\sigma \in \Delta^{(p)}, \tau \in \Delta^{(k-p)} \\ \gamma \subset \sigma, \gamma \subset \tau}} m^{\gamma}_{\sigma,\tau} \cdot c(\sigma) \cdot [V(\tau)] \in A_{k-p}(X).$$

- *If $c \in MW^p(X)$ and $\tilde{c} \in MW^q(X)$ then the cup product is given by*

$$(c \cup \tilde{c}) = \sum_{\substack{\sigma \in \Delta^{(p)}, \tau \in \Delta^{(q)} \\ \gamma \subset \sigma, \gamma \subset \tau}} m^{\gamma}_{\sigma,\tau} \cdot c(\sigma) \cdot \tilde{c}(\tau) \in MW^{p+q}(X),$$

  for every $\gamma \in \Delta^{(p+q)}$.

Proof. [WF95, 3, theorem 4] □

After having introduced weighted projective spaces in the next chapter we will compute the Chow groups and rings, as well as the homomorphisms between them for a special class of weighted projective planes.

3 Weighted Projective Spaces

There are many papers, including [Dol82], [MB86], [Del75] and [Mor75], to name the most important sources, giving a completely non-toric introduction to the theory of weighted projective spaces. In this chapter we will review some important properties presented in those papers and supplement them from a toric point of view.

3.1 Alternative Descriptions and First Properties

In this section we give several different descriptions of weighted projective spaces and deduce some properties. We begin with the

Definition 3.1. Let $Q = (q_0, \dots, q_n) \in \mathbb{Z}^{n+1}$ with $\gcd(q_0, \dots, q_n) = 1$. We define the **weighted projective space** with respect to the weight Q to be

$$\mathbb{P}(Q) := \mathbb{P}(q_0, \dots, q_n) = (k^{n+1} \setminus \{0\})/\sim$$

where $(a_0, \dots, a_n) \sim (b_0, \dots, b_n)$ if there is a $\lambda \in k^*$ such that $(a_0, \dots, a_n) = (\lambda^{q_0} b_0, \dots, \lambda^{q_n} b_n)$.

We will now give four equivalent

Descriptions

- The space $\mathbb{P}(Q)$ is a toric variety for every Q. This can be seen by letting the torus

  $$T = (k^*)^{n+1}/\sim$$

  with the same relation as above act on $\mathbb{P}(Q)$ via the well defined map

  $$(\langle t_0, \dots, t_n \rangle, \langle a_0, \dots, a_n \rangle) \mapsto \langle t_0^{q_0} a_0, \dots, t_n^{q_n} a_n \rangle$$

  which embeds T as an open and dense subset of $\mathbb{P}(Q)$.
  The fan of $\mathbb{P}(Q)$ can be constructed as follows. Let e_i be the standard basis vectors of $\mathbb{Z}^{n+1}$ and denote by $v_i := \overline{e_i}$ their image in

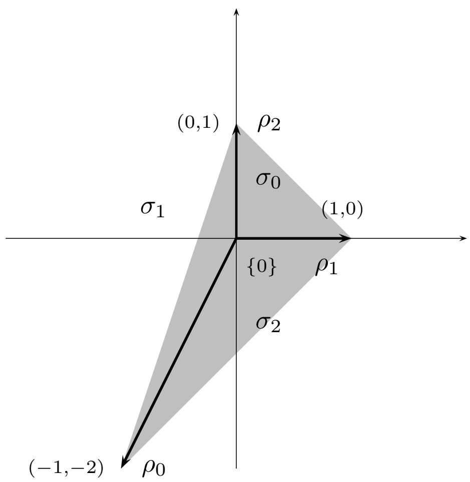
  $$N := \mathbb{Z}^{n+1}/\mathbb{Z} \cdot (q_0, \dots, q_n) \cong \mathbb{Z}^n.$$

  By construction we have the relation

  $$\sum_{i=0}^{n} q_i v_i = 0$$

  and it can be shown that the fan consisting of all cones generated by proper subsets of $\{v_0, \dots, v_n\}$ corresponds to the variety $\mathbb{P}(Q)$ (see [Cox]). Since this fan is complete and simplicial, we conclude that weighted projective spaces are complete simplicial toric varieties. Note that we defined the fan of $\mathbb{P}(Q)$ in $N := \mathbb{Z}^{n+1}/\mathbb{Z}(q_0, \dots, q_n)$ which is isomorphic to $\mathbb{Z}^n$, but it is in general not clear how the rays v_i are embedded. However in examples this can mostly be done.

  Example 3.2. The embedded fan of the weighted projective plane $\mathbb{P}(1, 1, 2)$ in $N = \mathbb{Z}^2$ looks as follows.



where the ρ_i are primitive vectors that clearly satisfy the relation $\sum_{i=0}^{2} q_i\rho_i = 0$. In example 2.17 we explicitly computed the affine toric varieties associated to the σ_i and saw that $X_{\sigma_0} = X_{\sigma_1} = k^2$ and $X_{\sigma_2} = V(x^2 - yz)$ is a cone in k^3. Since $\mathbb{P}(1,1,2)$ is complete, we see that this weighted projective plane can be seen as the compactification of this cone.

- We can also describe $\mathbb{P}(Q)$ as quotient obtained from the homogeneous Cox coordinate ring. So far we know $N = \mathbb{Z}^{n+1}/\mathbb{Z}(q_0,\ldots,q_n)$ and hence $M = \mathrm{Hom}(N,\mathbb{Z}) \cong \mathbb{Z}(q_0,\ldots,q_n)^{\perp}$. Thus, using that $A_{n-1}(\mathbb{P}(Q)) \cong \mathbb{Z}$ (see for example [Cox]), the short exact sequence from the last chapter
$$0 \to M \longrightarrow \bigoplus_{\rho\in\Delta(1)} \mathbb{Z}\cdot D_\rho \xrightarrow{\deg_{\mathbb{P}(Q)}} A_{n-1}(\mathbb{P}(Q)) \to 0$$
reduces to
$$0 \to \mathbb{Z}(q_0,\ldots,q_n)^{\perp} \longrightarrow \mathbb{Z}^{n+1} \xrightarrow{\deg_{\mathbb{P}(Q)}} \mathbb{Z} \to 0,$$
where the first map is given by inclusion and $\deg_{\mathbb{P}(Q)}$ has to be dot product by the vector $(q_0,\ldots,q_n)$ to make the sequence exact. Therefore the homogeneous coordinate ring $S' := S(Q) := S(\mathbb{P}(Q))$ is just $k[y_0,\ldots,y_n]$ where y_i has the $\mathbb{Z}$-grading $\deg_{\mathbb{P}(Q)}(y_i) = q_i$. The irrelevant ideal is $B = \langle x_{\rho_0},\ldots,x_{\rho_n}\rangle$, thus $V(B)$ is just $0 \in k^{n+1}$. Moreover
$$G = \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(\mathbb{P}(Q)), k^*) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, k^*) \cong k^*$$
is embedded in $k^{\Delta(1)} \cong k^{n+1}$ via the dual map of $\deg_{\mathbb{P}(Q)}$, which is, as we said before, scalar product by the weight vector $(q_0,\ldots,q_n)$. Therefore the image of G in k^{n+1}, which we also denote by G is just
$$G = \{(\lambda^{q_0},\ldots,\lambda^{q_n}) | \lambda \in k^*\}.$$
Thus we recover by
$$(k^{n+1} \setminus V(B))/G \cong (k^{n+1} \setminus \{0\})/k^*$$
the above definition of the weighted projective space $\mathbb{P}(Q)$. Using the coordinates from this construction, we have the following

Example 3.3. Let y_0, y_1 and y_2 be the coordinates of $X = \mathbb{P}(1,1,2)$. It is clear that they have $\mathbb{Z}$-degree $1,1$ and 2 respectively. We claim that there is an embedding of X into $\mathbb{P}^3$. Note that the homogeneous parts of degree 2 of $\mathbb{P}(1,1,2)$, i.e. the generators of S_2, are given by y_0^2, y_0y_1, y_1^2 and y_2. We define a map
$$\begin{aligned}\varphi\colon \mathbb{P}(1,1,2) &\to \mathbb{P}^3\\ \langle a_0,a_1,a_2\rangle &\mapsto \langle a_0^2, a_0a_1, a_1^2, a_2\rangle,\end{aligned}$$

which is well defined, injective and $\varphi(X) = \{x_0x_2 - x_1^2 = 0\} \subset \mathbb{P}^3$ for coordinates x_i on $\mathbb{P}^3$. In the last example we saw that $P(1,1,2)$ can be seen as the compactification of an affine cone, here we computed the concrete embedding in $\mathbb{P}^3$.

- Note that from the last description one sees that for any $Q = (q_0, \dots, q_n), q_i \in \mathbb{N}$, we have an isomorphism
$$\mathbb{P}(Q) \cong \mathbf{Proj}(k[y_0, \dots, y_n]),$$
for coordinates y_i of degree q_i, using the Proj-construction. This differs only slightly from the previous, where we had the same ring $k[y_0, \dots, y_n]$, but presented $\mathbb{P}(Q)$ as quotient.

- In our last description we show that $\mathbb{P}(Q)$ can also be realized as a finite quotient of $\mathbb{P}^n$. To do so we introduce the following short hand notations:

Notation. For any two elements $a, b \in \mathbb{Z}^{n+1}$ we write
$$\begin{aligned} ab &:= (a_0b_0, \dots, a_nb_n) \\ \text{and } |ab| &:= a_0b_0 + \dots + a_nb_n. \end{aligned}$$
We will also use this in the case of a single argument, i.e.
$$|a| := a_0 + \dots + a_n.$$
Note that there is a ring homomorphism
$$\begin{aligned} \Psi\colon\ S' = \bigoplus_{\alpha \in \mathbb{Z}} S'_\alpha &\to \bigoplus_{\beta \in \mathbb{Z}} S_\beta = S \\ y^a := y_0^{a_0} \cdot \ldots \cdot y_n^{a_n} &\mapsto x_0^{a_0q_0} \cdot \ldots \cdot x_n^{a_nq_n} =: x^{aQ} \end{aligned}$$
which preserves $\mathbb{Z}$-degrees, since the monomial y^a has degree $|aQ| := a_0q_0 + \dots + a_nq_n$, which is the same as the degree of x^{aQ}. The rings S and S' also have the above introduced $\mathbb{Z}^{n+1}$ or fine grading, thus we can write down the morphism Ψ on the level of the fine graded parts. Let $a \in \mathbb{Z}^{n+1}$, then there is an induced isomorphism
$$S'_a = k \cdot y^a = k \cdot y_0^{a_0} \dots y_n^{a_n} \to k \cdot x_0^{a_0q_0} \dots x_n^{a_nq_n} = k \cdot x^{aQ} = S_{aQ}$$
which unfortunately does not preserve the $\mathbb{Z}^{n+1}$-degree. Note however that we can fill the "gaps" that this maps leaves by finitely many fine graded parts, i.e. we have an isomorphism
$$S = \bigoplus_{\substack{p=(p_0,\dots,p_n) \in \mathbb{Z}^{n+1} \\ 0 \le p_i < q_i}} x^p S',$$
which makes S into a finitely generated graded S'-module. This ring homomorphism Ψ now induces a morphism of varieties
$$\pi\colon\ \mathbb{P}^n \to \mathbb{P}(Q).$$
We want to study further properties of this map, therefore we introduce the following group:

Definition 3.4. We define $\mu_Q := \mu_{q_0} \times \dots \times \mu_{q_n}$, where
$$\mu_{q_i} := \{\xi \in k | \xi^{q_i} = 1\}$$
is the group of q_i-th roots of zeros in k.

Note that we have an action of μ_Q on $\mathbb{P}^n$ given by
$$\begin{aligned} \mu_q \times \mathbb{P}^n &\to \mathbb{P}^n \\ ((\xi_0, \dots, \xi_n), \langle x_0, \dots, x_n \rangle) &\mapsto \langle \xi_0x_0, \dots, \xi_nx_n \rangle \end{aligned}$$

as well as an action on the coordinate ring S by

$$\begin{aligned} \mu_q \times S &\to S \\ ((\xi_0, \ldots, \xi_n), f(x_0, \ldots, x_n)) &\mapsto f(\xi_0 x_0, \ldots, \xi_n x_n). \end{aligned}$$

Thus it is clear that S' is isomorphic to the ring of invariants of this action, i.e.

$$S' \cong S^{\mu_Q} \hookrightarrow S.$$

Hence the composed map is given by sending y_i to $x_i^{q_i}$, as is easily seen. So we get the following result.

Proposition 3.5. *The weighted projective spaces $\mathbb{P}(Q)$ are just the quotients of $\mathbb{P}^n$ by the group μ_Q, i.e.*

$$\mathbb{P}(Q) \cong \mathbb{P}^n/\mu_Q.$$

Moreover the quotient map $\mathbb{P}^n \to \mathbb{P}(Q)$ which can naturally be identified with π is a finite, toric morphism.

Proof. By the above considerations the only thing left to proof is that this morphism is toric. Let

$$\langle t_0, \ldots, t_n \rangle_{\mathbb{P}^n} \in T_{\mathbb{P}^n}$$

be any torus element. Then we have the following chain of equations

$$\begin{aligned} \pi(\langle t_0, \ldots, t_n \rangle_{\mathbb{P}^n} \cdot \langle x_0, \ldots, x_n \rangle_{\mathbb{P}^n}) &= \pi(\langle t_0 x_0, \ldots, t_n x_n \rangle_{\mathbb{P}^n}) \\ &= \langle t_0^{q_0} x_0^{q_0}, \ldots, t_n^{q_n} x_n^{q_n} \rangle_{\mathbb{P}(Q)} \\ &= \langle t_0, \ldots, t_n \rangle_{\mathbb{P}(Q)} \cdot \pi(\langle x_0, \ldots, x_n \rangle_{\mathbb{P}^n}) \end{aligned}$$

and the torus action therefore commutes with π. □

We define $\mu_Q^* := \mathrm{Hom}_k(\mu_Q, k^*)$ to be the dual of μ_Q. In the next section it turns out that this group is extremely useful for the description of sheaves on $\mathbb{P}(Q)$. Clearly this group decomposes into characters

$$\mu_Q^* \cong \bigoplus_{i=0}^{n} \mu_{q_i}^*$$

where $\mu_{q_i}^* \cong \mathbb{Z}/q_i\mathbb{Z}$ and $\mu_{q_i}^* \ni p_i \colon \mu_{q_i} \to k^*$ is given by $\xi \mapsto \xi^{p_i}$. Therefore we have an identification

$$\mu_Q^* \cong \{p := (p_0, \ldots, p_n) | 0 \le p_i < q_i \text{ for all } i\}$$

and we will by abuse of notation write $0 \le p < Q$ for any such element.

Remark. Since the order of $\mu_Q = \sum_{i=0}^{n} q_i$ is invertible in k we have for any action of μ_Q on a k-vector space V an eigenspace decomposition

$$V = \bigoplus_{p \in \mu_Q^*} V^p$$

where $V^p = \{v \in V | \xi \cdot v = p(\xi) \cdot v \quad \forall \xi \in \mu_Q\}$.

Example 3.6. We rediscover the composition

$$S = \bigoplus_{\substack{p=(p_0,\ldots,p_n)\in\mathbb{Z}^{n+1} \\ 0 \le p_i < q_i}} x^p S^{\mu_Q}$$

by the observation $S^p \cong x^p S^{\mu_Q}$ for any character $p \in \mu_Q^*$. Note that in particular $S^0 \cong S^{\mu_Q}$, which already shows that the zero character is extremely important for further considerations of weighted projective spaces and motivates why we will usually write μ_Q for this character.

3.2 Sheaves

Structure Sheaves and the Relation to $\mathbb{P}^n$

Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_{\mathbb{P}^n}$-modules on $\mathbb{P}^n$ on which μ_Q acts (compatible with the action of μ_Q on $\mathbb{P}^n$). Then for all open subsets $U \subset \mathbb{P}(Q)$ the module $\pi_*\mathcal{F}(U) = \mathcal{F}(\pi^{-1}(U))$ is a k-vector space. Hence, as described above, the direct image sheaf decomposes as

$$\pi_*\mathcal{F} = \bigoplus_{0 \leq p < Q} \pi_*^p \mathcal{F},$$

where on an open set $U \subset \mathbb{P}(Q)$ the module $\pi_*^p\mathcal{F}(U)$ for the character p is given by

$$\pi_*^p\mathcal{F}(U) = [\mathcal{F}(\pi^{-1}(U))]^p = \{f \in \mathcal{F}(\pi^{-1}(U)) | \xi \cdot f = p(\xi) \cdot f, \forall \xi \in \mu_Q\}$$

For the zero character we have in particular that

$$\pi_*^0\mathcal{F}(U) = [\mathcal{F}(\pi^{-1}(U))]^0 = \{f \in \mathcal{F}(\pi^{-1}(U)) | \xi \cdot f = 0(\xi) \cdot f = f, \forall \xi \in \mu_Q\}$$

is simply the set of μ_Q-stable sections, therefore we will write $\pi_*^{\mu_Q} := \pi_*^0$ (having the corresponding fact for the coordinate ring in mind).

Definition 3.7. We denote by $\boldsymbol{\mu_Q}$**-Coh(**$\mathbb{P}$**(Q))** the category of coherent sheaves on $\mathbb{P}(Q)$ on which μ_Q acts. The morphisms of this category morphisms of $\mathrm{Coh}(\mathbb{P}(Q))$ preserving this action.

Remark. The morphism π is finite, so

$$\pi_* = \bigoplus_{p \in \mu_Q^*} \pi_*^p$$

is an exact functor, which is also true for any of the parts π_*^p.
Unfortunately it is not flat outside of the singularities of $\mathbb{P}(Q)$, thus when we push down a vector bundle via π_*^p we cannot expect to obtain a vector bundle on $\mathbb{P}(Q)$. However we have the following

Lemma 3.8. *Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^n$. Then for all $p \in \mu_Q^*$ the sheaf $\pi_*^p\mathcal{E}$ is reflexive.*

Proof. Denote by $X_{reg} \overset{i}{\hookrightarrow} \mathbb{P}(Q)$ the set of regular points of $\mathbb{P}(Q)$, set

$$X'_{reg} := \pi^{-1}(X_{reg}) \overset{i'}{\hookrightarrow} \mathbb{P}^n$$

and consider the restricted map $\overline{\pi} := \pi|_{X'_{reg}} \colon X'_{reg} \to X_{reg}$, which induces the commutative diagram

$$\begin{tikzcd} X'_{reg} \arrow[r, "\overline{\pi}"] \arrow[d, "i'"'] & X_{reg} \arrow[d, "i"] \\ \mathbb{P}^n \arrow[r, "\pi"'] & \mathbb{P}(Q). \end{tikzcd}$$

Since the toric varieties $\mathbb{P}^n$ and $\mathbb{P}(Q)$ are normal we know that

$$\mathrm{codim}(\mathbb{P}(Q) \setminus X_{reg}) \geq 2 \leq \mathrm{codim}(\mathbb{P}^n \setminus X'_{reg}).$$

Hence $\mathcal{E} \cong i'_*(\mathcal{E}|_{X'_{reg}})$ and so

$$\pi_*^p\mathcal{E} \cong \pi_*^p i'_*(\mathcal{E}|_{X'_{reg}}) \cong i_*\overline{\pi}_*^p(\mathcal{E}|_{X'_{reg}}).$$

$\overline{\pi}_*^p(\mathcal{E}|_{X'_{reg}})$ is locally free since $\overline{\pi}$ is flat, thus $\pi_*^p\mathcal{E}$ is reflexive. $\square$

Lemma 3.9. *For every coherent μ_Q-sheaf $\mathcal{F}$ on $\mathbb{P}^n$ there is a natural action of μ_Q on $H^i(\mathbb{P}^n, \mathcal{F})$ for all $i \in \mathbb{N}$ and*

$$[H^i(\mathbb{P}^n, \mathcal{F})]^p \cong H^i(\mathbb{P}(Q), \pi_*^p \mathcal{F})$$

for all characters $p \in \mu_Q^*$.

Proof. Consider the Čech complex $C^\bullet(\mathcal{U}, \mathcal{F})$ of $\mathcal{F}$ on the standard open covering $\mathcal{U} = \{E_0, \ldots, E_n\}$ given by the divisors $E_i = \{x_i = 0\}$. All the E_i and their common intersections are μ_Q-invariant, so μ_Q acts on $C^i(\mathcal{U}, \mathcal{F})$. Since also all the maps between the Čech groups are μ_Q-morphisms, μ_Q acts naturally on $\check{\mathrm{H}}^i(\mathcal{U}, \mathcal{F}) := \mathrm{h}^i(C^\bullet(\mathcal{U}, \mathcal{F})$. By [Har77] there is an isomorphism

$$\check{\mathrm{H}}^i(\mathcal{U}, \mathcal{F}) \cong \mathrm{H}^i(\mathbb{P}^n, \mathcal{F}),$$

which shows the first part. Let analogously $\mathcal{V} = \{D_0, \ldots, D_n\}$ be the affine open covering of $\mathbb{P}(Q)$ defined by $D_i = \{y_i = 0\}$, then since $U_i = \pi^{-1}(D_i)$ we see that $C^\bullet(\mathcal{U}, \mathcal{F}) \cong C^\bullet(\mathcal{V}, \pi_*\mathcal{F})$ and therefore

$$\mathrm{H}^i(\mathbb{P}^n, \mathcal{F}) \cong \mathrm{H}^i(\mathbb{P}(Q), \pi_*\mathcal{F}).$$

Now we only have to consider the eigenspace of a single $p \in \mu_Q^*$ to obtain the last equation, i.e.

$$[\mathrm{H}^i(\mathbb{P}^n, \mathcal{F})]^p \cong \mathrm{H}^i(\mathbb{P}(Q), \pi_*^p \mathcal{F}).$$

□

Lemma 3.10. *There is a natural isomorphism*

$$\pi_* \mathcal{O}_{\mathbb{P}^n} \cong \bigoplus_{p \in \mu_Q^*} \mathcal{O}_{\mathbb{P}(Q)}(-|p|).$$

Proof. During this proof we will always write the $\sim$-functor together with the corresponding ring to make clear over which ring we work. First of all we see that

$$\pi_* \mathcal{O}_{\mathbb{P}^n} \cong \pi_* \widetilde{S}^S \cong (\bigoplus_{0 \leq p < Q} x^p S^{\mu_Q})^{\sim S'}.$$

From [Gro61] we know more generally for any S-module M there is an isomorphism

$$\pi_* \widetilde{M}^S \cong \widetilde{M_{S'}}^{S'}.$$

The Grothendieck-functor commutes with direct sums, so we continue with

$$\pi_* \mathcal{O}_{\mathbb{P}^n} \cong \bigoplus_{0 \leq p < Q} \widetilde{x^p S^{\mu_Q}} \cong \bigoplus_{0 \leq p < Q} \widetilde{S'(-|p|)} \cong \bigoplus_{0 \leq p < Q} \mathcal{O}_{\mathbb{P}(Q)}(-|p|).$$

Here the second isomorphism comes from

$$\begin{aligned} S'(-|p|)_\alpha = S'_{\alpha - |p|} &= \bigoplus_{|aQ| = \alpha - |p|} S'_\alpha = \bigoplus_{|aQ| + |p| = \alpha} k \cdot y^a \\ &\cong \bigoplus_{|aQ| + |p| = \alpha} k \cdot x^{aQ+p} = \bigoplus_{|b| = \alpha} (x^p S^{\mu_Q})_b = (x^p S^{\mu_Q})_\alpha \end{aligned}$$

for all $\alpha \in \mathbb{Z}$ and the isomorphism in the middle is given by multiplication with x^p and replacing y^a by x^{aQ}. □

Remark. Clearly the proof does not change if we twist by some $j \in \mathbb{Z}$, thus we also obtain

$$\pi_* \mathcal{O}_{\mathbb{P}^n}(j) \cong \bigoplus_{p \in \mu_Q^*} \mathcal{O}_{\mathbb{P}(Q)}(j - |p|).$$

Proposition 3.11. *More generally for all S'-modules M interpreted as S-module via the embedding $S' \cong S^{\mu_Q} \hookrightarrow S$ and thus as μ_Q-invariant module, we have an isomorphism*

$$\pi_*(\widetilde{M}^S) \cong \bigoplus_{p \in \mu_Q^*} \widetilde{x^p M}^{S'}$$

Proof. Let U be an affine open subset of $\mathbb{P}(Q)$ of the form $U = \mathrm{Spec} S'_{(f)}$ for some homogeneous polynomial $f \in S'$. Then it follows by [Gro71] that

$$\begin{aligned}
\pi_*(\widetilde{M}^S)|_U &\cong \pi_*|_{\pi^{-1}(U)}(\widetilde{M}^S|_{\pi^{-1}(U)}) \\
&\cong \pi_*|_{\pi^{-1}(U)}(\widetilde{M_{(f)}}^S) \\
&\cong \bigoplus_{0 \le p < Q} (x^p (M_{S'})_{(f)})^{\sim S'} \\
&\cong (\bigoplus_{0 \le p < Q} (\widetilde{x^p M_{S'}}^{S'}))|_U.
\end{aligned}$$

□

Theorem 3.12. *For any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$, any μ_Q-sheaf $\mathcal{G}$ on $\mathbb{P}(Q)$ and an arbitrary character $p \in \mu_Q^*$ there is an isomorphism*

$$\pi_*^p(\pi^*\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{G}) \cong \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}(Q)}} \pi_*^p \mathcal{G}.$$

Proof. See [Can00, Proposition 1.5] for a proof. □

Corollary 3.13. $\pi_*^{\mu_Q} \circ \pi^* \cong id_{Coh(\mathbb{P}(Q))}$.

Proof. For any coherent sheaf $\mathcal{F}$ on $\mathbb{P}(Q)$ there is an isomorphism

$$\begin{aligned}
\pi_*^{\mu_Q} \circ \pi^*\mathcal{F} &\cong \pi_*^{\mu_Q}(\pi^*\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}) \\
&\cong \pi_*^{\mu_Q}\pi^*\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}(Q)}} \pi_*^{\mu_Q}\mathcal{O}_{\mathbb{P}^n} \\
&\cong \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}(Q)}} \mathcal{O}_{\mathbb{P}(Q)}(-|0|) \cong \mathcal{F},
\end{aligned}$$

using the last theorem. □

Remark. Unfortunately the the functors

$$\begin{aligned}
\pi_*^{\mu_Q}\colon\ &\mu_Q\text{-Coh}(\mathbb{P}^n) \to \mathrm{Coh}(\mathbb{P}(Q)) \\
\text{and } \pi^*\colon\ &\mathrm{Coh}(\mathbb{P}(Q)) \to \mu_Q\text{-Coh}(\mathbb{P}^n)
\end{aligned}$$

do not define equivalences of categories, see [Can00].

Sheaves of Regular Differentials

Following Dolgachev, respectively Canonaco we define $\Omega_{S'} := \Omega_{S'/k}$ to be the module of k-differentials, for all $i \in \mathbb{N}$ we put $\Omega^j_{S'} := \Lambda^j \Omega_{S'}$ with the usual converntion $\Omega^0_{S'} := S'$. Then of course $\Omega^j_{S'}$ is a free S'-module via the canonical derivation d: $S' \to \Omega_{S'}$ with basis

$$\{dy_{i_1} \wedge \ldots \wedge dy_{i_j} | 0 \le i_1 < \ldots < i_j \le n\}.$$

These modules have a natural $\mathbb{Z}$-grading induced by the weights of the y_i:

$$\deg(dy_{i_1} \wedge \ldots \wedge dy_{i_j}) := q_{i_1} + \ldots + q_{i_j}.$$

Remark. Since the modules are free with the introduced grading we obtain a graded isomorphism of degree zero by

$$\Omega^j_{S'} = \bigoplus_{0\le i_1<\ldots i_j\le n} S'(-q_{i_1} - \ldots - q_{i_j}).$$

Now for each $j < 0$ we define the degree zero morphisms

$$\begin{aligned}\Delta^j\colon\ \Omega^j_{S'} &\to \Omega^{j-1}_{S'}\\ dy_{i_1}\wedge\ldots\wedge dy_{i_j} &\mapsto \sum_{l=1}^{j}(-1)^{l-1}q_{i_l}\cdot y_{i_l}\cdot dy_{i_1}\wedge\ldots\wedge\widehat{dy_{i_l}}\wedge\ldots\wedge dy_{i_j}.\end{aligned}$$

Hence by definition of the maps we obtain a complex

$$0\to\Omega^{n+1}_{S'}\xrightarrow{\Delta^{n+1}}\Omega^n_{S'}\xrightarrow{\Delta^n}\ldots\xrightarrow{\Delta^2}\Omega^1_{S'}\xrightarrow{\Delta^1}S'\to 0$$

which is just the Koszul complex of the regular sequence $(q_0t_0,\ldots q_nt_n)$ and so it is exact everywhere except for the first position, where we have $\operatorname{coker}(\Delta^1) = S'/(q_0t_0,\ldots,q_nt_n)\cong k$.
Since it turns out to be more useful we set $\overline{\Omega}^j_{S'} := \operatorname{im}(\Delta^{i+1}) = \ker(\Delta^i)$ and will mainly work with the following

Definition 3.14. For every j the sheaves $\overline{\Omega}^j_{\mathbb{P}(Q)} := (\overline{\Omega}^j_{S'})^\sim$ with the induced grading are called **sheaves of regular differential j-forms**. Moreover we put $\overline{\Omega}^j_{\mathbb{P}(Q)}(l) := (\overline{\Omega}^j_{S'}(l))^\sim$ for the twist by $l\in\mathbb{Z}\cong A_{n-1}(\mathbb{P}(Q))$.

Remark. Since $\overline{\Omega}^{n+1}_{S'} = 0$ and by the above remark we have

$$\overline{\Omega}^n_{\mathbb{P}(Q)}\cong(\overline{\Omega}^n_{S'})^\sim\cong(S'(-q_0-\ldots-q_n))^\sim\cong\mathcal{O}_{\mathbb{P}(Q)}(-|Q|).$$

Since the spaces $\mathbb{P}(Q)$ are normal there is another natural way to define differential sheaves:

Definition 3.15. Denote by $i\colon X_{reg}\to X$ the set of regular points of $X=\mathbb{P}(Q)$. Then for every $j>0$ we define the **Zariski sheaf of germs of j-forms** to be

$$\widetilde{\Omega}^j_X := i_*\Omega^j_{X_{reg}}.$$

We refer to [Oda88] for more about those sheaves, which can be defined on every toric variety. The important result is that on $\mathbb{P}(Q)$ these sheaves coincide with the sheaves of regular differentials, i.e.

$$\widetilde{\Omega}^j_{\mathbb{P}(Q)}\cong\overline{\Omega}^j_{\mathbb{P}(Q)}.$$

See [Dol82, 2.2.4] for a proof of this statement. This particularly means that $\overline{\Omega}^j_{\mathbb{P}^n}\cong\widetilde{\Omega}^j_{\mathbb{P}^n}\cong\Omega^j_{\mathbb{P}^n}$ and so the our new sheaves really generalize the differential sheaves on $\mathbb{P}^n$.

Note that we also have the sheaf of regular j-forms in the special case of $\mathbb{P}(Q)=\mathbb{P}^n$, where μ_Q naturally acts on Ω^j_S by

$$((\xi_0,\ldots,\xi_n), dx_{i_1}\wedge\ldots\wedge dx_{i_j})\mapsto\xi_{i_1}\ldots\xi_{i_j}dx_{i_1}\wedge\ldots\wedge dx_{i_j},$$

and so all maps Δ^j are μ_Q-morphisms, hence $\overline{\Omega}^j_{\mathbb{P}^n}(l)\in\mu_Q\text{-}Coh(\mathbb{P}^n)$.

Generalized Euler and Koszul Sequences

Theorem 3.16 (Generalized Euler short exact sequence)**.** *Let $X=\mathbb{P}(Q)$ a weighted projective space of dimension n with toric divisors $D_0,\ldots,D_n$. Then there is an exact sequence*

$$0\longrightarrow\overline{\Omega}^1_{\mathbb{P}(Q)}\longrightarrow\bigoplus_{i=0}^{n}\mathcal{O}_{\mathbb{P}(Q)}(-D_i)\xrightarrow{\begin{pmatrix}q_0y_0\\ \vdots\\ q_ny_n\end{pmatrix}}\mathcal{O}_{\mathbb{P}(Q)}\longrightarrow 0.$$

Proof. See [VB94, Section 12] for a more general result for complete simplicial toric varieties. □

Identifying the divisors D_i with elements of $\mathbb{Z}^{n+1}$ the generalized Euler sequence basically comes from the following surjection of S'-modules

$$\bigoplus_{i=0}^{n} S'(-D_i) \xrightarrow{\begin{pmatrix} q_0y_0 \\ \vdots \\ q_ny_n \end{pmatrix}} S' \to 0.$$

We therefore obtain the Koszul complex of this matrix:

$$0 \longrightarrow S'(\sum_{i=0}^{n} -D_i) \longrightarrow \ldots \longrightarrow \bigoplus_{i<j}^{n} S'(-D_i - D_j) \bigoplus_{i=0}^{n} S'(-D_i) \xrightarrow{\begin{pmatrix} q_0y_0 \\ \vdots \\ q_ny_n \end{pmatrix}} S' \to 0.$$

and so by tensoring with $\mathcal{O}_{\mathbb{P}(Q)}$ we get a generalized Koszul complex on $\mathbb{P}(Q)$:

$$\ldots \to \bigoplus_{i<j}^{n} \mathcal{O}_{\mathbb{P}(Q)}(-D_i - D_j) \to \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}(Q)}(-D_i) \xrightarrow{\begin{pmatrix} q_0y_0 \\ \vdots \\ q_ny_n \end{pmatrix}} \mathcal{O}_{\mathbb{P}(Q)} \to 0.$$

To simplify this a little bit we can delete the weights from the coordinate vector by the following transformation

$$\begin{tikzcd}
\bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}(Q)}(-D_i) \arrow[r, "\begin{pmatrix} q_0y_0 \\ \vdots \\ q_ny_n \end{pmatrix}"] \arrow[d, "\cong"', "\begin{pmatrix} q_0 & & 0 \\ & \ddots & \\ 0 & & q_n \end{pmatrix}"'] & \mathcal{O}_{\mathbb{P}(Q)} \arrow[r] \arrow[d, equal] & 0 \\
\bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}(Q)}(-D_i) \arrow[r, "\begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix}"'] & \mathcal{O}_{\mathbb{P}(Q)} \arrow[r] & 0.
\end{tikzcd}$$

We will use these sequences several times in the next chapters.

The Cohomology of $\mathcal{O}_{\mathbb{P}(Q)}(i)$

Recall that we defined $\mathcal{O}_{\mathbb{P}(Q)}(i) := \widetilde{S'(i)}$. For $f \in S'$ there is a natural homomorphism $S'_i \to S'(i)_{(f)}$ by $a \mapsto \frac{a}{1}$ and inducing the so called **Serre homomorphism**

$$\alpha_i\colon S'_i \to \mathrm{H}^0(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(i)).$$

Theorem 3.17. *The Serre homomorphism $\alpha_i\colon S'_i \to H^0(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(i))$ is an isomorphism for all $i \in \mathbb{N}$. Moreover*

$$H^j(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(i)) = \begin{cases} S'_{-n-|Q|} & , j \neq 0, n \\ 0 & , \textit{ else} \end{cases}$$

Proof. See [Dol82, section 1.4] for a proof. □

Let $P_{S'}(t)$ denote the **Poincaré series** of S' with coefficients a_i determined by

$$P_{S'}(t) := \sum_{i=0}^{\infty} a_i \cdot t^i := \prod_{j=0}^{n} (1 - t^{q_j})^{-1}.$$

Then we have the following Corollary as a direct consequence of the theorem:

Corollary 3.18. *The dimension of the cohomology groups can easily be read of from the Poincaré series, i.e.*

$$dim_k H^j(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(i)) = \begin{cases} a_i & , j = 0 \\ 0 & , j \neq 0, n \\ a_{-i-|Q|} & , j = n \end{cases}$$

Proof. This follows directly from $a_i = \dim_k S'_i$ and gives us a good way to compute the dimensions. ☐

The Cohomology of $\overline{\Omega}^i_{\mathbb{P}(Q)}(l)$

The proofs of the results presented in this section can be found in [Dol82, section 2.3].

Notation. For $\varnothing \neq I \subset \{0, \dots, n\}$ we denote by $|Q_I|$ the sum $\sum_{i \in I} q_i$ and define

$$h(j, i; l) := \dim_k \mathrm{H}^j(\mathbb{P}(Q), \overline{\Omega}^i_{\mathbb{P}(Q)}(l)),$$

where as before a_i denotes the dimension of the i-th graded part of S'.

With these useful notations we can compactly formulate the next

Theorem 3.19. *For all twists $l \in \mathbb{Z}$ we have*

- $h(0, i; l) = \sum\limits_{\#I=i} a_{l-|Q_I|} - h(0, i-1; l) \quad \forall i \geq 1,$
- $h(j, i; l) = 0 \quad$ *, if* $j \neq 0, i, n,$
- $h(i, i; 0) = 1 \quad , i = 0, \dots, n,$
- $h(n, i; l) = \sum\limits_{\#I=n+1-i} a_{-n-|Q_I|} - h(n, i-1; l) \quad \forall i \geq 1.$

Corollary 3.20 (Bott-Steenbrink)**.** *Most of the cohomology groups vanish:*

- $H^0(\mathbb{P}(Q), \overline{\Omega}^i_{\mathbb{P}(Q)}(l)) = 0$*, if* $l < min\{|Q_I|\colon \#I = i\}$ *and*
- $H^n(\mathbb{P}(Q), \overline{\Omega}^i_{\mathbb{P}(Q)}(l)) = 0$*, if* $n > -min\{|Q_I|\colon \#I = n+1-i\}$
- *If* $l > 0$ *then the group* $H^j(\mathbb{P}(Q), \overline{\Omega}^i_{\mathbb{P}(Q)}(l))$ *is only in the case where* $j = 0$ *and* $l > min\{|Q_I|\colon \#I = i\}$ *is nonzero.*

3.3 Induced Morphisms

Fix $Q = (q_0, \ldots, q_n)$. Given a morphism $\mathcal{O}_{\mathbb{P}^n}(-|r|) \xrightarrow{\cdot x^r} \mathcal{O}_{\mathbb{P}^n}$ for an arbitrary $r \in \mathbb{N}^{n+1}$, we would like to know how the induced morphism $\pi_* x^r$ for the map $\pi\colon \mathbb{P}^n \to \mathbb{P}(Q)$ looks like, i.e. which morphism makes the following diagram commutative:

$$\begin{tikzcd}
\pi_*\mathcal{O}_{\mathbb{P}^n}(-|r|) \arrow[r, "\pi_* x^r"] \arrow[d, "\cong"'] & \pi_*\mathcal{O}_{\mathbb{P}^n} \arrow[d, "\cong"] \\
\bigoplus\limits_{0\le p<Q} \mathcal{O}_{\mathbb{P}(Q)}(-|r|-|p|) \arrow[r, dotted] & \bigoplus\limits_{0\le p'<Q} \mathcal{O}_{\mathbb{P}(Q)}(-|p'|).
\end{tikzcd}$$

Therefore we consider the corresponding maps on the homogeneous coordinate rings. There we have:

$$\begin{tikzcd}
S(-|r|) \arrow[r, "\cdot x^r"] \arrow[d, "\cong"'] & S \arrow[d, "\cong"] \\
\bigoplus\limits_{0\le p<Q} x^p S^Q(-|r|) \arrow[r, "\cdot x^r"'] & \bigoplus\limits_{0\le p'<Q} x^{p'} S^Q,
\end{tikzcd}$$

thus for fixed p and $\alpha \in \mathbb{Z}$ we obtain for the α-graded parts

$$\begin{tikzcd}
x^p S^Q(-|r|)_\alpha \arrow[r, "\cdot x^r"] \arrow[d, "\cong"'] & \bigoplus\limits_{0\le p'<Q} (x^{p'} S^Q)_\alpha \arrow[d, "\cong"] \\
\bigoplus\limits_{\substack{a\in\mathbb{Z}^{n+1} \\ |p|+|aQ|=\alpha-|r|}} k \cdot x^{p+aQ} \arrow[r, "\cdot x^r"'] & \bigoplus\limits_{\substack{0\le p'<Q \\ |p'|+|cQ|=\alpha}} (\bigoplus_c k \cdot x^{p'+cQ}).
\end{tikzcd}$$

Hence, going another step down, for a single monomial the morphism we are looking for has to look like this

$$\begin{tikzcd}
x^{p+aQ} \arrow[r, "x^r"] & x^{(r+p)+aQ} \\
y^a \arrow[u] \arrow[r, "y^b"'] & y^{a+b} \arrow[u]
\end{tikzcd}$$

and we can write

$$x^{(r+p)+aQ} = x^{p'+(a+b)Q}$$

where b and $p' < Q$ are uniquely determined by division with remainder. So for the modules this means

$$\begin{tikzcd}
\bigoplus\limits_{0\le p<Q} (x^p S^Q(-|r|))_\alpha \arrow[r, "x^r"] & \bigoplus\limits_{0\le p'<Q} (x^{p'} S^Q)_\alpha \\
\bigoplus\limits_{0\le p<Q} (x^p S'(-|r|-|p|))_\alpha \arrow[u, "\cong"] \arrow[r, "(f_{p,p'})_{p,p'}"'] & \bigoplus\limits_{0\le p'<Q} (x^{p'} S'(-|p|))_\alpha, \arrow[u, "\cong"']
\end{tikzcd}$$

where we set

$$f_{p,p'} = y^b \Leftrightarrow r + p = bQ + p \Leftrightarrow p' = r + p - bq$$

and zero else. It follows directly that the matrix $(f_{p,p'})_{p,p'}$ is an elementary matrix. The map does not depend on α, which means we can write the whole diagram down again without the subindex α.

Let us now consider a finitely generated S- module F, which by definition has a finite representation, i.e for some α_i and β_j we have an exact sequence

$$\bigoplus_j S(\beta_j) \xrightarrow{(f_{ij})_{i,j}} \bigoplus_i S(\alpha_i) \to F \to 0$$

with $f_{i,j} = \lambda_{i,j} x^{r_{i,j}}$ for $\lambda_{i,j} \in k$ possibly zero. The exactness of the Grothendieck functor gives us a sequence of sheaves

$$\bigoplus_j \mathcal{O}_{\mathbb{P}^n}(\beta_j) \xrightarrow{(f_{ij})_{i,j}} \bigoplus_i \mathcal{O}_{\mathbb{P}^n}(\alpha_i) \to \tilde{F} =: \mathcal{F} \to 0.$$

Now we apply the exact functor π_* to this sequence and obtain

$$\bigoplus_j \bigoplus_{0 \le p < Q} \mathcal{O}_{\mathbb{P}(Q)}(\beta_j - |p|) \xrightarrow{(\tilde{f}_{i,j})_{i,j}} \bigoplus_i \bigoplus_{0 \le p' < Q} \mathcal{O}_{\mathbb{P}(Q)}(\alpha_i - |p'|) \to \pi_* \mathcal{F} \to 0,$$

where $(\tilde{f}_{i,j})_{i,j}$ is the induced block-matrix with entries $f_{i,j} = (f_{i,j}^{p,p'})_{p,p'}$ and

$$f_{i,j}^{p,p'} = \begin{cases} y^{b_{i,j}^{p,p'}} & \text{, if } r_{i,j} + p = b_{i,j}^{p,p'} Q + p' \\ 0 & \text{, else} \end{cases}.$$

Example 3.21. Let $Q = (1,1,2)$ and $r = (2,3,5)$. Then $|r| = 10$, $p, p' \in \{(0,0,0),(0,0,1)\}$ and we get the induced morphism

$$\mathcal{O}_{\mathbb{P}(Q)}(-11) \oplus \mathcal{O}_{\mathbb{P}(Q)}(-10) \xrightarrow{\pi_*(x^r)} \mathcal{O}_{\mathbb{P}(Q)}(0) \oplus \mathcal{O}_{\mathbb{P}(Q)}(-1) \to \bigoplus_{\chi} \pi_*^{\chi} \mathcal{F} \to 0$$

with

$$\pi_*(x^r) = \begin{pmatrix} 0 & y^{(2,3,2)} \\ y^{(2,2,3)} & 0 \end{pmatrix}.$$

Example 3.22. In the same way as before we can push down the Euler-sequence on $\mathbb{P}^n$ to obtain

$$\pi_* \Omega^1_{\mathbb{P}^n} \to (n+1) \left(\bigoplus_{0 \le p < Q} \mathcal{O}_{\mathbb{P}(Q)}(-1 - |p|) \right) \xrightarrow{M} \bigoplus_{0 \le p' < Q} \mathcal{O}_{\mathbb{P}(Q)}(-|p'|) \to 0,$$

where M is a matrix consisting of $n+1$ blocks and the i-th block is an elementary matrix, whose nonzero elements are either y_i or 1 in case of $b = 0$.

Example 3.23. We consider again $\mathbb{P}(Q) = \mathbb{P}(1,1,2)$. Then the induced Euler sequence with three 2×2-blocks is given by

$$0 \to \pi_* \Omega^1_{\mathbb{P}^2} \to 3(\mathcal{O}_{\mathbb{P}(Q)}(-1) \oplus \mathcal{O}_{\mathbb{P}(Q)}(-2)) \xrightarrow{\begin{pmatrix} y_0 & 0 \\ 0 & y_0 \\ y_1 & 0 \\ 0 & y_1 \\ 0 & 1 \\ y_2 & 0 \end{pmatrix}} \mathcal{O}_{\mathbb{P}(Q)} \oplus \mathcal{O}_{\mathbb{P}(Q)}(-1) \to 0.$$

Now it is easy to see that each column corresponds to the sequence for a single character, i.e.

$$0 \to \pi_*^{\mu_Q} \Omega^1_{\mathbb{P}^2} \cong \overline{\Omega}^1_{\mathbb{P}(Q)} \to 2\mathcal{O}_{\mathbb{P}(Q)}(-1) \oplus \mathcal{O}_{\mathbb{P}(Q)}(-2) \xrightarrow{\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}} \mathcal{O}_{\mathbb{P}(Q)} \to 0$$

and

$$0 \to \pi_*^{(0,0,1)} \Omega^1_{\mathbb{P}^2} \to 2\mathcal{O}_{\mathbb{P}(Q)}(-2) \oplus \mathcal{O}_{\mathbb{P}(Q)}(-1) \xrightarrow{\begin{pmatrix} y_0 \\ y_1 \\ 1 \end{pmatrix}} \mathcal{O}_{\mathbb{P}(Q)}(-1) \to 0.$$

Remark. The rows of the form $(0, \ldots, 0, 1, 0 \ldots, 0)$ can always be deleted by a process similar to the Gaussian elimination known from linear algebra, since such a row does not have any influence on the cokernel.

Example 3.24.

$$\begin{tikzcd}
0 \arrow[r] & \pi_*^{(0,0,1)}\Omega^1_{\mathbb{P}^2} \arrow[r] & 2\mathcal{O}_{\mathbb{P}(Q)}(-2) \oplus \mathcal{O}_{\mathbb{P}(Q)}(-1) \arrow[r, "\begin{pmatrix} y_0 \\ y_1 \\ 1 \end{pmatrix}"] & \mathcal{O}_{\mathbb{P}(Q)}(-1) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & 2\mathcal{O}_{\mathbb{P}(Q)}(-2) \arrow[r] \arrow[u, dotted, "\cong"] & 2\mathcal{O}_{\mathbb{P}(Q)}(-2) \oplus \mathcal{O}_{\mathbb{P}(Q)}(-1) \arrow[r, "\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}"'] \arrow[u, "\cong", "\begin{pmatrix} 1 & 0 & -y_0 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{pmatrix}"'] & \mathcal{O}_{\mathbb{P}(Q)}(-1) \arrow[r] & 0
\end{tikzcd}$$

So we get

$$\pi_*^{(0,0,1)}\Omega^1_{\mathbb{P}^2} \cong 2\mathcal{O}_{\mathbb{P}(Q)}(-2)$$

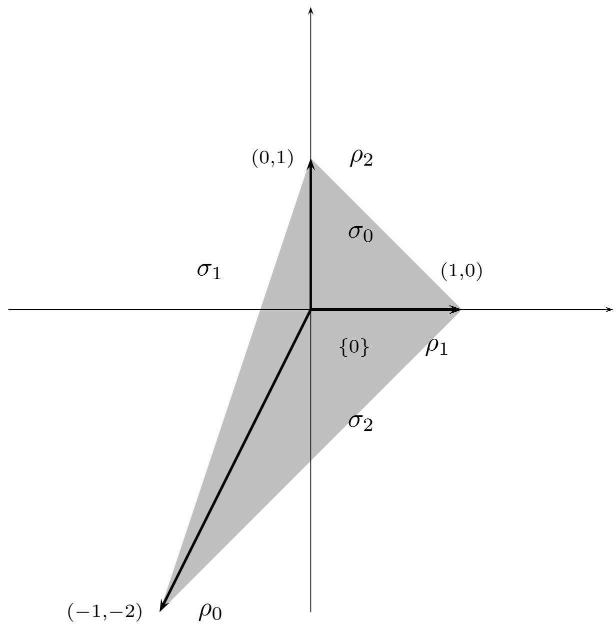
as a byproduct. We will see in the next chapter that this is not just good luck, i.e. all the sheaves of the form $\pi_*^X\Omega^i_{\mathbb{P}^n}$ are isomorphic to a direct sum of twisted structure sheaves.

In the next chapter we will see how we can get concrete resolutions in terms of structure and differential sheaves, which is the main tool we need to classify some toric sheaves on weighted projective planes.

3.4 Chow Rings

In this section we compute some Chow Rings and morphisms between them for weighted projective spaces by using the intersection theory on toric varieties which we have introduced in the last chapter.

Example 3.25. We will calculate the Chow ring $A^*(\mathbb{P}(1,1,2)) = A^*(X)$, so we first of all have to look at the Chow groups $A_i(X)$ in detail, since they determine the relations we need to know for the computation of the Minkowksi weights. Let the rays and cones be denoted as in the picture below



$A_0(X)$: We know that $A_0(X)$ is generated by the three cones σ_0, σ_1 and σ_2 with three relations given by the rays ρ_i. This gives rise to a lengthy computation:

- ρ_0: We have $M(\rho_0) = \{u = (0, u_2) | u_2 \in \mathbb{Z}\}$ and $n_{\sigma_1,\rho_0} = (0,1)$, $n_{\sigma_2,\rho_0} = (0,-2)$ for the generators (note here that $\sigma_0 \not\supseteq \rho_0$ and therefore σ_0 does not play a role in the computation). Thus

$$\mathrm{div}(\chi(u)) = \mathrm{div}(\chi(0,u_2)) = u_2[V(\sigma_1)] - 2u_2[V(\sigma_2)].$$

- ρ_1: Here we have $M(\rho_1) = \{u = \lambda \cdot (2,-1) | \lambda \in \mathbb{Z}\}$, $n_{\sigma_0,\rho_1} = (0,1)$ and $n_{\sigma_2,\rho_1} = (1,0)$, so

$$\mathrm{div}(\chi(u)) = \mathrm{div}(\chi(2\lambda, -\lambda)) = -\lambda[V(\sigma_0)] + 2\lambda[V(\sigma_2)].$$

- ρ_2: Analogously for the third case $M(\rho_2) = \{u = (u_1, 0) | u_1 \in \mathbb{Z}\}$, $n_{\sigma_0,\rho_2} = (-1,0)$,$n_{\sigma_1,\rho_2} = (1,0)$ and therefore

$$\mathrm{div}(\chi(u)) = \mathrm{div}(\chi(u_1,0)) = -u_1[V(\sigma_0)] + u_1[V(\sigma_2)].$$

$A_1(X)$: The group $A_1(X)$ is generated by the ρ_i with one relation coming from the zero cone, namely (since $N_{\rho_i}/N_{\{0\}} = N_{\rho_i}/N \cap \{0\} = N_{\rho_i} \ni n_{\rho_i}$ is simply the primitive vector in ρ_i)

$$\mathrm{div}(\chi(u)) = u_1 \cdot [V(\rho_0)] + (-u_1 - 2u_2) \cdot [V(\rho_1)] + u_2 \cdot [V(\rho_2)],$$

where $u = (u_1, u_2) \in M(\{0\}) = M \cap \{0\}^\perp = M \cong \mathbb{Z}^2$.

$A_2(X)$: $A_2(X)$ is easy to determine, since it is generated by $\{0\}$ and has no relations, it is isomorphic to $\mathbb{Z}$.

As a direct consequence of the last calculation we obtain

$A^2(X)$: $A^2(X) \cong \mathrm{Hom}(A_2(X), \mathbb{Z}) \cong \mathbb{Z}$. Note that the direct calculation of $A^2(X)$ is not much more complicated: $A^2(X) \cong MW^2(X) = \{c\colon \{0\} \to \mathbb{Z}\} \cong \mathbb{Z}$, since $\Delta^{(2)} = \{\{0\}\}$.

$A^1(X)$: The group $A^1(X)$ is isomorphic to all functions $c\colon \Delta^{(1)} \to \mathbb{Z}$ that vanish on the one given relation, i.e.

$$\begin{aligned} 0 &= u_1 \cdot c(\rho_0) + (-u_1 - 2u_2) \cdot c(\rho_1) + u_2 \cdot c(\rho_2) \\ &= u_1 \cdot (c(\rho_0) - c(\rho_1)) + u_2 \cdot (c(\rho_2) - 2c(\rho_1)) \qquad \forall u \in M. \end{aligned}$$

This condition is equivalent to say that $c(\rho_0) = c(\rho_1)$ and $c(\rho_2) = 2c(\rho_0)$, i.e. $c(\rho_0) \in \mathbb{Z}$ is arbitrary and the rest uniquely determined, so $A^1(X) \cong \mathbb{Z}$.

$A^0(X)$: For $A^0(X)$ we have a similar computation, consider maps $c\colon \Delta^{(0)} \to \mathbb{Z}$ such that for all u, λ as given above

$$\begin{aligned} u_2(c(\sigma_1) - 2c(\sigma_2)) &= 0 \\ \lambda(-c(\sigma_0) + 2c(\sigma_2)) &= 0 \\ u_1(c(\sigma_1) - c(\sigma_0)) &= 0. \end{aligned}$$

Hence the conditions are $c(\sigma_0) = c(\sigma_1)$ and $c(\sigma_0) = 2c(\sigma_2)$ which simply means that $c(\sigma_2)$ can be chosen arbitrary and so $A^0(X) \cong \mathbb{Z}$. To summarize all this we remark that the Chow ring $A^*(X) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ of $\mathbb{P}(1, 1, 2)$ is the same as the Chow ring of $\mathbb{P}^2$.

After having computed the last example very explicitly, we will be a little bit sketchier in the next one

Example 3.26. Let $p, q \in \mathbb{Z}$ with $\gcd(p, q) = 1$ and consider the weighted projective space $X = \mathbb{P}(1, p, q)$. Since

$$0 \to \mathbb{Z}^2 \xrightarrow{\left(\begin{smallmatrix} p & -1 & 0 \\ p+q & -1 & -1 \end{smallmatrix}\right)} \mathbb{Z}^3 \xrightarrow{\left(\begin{smallmatrix} 1 \\ p \\ q \end{smallmatrix}\right)} \mathbb{Z} \to 0$$

is exact and the matrices have primitive columns, the fan of the complete toric variety $\mathbb{P}(1, p, q)$ is spanned by the rays

$$\rho_0 := \begin{pmatrix} p \\ p+q \end{pmatrix} \quad \rho_1 := \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad \rho_2 := \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Then in the same way as before we see that

$A^1(X)$: the group $A^1(X)$ is given by all Minkowski weights c such that

$$(pu_1 + (p+q)u_2) \cdot c(\rho_0) + (-u_1 - u_2) \cdot c(\rho_1) + (-u_2) \cdot c(\rho_2) = 0,$$

which is equivalent to $c(\rho_1) = p \cdot c(\rho_0)$ and $(p+q) \cdot c(\rho_0) = c(\rho)_1 + c(\rho_2)$. This means that $c(\rho_0)$ is arbitrary and the other values are fixed, hence $A^1(X) \cong \mathbb{Z}$.

$A^2(X)$: The fact $A^2(X) \cong \mathbb{Z}$ is again easily checked, as in the case of

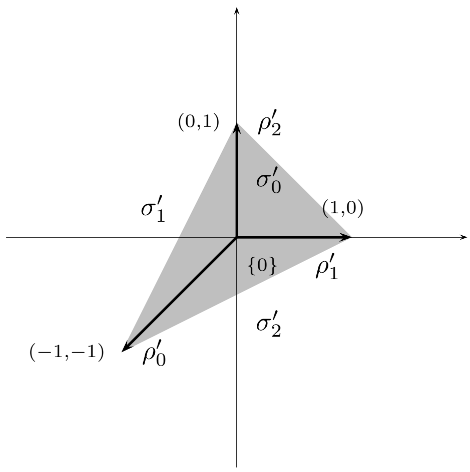
$A^0(X)$: where we have $A^0(X) \cong \mathbb{Z}$ by a computation along the same lines as before.

Next we will compute how the dominant toric morphism $\pi\colon \mathbb{P}^n \to \mathbb{P}(Q)$ looks like on the level of Chow rings, i.e. we want to know more about the map

$$\pi^*\colon A^*(\mathbb{P}(Q)) \to A^*(\mathbb{P}^n)$$

in some special cases. We use proposition 2.64 in order to do so.

Example 3.27. Let $X(\Delta) = \mathbb{P}(1,1,2)$ with rays and cones denoted as in the example above and let $X(\Delta') = \mathbb{P}^2$ given by the fan



Moreover let $\pi\colon X(\Delta') = \mathbb{P}^2 \to \mathbb{P}(1,1,2) = X(\Delta)$ and $c\colon \Delta^{(1)} \to \mathbb{Z}$ a Minkowski weight.
To see how π^*c looks like we assume without loss of generality that $c \in A^1(X) \cong \mathbb{Z}$ corresponds to the positive generator 1, i.e. $c(\rho_0) = 1$. Then by proposition 2.64 we see that

$$\begin{aligned}(\pi^*c)(\rho_i') &= c(\rho_i) \cdot [N : (\Psi(N') + N_{\rho_i})] \\ &= \begin{cases} 1 \cdot 2 & \text{, if } i = 0,1 \\ 2 \cdot 1 & \text{, if } i = 2 \end{cases} \\ &= 2\end{aligned}$$

which can directly be read of from the toric morphism π by calculating the index of the lattices. Thus the map is multiplication by 2. Clearly for $c\colon \Delta^{(2)} \cong \{\{0\}\} \to \mathbb{Z}$ with $c(\{0\}) = 1$ we get the same result, i.e.

$$(\pi^*c)(\{0'\}) = c(\{0\}) \cdot [N : (\Psi(N') + N_{\rho_i})] = 1 \cdot 2 = 2.$$

The only thing left to show is that $f*$ acts on the positive generator of $A^0(X)$, which is the weight c determined by $c(\sigma_2) = 1$ and $c(\sigma_0) = 2 = c(\sigma_1)$, also as multiplication by two, but this is easy to see:

$$\begin{aligned}(\pi^*c)(\sigma_i') &= c(\sigma_i) \cdot [N : (\Psi(N') + N_{\sigma_i})] \\ &= \begin{cases} 2 \cdot 1 & \text{, if } i = 0,1 \\ 1 \cdot 2 & \text{, if } i = 2 \end{cases} \\ &= 2.\end{aligned}$$

Hence

$$\begin{aligned}\pi^*\colon A^*(\mathbb{P}(1,1,2)) &\to A^*(\mathbb{P}^2) \\ c &\mapsto 2 \cdot c.\end{aligned}$$

Example 3.28. This time we consider the weighted projective surfaces $\mathbb{P}(1,p,q) = X(\Delta)$ for some positive integers p, q with $\gcd(p,q) = 1$. Let $c\colon \Delta^{(1)} \to \mathbb{Z}$ be the generating Minkowski weight with

$c(\rho_0) = 1$, $c(\rho_1) = p$ and $c(\rho_2) = q$. Then

$$\begin{aligned}
(\pi^* c)(\rho_i') &= c(\rho_i) \cdot [N : (\Psi(N') + N_{\rho_i})] \\
&= \begin{cases} 1 \cdot (p \cdot q) & \text{, if } i = 0 \\ p \cdot q & \text{, if } i = 1 \\ q \cdot p & \text{, if } i = 2 \end{cases} \\
&= p \cdot q
\end{aligned}$$

Moreover, for the weight $c\colon \{0\} \to \mathbb{Z}$ with $c(\{0\}) = 1$ we see that

$$(\pi^* c)(\{0'\}) = c(\{0\}) \cdot [N : (\Psi(N') + N_{\rho_i})] = p \cdot q.$$

4 The Theorem of Beilinson

The theorem of Beilinson, giving explicit bounded resolutions in terms of vector bundles for every coherent sheaf on projective space, was discussed and generalized in many ways since it has been published in 1978. For example Ancona and Ottaviani [VA89] extended it to Schubert varieties and Kapranov [Kap88], respectively [Kap84] to certain quadrics. Recently Alberto Canonaco [Can00] introduced similar Beilinson-resolutions on weighted projective spaces. Since the proofs of those theorems and the strongest formulations are based on the language of derived categories, we start this chapter by recalling some basics from category theory. This will enable us to give the proofs and state the main ideas of Beilinson and Canonaco, before closing with some examples.

4.1 Category Theory

In this section we introduce derived categories to give some ideas what they are used for and to present some properties we will use later on. Everything presented here can be found in the standard literature, see for example [Ver94] or [Huy06] for good introductions. Since we have to fix the notation anyway it seems reasonable to recall what we need. We assume familiarity with the basic notions such as additive and abelian categories, functors, natural transformations etc.

Derived Categories

Definition 4.1. Let $\mathfrak{A}$ be an abelian category. A **complex** X of objects in $\mathfrak{A}$ is an ordered set of pairs $(X^n, d_X^n)_{n\in\mathbb{Z}}$ where $X^n \in Ob(\mathfrak{A})$ are objects for all n and $d^n\colon X^n \to X^{n+1}$ morphisms with the property $d_X^{n+1} \circ d_X^n = 0$ for all n, so it can be seen as a sequence

$$\ldots \to X^{n-1} \xrightarrow{d^n} X^n \xrightarrow{d^{n+1}} X^{n+1} \to \ldots \tag{4.2}$$

such that the composition of two morphisms is zero. We call $d := (d^n)_{n\in\mathbb{Z}} := (d_X^n)_{n\in\mathbb{Z}}$ the **differential** of X. A morphism of complexes $f\colon X \to Y$ is a sequence of morphisms

$$(f^n\colon X^n \to Y^n)_{n\in\mathbb{Z}},$$

such that the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] & X^{n-1} \arrow[r, "d_X^{n-1}"] \arrow[d, "f^{n-1}"'] & X^n \arrow[r, "d_X^n"] \arrow[d, "f^n"'] & X^{n+1} \arrow[r, "d_X^{n+1}"] \arrow[d, "f^{n+1}"'] & \cdots \\
\cdots \arrow[r] & Y^{n-1} \arrow[r, "d_Y^{n-1}"'] & Y^n \arrow[r, "d_Y^n"'] & Y^{n+1} \arrow[r, "d_Y^{n+1}"'] & \cdots .
\end{tikzcd}$$

Having defined objects and morphisms we can form the **category $\mathbf{C}(\mathfrak{A})$ of complexes** of $\mathfrak{A}$. Similarly we define $C^b(\mathfrak{A})$ to be the category of bounded complexes of $\mathfrak{A}$, that is the full subcategory of $C(\mathfrak{A})$ whose objects consist of all X with all but finitely many X^n zero.

Remark. Every object X in $\mathfrak{A}$ can be seen as a complex in its own rights by setting $X^0 := X$, $X^n := 0$ for all $n \neq 0$ and with differential the zero map. Hence we can embed $\mathfrak{A}$ as a full subcategory of $C(\mathfrak{A})$.

Definition 4.3. We say that two morphisms of complexes $f, g\colon X \to Y$ are **homotopic**, and we will write $f \sim g$, if there is a sequence of maps $k := (k^n)_{n\in\mathbb{Z}}$, called a **homotopy**, such that $k^n \in Hom_{\mathfrak{A}}(X^n, Y^{n-1})$ and

$$f^n - g^n = d_Y^{n-1} \circ k^n + k^{n+1} \circ d_X^n$$

for all n. This can be visualized by the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & X^{n-1} \arrow[r, "d_X^{n-1}"] \arrow[d, shift right, "f^{n-1}"'] \arrow[d, shift left, "g^{n-1}"] & X^n \arrow[r, "d_X^n"] \arrow[d, shift right, "f^n"'] \arrow[d, shift left, "g^n"] \arrow[dl, "k^n"] & X^{n+1} \arrow[r, "d_X^{n+1}"] \arrow[d, shift right, "f^{n+1}"'] \arrow[d, shift left, "g^{n+1}"] \arrow[dl, "k^{n+1}"] & \cdots \\
\cdots \arrow[r] & Y^{n-1} \arrow[r, "d_Y^{n-1}"'] & Y^n \arrow[r, "d_Y^n"'] & Y^{n+1} \arrow[r, "d_Y^{n+1}"'] & \cdots .
\end{tikzcd}$$

The relation $\sim$ on $C(\mathfrak{A})$ is in fact an equivalence relation and moreover compatible with the composition of morphisms, i.e. if

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] & Z
\end{tikzcd}$$

and $f \sim g$, $f' \sim g'$ then we automatically have $f' \circ f \sim g' \circ g$. This leads to a well defined next

Definition 4.4. The **homotopy category** $K(\mathfrak{A})$ is the category obtained from $C(\mathfrak{A})$ by setting $Ob(K(\mathfrak{A})) = Ob(C(\mathfrak{A}))$ and

$$Hom_{K(\mathfrak{A})}(X, Y) = Hom_{C(\mathfrak{A})}(X, Y)/\sim .$$

Again, by $K^b(\mathfrak{A})$ we denote the subcategory of bounded complexes.

Remark. If $\mathfrak{A}$ is an additive category, then so is $K(\mathfrak{A})$, but if $\mathfrak{A}$ is abelian, it might happen that $K(\mathfrak{A})$ looses this property.

Definition 4.5. As in topology we can now define for every complex $X \in Ob(C(\mathfrak{A}))$ and every integer n the **n-cycle** $Z^n(X) := \mathrm{Ker}(d_X^n)$, the **n-boundary** $B^n(X) := \mathrm{Im}(d_X^{n-1})$ and the **n-th cohomology** to be

$$H^n(X) := Z^n(X)/B^n(X) := \mathrm{Coker}(\mathrm{Im}(d_X^{n-1}) \to \mathrm{Ker}(d_X^n)).$$

Lemma 4.6. *Every morphism $f\colon X \to Y$ gives rise to a morphism on n-th cohomology $H^n(f)\colon H^n(X) \to H^n(Y)$ for every n. Moreover, if g is a morphism homotopic to f, then $H^n(f) = H^n(g)$ for all $n \in \mathbb{Z}$ This means that applying the functor H^n factors through the homotopy category, i.e. the diagram*

$$\begin{tikzcd}
C(\mathfrak{A}) \arrow[rr, "H^n"] \arrow[dr, "\pi"'] & & \mathfrak{A} , \\
& K(\mathfrak{A}) \arrow[ur, dotted, "\exists"'] &
\end{tikzcd}$$

where π denotes the natural map taking a morphism to its class, commutes.

Lemma 4.7. *Suppose that there is an exact sequence of complexes*

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

i.e. exact at every position, then there is an induced long exact cohomology sequence

$$\ldots \longrightarrow H^n(X) \xrightarrow{H^n(f)} H^n(Y) \xrightarrow{H^n(g)} H^n(Z) \xrightarrow{\delta^n} H^{n+1}(X) \xrightarrow{H^{n+1}(f)} \ldots .$$

Proof. For the proofs of the last two statements see for example [Wei94] □

So we see that cohomology behaves well on the homotopy category and we can make use of the long exact cohomology sequence. Thus we can compare two different complexes by studying their behavior on the level of cohomology. If two complexes have isomorphic cohomology they are considered quite similar and we would like to identify them in another category, which we will then call the derived category of $\mathfrak{A}$. To make this precise and to describe derived categories effectively we need one last notion, namely the following

Definition 4.8. A morphism $f\colon X \to Y$ in $C(\mathfrak{A})$ is called a **quasi isomorphism** or by abuse of notation simply a **qis** if $H^n(f)\colon H^n(X) \to H^n(Y)$ is an isomorphism for each n.

Since the composition of two quasi isomorphisms is again a quasi isomorphism and by the last remarks the notion of a qis also makes sense in the homotopy category. The next theorem now ensures us that the derived category, the category which makes qis in $K(\mathfrak{A})$ invertible, exists unique up to isomorphism:

Theorem 4.9. *For every abelian category $\mathfrak{A}$ there exists a category $D(\mathfrak{A})$, called the* **derived category**, *and a functor $Q\colon C(\mathfrak{A}) \to D(\mathfrak{A})$ such that for every qis f the morphism $Q(f)$ is an isomorphism in $D(\mathfrak{A})$ and we have the following universal property:*
Let $F\colon C(\mathfrak{A}) \to \mathfrak{D}$ be an arbitrary functor into a category $\mathfrak{D}$ which makes qis invertible, then F factors uniquely through Q, i.e.

$$\begin{tikzcd} C(\mathfrak{A}) \arrow[rr, "Q"] \arrow[dr, "F"'] & & D(\mathfrak{A}) \arrow[dl, dotted, "\exists!"] \\ & \mathfrak{D} & \end{tikzcd} .$$

Proof. A proof can be found in See [Huy06, Theorem 2.7] ☐

This pure existence statement is almost useless for applications, so let us at least characterize the morphisms in our new category.

Proposition 4.10. *A morphism in $D(\mathfrak{A})$ between two objects X and Y, denoted by $X \dashrightarrow Y$ to distinguish it from a morphism in $K(\mathfrak{A})$, is given by a triple (g, h, Z), where Z is a complex, $g\colon Z \to X$, $h\colon Z \to Y$ are morphisms and g additionally is a qis. Roughly speaking the morphism f is obtained by composition of the inverse of the qis g and the morphism of complexes h.*

Remark. Choosing $Z = Y$ and $h = id_Y$ one sees, using this description, that quasi isomorphism are in fact invertible in $D(\mathfrak{A})$. Moreover we can now declare two complexes X and Y to be equivalent and write $X \sim Y$, if they are isomorphic in $D(\mathfrak{A})$. Note however that in this case their cohomology is isomorphic, but in general there is no quasi isomorphism between them!

Objects of the derived category are again the same as the objects in the homotopy category, but morphisms in $D(\mathfrak{A})$ do not correspond to "real" morphisms of complexes in general. If so, then we loose the notion of kernels and cokernels, which means that $D(\mathfrak{A})$ is not abelian and in particular we can not speak about exactness anymore. This defect can partially be supplied by the triangulated structure every derived category inherits and which we will introduce next.

Triangulated Categories

Definition 4.11. For every category of complexes $C(\mathfrak{A})$ we define the shift functor of degree i to be the functor $T^i\colon C(\mathfrak{A}) \to C(\mathfrak{A})$ which assigns to every complex X and every integer n the complex $T(X)$ with

$$(T^i(X))^n := X^{n+i} \tag{4.12}$$
$$d^n_{T^i(X)} := (-1)^i d_X^{n+i} \tag{4.13}$$

and to every morphism f a corresponding $T^i(f)$ with

$$(T^i(f))^n := f^{n+i}.$$

We will mainly use this definition in the case of $i = 1$, where we will simply write $T := T^1$. This is especially important for the following

Definition 4.14. Let $f\colon X \to Y$ be a morphism of complexes. Then we denote by $\mathrm{Cone}(f)$ or $\mathrm{Cone}(X \to Y)$ the **mapping cone of f**, which is a complex in $C(\mathfrak{A})$ defined by

$$\mathrm{Cone}(f) := T(X) \oplus Y \tag{4.15}$$
$$d_{\mathrm{Cone}(f)} := \begin{pmatrix} T(d_x) & T(f) \\ 0 & d_Y \end{pmatrix} \tag{4.16}$$

Remark. For any given cone we have canonical morphisms of complexes

$$i_f\colon Y \to \mathrm{Cone}(f) \qquad i_f^n = \begin{pmatrix} 0 & id_{Y^n} \end{pmatrix}$$

and

$$p_f\colon \mathrm{Cone}(f) \to X[1] \qquad p_f^n = \begin{pmatrix} id_{X^{n+1}} \\ 0 \end{pmatrix}.$$

Definition 4.17. We call a sextuple (A, B, C, u, v, w) given by a sequence of the form

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A)$$

for $A, B, C \in \mathrm{Ob}(C(\mathfrak{A}))$ and $u, v, w \in \mathrm{Hom}(C(\mathfrak{A}))$ a **triangle**. Such a triangle is called **distinguished** if it is isomorphic to

$$X \xrightarrow{f} Y \xrightarrow{i_f} \mathrm{Cone}(f) \xrightarrow{p_f} T(X)$$

for some cone in $C(\mathfrak{A})$. A **morphism of triangles** is given by a triple (f, g, h) such that the following diagram is commutative

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & T(X) \arrow[d, "T(f)"] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & T(X'). \end{tikzcd}$$

Definition 4.18. We say that $K(\mathfrak{A})$, together with the distinguished triangles given by the shift operator and morphisms of triangles, is a **triangulated category**. We say that $D(\mathfrak{A})$ is triangulated or has a triangulated structure.
Moreover, we call every subcategory $\mathfrak{S} \subset K(\mathfrak{A})$ that is closed with respect to T and has the property that for every $f\colon X \to Y$ in $\mathrm{Hom}_{\mathfrak{S}}(X, Y)$ the morphisms $Cone(f)$, i_f and p_f also belong to $\mathfrak{S}$ a **triangulated subcategory**.

Remark. The usual definition of a triangulated category starts with a more general T and is defined by four axioms on the distinguished triangles with respect to this T. Since we only need this language in the special case of the derived category of coherent sheaves on $\mathbb{P}^n$ with the above defined shift functor, this definition will be sufficient for us to proceed.

Derived Functors

The idea of derived functors is the following: Given two abelian categories $\mathfrak{A}$ and $\mathfrak{B}$ and an additive functor F between them, it is always possible to extend F to a ∂-functor

$$F\colon K^b(\mathfrak{A}) \to K(\mathfrak{B}),$$

i.e. a functor that commutes with the shift functor T and preserves triangles, but we would like to go on and carry this functor over to the derived category, i.e. to an

$$\hat{F}\colon D^b(\mathfrak{A}) \to D(\mathfrak{B}),$$

which also preserves triangles and, applied to a morphism of complexes, quasi-isomorphisms. In general, without further assumptions, this is unfortunately too much to ask for. In some cases we can achieve a functor RF called the **right derived functor** (of course there is also an analogously defined left derived functor), which is in some sense pretty close to the original F. We will not give a general definition of RF (you might look this up in [Har66] or [PH71]), but the next theorem at least ensures us the existence in the cases we need them and even more important it tells us when it coincides with F. We need a last

Definition 4.19. Assume that RF exists, then we define for every object X in $\mathfrak{A}$

$$R^n(F(X)) := H^n(RF(X))$$

and call this object F**-acyclic**, if

$$R^n(F(X)) = 0 \quad \forall n \neq 0.$$

Recall that an element $X \in Ob(K(\mathfrak{A}))$ is called exact if all its cohomology is zero, i.e. $\mathrm{H}^n(X) = 0$ for all $n \in \mathbb{N}$. Then we have the following

Theorem 4.20. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be abelian categories, $F\colon K^b(\mathfrak{A}) \to K(\mathfrak{B})$ a ∂-functor and $\mathfrak{S}$ be a triangulated subcategory of $K^b(\mathfrak{A})$, such that every object of $K^b(\mathfrak{A})$ is quasi-isomorphic to an object of $\mathfrak{S}$ and for all exact objects $S \in Ob(\mathfrak{S})$ the object $F(S)$ is also exact, then:*

1. *F has a derived functor RF.*

2. *Suppose that there is an n_0 such that for all $n \geq n_0$ and all $Y \in Ob(A)$ we have $R^n(F(Y)) = 0$. If X is a complex of F-acyclic objects, that is X^i is F-acyclic for every i, if follows that $RF(X) = F(X)$.*

Proof. The proof can be found in [Har66, Corollary 5.3, p.56-58] $\square$

Recall that we say a category has **enough injectives**, respectively **projectives**, if every object of the category is isomorphic to a subobject of an injective, respectively to a factorobject of a projective object.

Lemma 4.21. *Let $\mathfrak{A}$ be an abelian category with enough injectives and projectives. Suppose for $M, N \in Ob(K(\mathfrak{A}))$ that all higher Ext-objects vanish, i.e.*

$$Ext^p(M^i, N^j) = 0 \quad \forall i, j \text{ and } p > 0.$$

Then every morphism in $D^b(\mathfrak{A})$ can be lifted to an actual morphism of complexes, to be precise

$$Hom_{D^b(\mathfrak{A})}(M, N) = Hom_{K^b(\mathfrak{A})}(M, N).$$

Proof. [Kap88, Lemma 1.6] $\square$

Note that the category of sheaves of $\mathcal{O}_X$-modules on a scheme X always has enough injectives , see for example [Har77, Proposition 2.2, p. 207], and the category of coherent sheaves on $\mathbb{P}^n$ has also enough projectives, since any sheaf in $Coh(\mathbb{P}^n)$ has a (finite) resolution in terms of vector bundles, which correspond to projective modules (see [Eis96, Appendix 3] or [Har77]). Before we can proof Beilinson's theorem we need another rather technical lemma.

Lemma 4.22. *Let $f\colon X \to Y$ be a separated morphism of schemes, $\mathcal{U}$ be an open affine covering and let $\mathcal{F}$ be a quasi-coherent. Then the Čech cohomology groups with respect to $\mathcal{U}$ are f_*-acyclic, i.e.*

$$R^n f_*(C^i(\mathcal{U}, \mathcal{F})) = 0 \quad \forall n, i$$

Proof. [Har66, Proposition 3.2, p.149] $\square$

4.2 The Classical Beilinson Theorem

In 1978 A.A. Beilinson published an only two pages long paper which reduced the problem of finding all coherent sheaves on $\mathbb{P}^n$ (Beilinson was mainly interested in finding all vector bundles between them) to a pure problem of linear algebra. However, it turned out that linear algebra can be very hard and classification was only possible in special cases, see for example [CO80], [Bar77] or [WB78] for more on these questions. Since we are only considering the subclass of toric sheaves it is reasonable, having good tools at hand to filter those sheaves, to hope that one should have less linear algebra problems and hopefully more success solving classifying problems .
Since the original paper of Beilinson is practically unreadable, Kapranovs terminology is inconsistent in the english translation and the very well written paper of Ancona and Ottaviani is practically unavailable, we will reprove the theorem to make clear what the ideas leading to this important result are.

We need two lemmata from Beilinson's work to proceed.

Lemma 4.23. *Let $X = \mathbb{P}^n$, then for all $i, j \in \mathbb{N}$ and all $p > 0$*

$$Ext^p_X(\Omega^i_X(i), \Omega^j_X(j)) = 0$$
$$Ext^p_X(\mathcal{O}_X(-i), \mathcal{O}_X(-j)) = 0.$$

Lemma 4.24. *Let $X = \mathbb{P}^n$, then for all $i, j \in \mathbb{N}$*

$$Hom_{Coh(X)}(\Omega^i_X(i), \Omega^j_X(j)) \cong \Lambda^{i-j}(V)$$
$$Hom_{Coh(X)}(\mathcal{O}_X(-i), \mathcal{O}_X(-j)) \cong S^{i-j}(V^*).$$

Proof. See [Bei78] for both proofs. □

To give a compact description of Beilinson's theorem we need to introduce another category

Definition 4.25. For every graded ring R and any natural number l we denote by $mod_{[0,l]}(R)$ the full subcategory of the category $mod(R)$ of R-modules, where each object $X \in \mathrm{Ob}(mod_{[0,l]}(R))$ is of the form

$$\bigoplus_{i=1}^{r} R(-j_i)$$

with $j_i \in \{0, \ldots, l\}$ for all $i = 1, \ldots, r$.

From the last lemma we get on $X = \mathbb{P}^n$ two induced additive functors

$$\begin{aligned} \hat{F}_S\colon mod_{[0,l]}(S(V^*)) &\to Coh(X) & \hat{F}_\Lambda\colon mod_{[0,l]}(\Lambda(V)) &\to Coh(X) \\ S(-j) &\mapsto \mathcal{O}_X(-j) & \Lambda(-j) &\mapsto \Omega^j_X(j) \end{aligned}$$

and the theorem of Beilinson states that these two functors extend to functors in the bounded homotopy, respectively derived category, i.e.

Theorem 4.26 (Theorem of Beilinson; abstract version)**.** *Both functors introduced above extend to exact functors of triangulated categories*

$$F_S\colon K^b(mod_{[0,l]}(S(V^*))) \to D^b(Coh(X))$$
$$F_\Lambda\colon K^b(mod_{[0,l]}(\Lambda(V))) \to D^b(Coh(X)).$$

Moreover F_S and $F_\wedge$ are equivalences of categories.

We will not proof the theorem in this generality, the interested reader might look it up in [Bei78], but should be aware that Beilinson's proof is only about two pages long and leaves out a lot of important steps, which makes the paper very hard to read. Since we want to use the theorem of Beilinson for classification problems we need a another, more concrete version of it:

Theorem 4.27 (Theorem of Beilinson; concrete version). *Let $X = \mathbb{P}^n$ and $\mathcal{F} \in Coh(X)$. Then for every $l \in \mathbb{Z}$ there are two bounded complexes B_I and B_{II} of vector bundles which we accordingly call the Beilinson I, respectively Beilinson II resolution of $\mathcal{F}$, having the following properties:*

1. *The complexes are exact, except for the* 0*-th position and we have*

$$H^i(B_I(l)) = H^i(B_{II}(l)) = \begin{cases} \mathcal{F}(l), & i = 0 \\ 0, & i \neq 0 \end{cases}.$$

2. *The resolutions are explicitly given by*

   (a) $B_I^k(l) = \sum_{j+k=i} H^i(\mathcal{F}(l) \otimes \mathcal{O}_X(-j)) \otimes_k \Omega_X^j(j)$

   (b) $B_{II}^k(l) = \sum_{j+k=i} H^i(\mathcal{F}(l) \otimes \Omega_X^j(j)) \otimes_k \mathcal{O}_X(-j)$

Proof. The idea of the proof is the following: Find a resolution in terms of the building blocks $\mathcal{O}_X(-j)$ and $\Omega_X^j(j)$, which is actually the easy part, then tensorize with $\mathcal{F}$ and work within the derived category using all the theoretical stuff we introduced in the last section in order to obtain the desired form and in "real" morphisms from theorem 4.21.
Let p, q: $X \times X \to X$ denote the first and second projection map, then with this notation the two dual Euler sequences

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow V \otimes \mathcal{O}_X \longrightarrow (\Omega_X^1(1))^* \longrightarrow 0$$
$$0 \longrightarrow \Omega_X^1(1) \longrightarrow V^* \otimes \mathcal{O}_X \longrightarrow \mathcal{O}_X(1) \longrightarrow 0$$

induce a morphism

$$\begin{tikzcd} p^*\Omega_X^1(1) \otimes q^*\mathcal{O}_X(-1) \arrow[r] \arrow[dr, dotted] & V^* \otimes V \otimes \mathcal{O}_{X\times X} \arrow[d] \\ & \mathcal{O}_{X\times X}, \end{tikzcd}$$

where the morphism on the right hand side is given by evaluating a functional at a vector, i.e. $V^* \otimes V \to k$ with $(f, v) \mapsto f(v)$.
Using the box-tensor notation, writing $\Omega_X^1(1) \boxtimes \mathcal{O}_X(-1)$ for $p^*\Omega_X^1(1) \otimes q^*\mathcal{O}_X(-1)$, we claim that this is contained in the kernel of the surjection $\mathcal{O}_X \boxtimes \mathcal{O}_X \cong \mathcal{O}_{X\times X} \to \mathcal{O}_\Delta$, where Δ denotes as usual the image of the diagonal morphism on X. Hence, we can factor through the ideal sheaf $\mathcal{I}_\Delta$ of Δ to obtain

$$\begin{tikzcd} & \Omega_X^1(1) \boxtimes \mathcal{O}_X(-1) \arrow[r] \arrow[d] \arrow[dr, dotted] & V^* \otimes V \otimes \mathcal{O}_{X\times X} \arrow[d] & & \\ 0 \arrow[r] & \mathcal{I}_\Delta \arrow[r] \arrow[d] & \mathcal{O}_{X\times X} \arrow[dr] & & \\ & 0 & & \mathcal{O}_\Delta \arrow[dr] & \\ & & & & 0 \end{tikzcd}$$

To see this, note that $\Omega_X^1 \cong \mathcal{I}_\Delta/\mathcal{I}_\Delta^2$ by [Har77, Section 2.5] and consider for a point $(\langle v\rangle, \langle w\rangle) \in X \times X$ the stalks of the building blocks:

$$\Omega_X^1(1)(\langle v\rangle) = (V/\langle v\rangle)^* \subset V^*$$
$$\mathcal{O}_X(-1)(\langle w\rangle) = \langle w\rangle \subset V$$

which give us the induced map

$$\begin{tikzcd}
(V/\langle v\rangle)^* \otimes \langle w\rangle \arrow[r] \arrow[dr, dotted, "\varphi"] & V^* \otimes V \arrow[d] \\
& k,
\end{tikzcd}$$

where $f \otimes \lambda w \mapsto f(\lambda w) = \lambda f(w)$. Since $(V/\langle v\rangle)^*$ contains all linear forms vanishing on $\langle v\rangle$, we see that

$$(\langle v\rangle, \langle w\rangle) \in \Delta \Leftrightarrow \langle v\rangle = \langle w\rangle \Rightarrow \varphi(\langle v\rangle, \langle w\rangle) = 0$$

and if $\langle v\rangle \neq \langle w\rangle$ there always exists an f and an element $x \in \langle w\rangle$ such that $f(x) \neq 0$. So in this case $\varphi \neq 0$. Therefore we get a Koszul complex, i.e. a finite resolution of $\mathcal{O}_\Delta$, by

$$\begin{aligned} 0 \longrightarrow \wedge^n(\Omega^1_X(1) \boxtimes \mathcal{O}_X(-1)) \longrightarrow \ldots \longrightarrow \wedge^1\, (\Omega^1_X(1) \boxtimes \mathcal{O}_X(-1)) \\ \longrightarrow \mathcal{O}_{X\times X} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0 \end{aligned}$$

which is, since both $\Omega^1_X(1)$ and $\mathcal{O}_X(-1)$ are locally free, isomorphic to

$$0 \longrightarrow \Omega^n_X(n) \boxtimes \mathcal{O}_X(-n) \longrightarrow \ldots \longrightarrow \Omega^1_X(1) \boxtimes \mathcal{O}_X(-1) \longrightarrow \mathcal{O}_{X\times X} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0.$$

We claim that this sequence is exact. This stems from the observation that first of all the complex is exact at every position except for the last one and $rk(\Omega^1_X(1) \boxtimes \mathcal{O}_X(-1)) = n$, $rk(\mathcal{O}_{X\times X}) = 1$.Thus we have $n = \operatorname{codim}(\Delta) \geq n-1$ and so the last morphism has to be surjective.
In the next step we tensorize with $q^*\mathcal{F}$ to obtain a resolution of $q^*\mathcal{F}|_\Delta$, namely

$$0 \longrightarrow \Omega^n_X(n) \boxtimes \mathcal{F}(-n) \xrightarrow{u_n} \ldots \xrightarrow{u_2} \Omega^1_X(1) \boxtimes \mathcal{F}(-1) \xrightarrow{u_1} q^*\mathcal{F} \xrightarrow{u_0} q^*\mathcal{F}|_\Delta \longrightarrow 0.$$

By abuse of notation, we will write M_i for $\Omega^i_X(i) \boxtimes \mathcal{F}(-i)$ and every i. Note that for any short exact sequence

$$0 \dashrightarrow X \dashrightarrow Y \dashrightarrow Z \dashrightarrow 0$$

Z is in the derived category equivalent to $\operatorname{Cone}(X \dashrightarrow Y)$ and therefore we can write $q^*\mathcal{F}$ as iteration of the cones of morphisms between the M_i. Starting with

$$\operatorname{Cone}(M_n \dashrightarrow M_{n-1}) = \operatorname{Cone}(u_n),$$

we go on inductively and arrive at

$$\begin{aligned} q^*\mathcal{F}|_\Delta = \operatorname{Cone}(\ldots \operatorname{Cone}(\operatorname{Cone}(\operatorname{Cone}(M_n \dashrightarrow M_{n-1}) \\ \dashrightarrow M_{n-2}) \dashrightarrow M_{n-3}) \ldots) \;. \end{aligned}$$

Since $D^b(\operatorname{Coh}(X))$ is triangulated and Rp_*, which exists by theorem 4.20, preserves cones of morphisms, we obtain an equivalence

$$\begin{aligned} \mathcal{F} \cong Rp_*(q^*\mathcal{F}) \sim \operatorname{Cone}(\ldots \operatorname{Cone}(\operatorname{Cone}(\operatorname{Cone}(Rp_*M_n \dashrightarrow Rp_*M_{n-1}) \\ \dashrightarrow Rp_*M_{n-2}) \dashrightarrow Rp_*M_{n-3}) \ldots) \;. \end{aligned} \tag{4.28}$$

Next we claim that

$$Rp_*M_j = Rp_*(\Omega^j_X(j) \boxtimes \mathcal{F}(-j)) \sim \Omega^j_X(j) \otimes \mathrm{H}(\mathcal{F}(-j))$$

in $D^b(\operatorname{Coh}(X))$ where on the right hand side the differentials are zero.
For the proof we choose an affine open covering $\mathcal{U}$ of X and consider the Čech complex $\mathcal{C} :=$

$C(\mathcal{U}, \mathcal{F}(-j))$. Since q is flat and $\Omega^j_X(j)$ locally free, applying q^* and tensoring by $\Omega^j_X(j)$ is exact, so it follows that $\Omega^j_X(j) \boxtimes \mathcal{F}(-j)$ and $\Omega^j_X(j) \boxtimes \mathcal{C}$ are equivalent in $D^b(\mathrm{Coh}(X \times X))$. Thus

$$Rp_*(\Omega^j_X(j) \boxtimes \mathcal{F}(-j)) \sim Rp_*(\Omega^j_X(j) \boxtimes \mathcal{C}).$$

Lemma 4.22 tells us now that each term $\mathcal{C}^l$ occurring in the Čech complex is p_*-acyclic, and using the Künneth formula (see for example [PH71, Theorem 2.1, p.172] or [Wei94, Theorem 3.6.1, p.87]):

$$\mathrm{H}^m(Rp_*(\Omega^j_X(j) \boxtimes \mathcal{C}^l)) \cong \bigoplus_{i+j=m} \mathrm{H}^i(Rp_*\Omega^j_X(j)) \otimes \mathrm{H}^j(Rp_*\mathcal{C}^l) = 0 \quad \forall m$$

we conclude that also $\Omega^j_X(j) \boxtimes \mathcal{C}^l$ is p_*-acyclic, so in this case we know from the second part of theorem 4.20 that Rp_* and p_* coincide, when applied to $\Omega^j_X(j) \boxtimes \mathcal{C}$, i.e. we have

$$Rp_*(\Omega^j_X(j) \boxtimes \mathcal{C}) = p_*(\Omega^j_X(j) \boxtimes \mathcal{C}) \sim \Omega^j_X(j) \otimes \mathrm{H}^0(X, \mathcal{C}).$$

Thus, we finally get

$$Rp_*M_j \sim \Omega^j_X(j) \otimes \mathrm{H}^0(X, \mathcal{C}) \sim \Omega^j_X(j) \otimes \mathrm{H}(X, \mathcal{F}(-j)),$$

where the last equivalence comes from the fact that the complex $\mathcal{C}$ is in $D^b(\mathrm{Coh}(X))$ isomorphic to the complex given by its cohomology, that is the cohomology of $\mathcal{F}(-j)$, with all differentials zero. Lemma 4.23 allows us to make use of lemma 4.21 and so we can lift the dotted arrows in (4.28) to actual morphisms of complexes in $K^b(\mathrm{Coh}(X))$. We use the notation $G_j := \mathrm{Ker}(u_j) = \mathrm{Im}(u_{j+1})$ to split the resolution of $q^*\mathcal{F}|_\triangle$ into short ones in order to correctly calculate the cones:

$$0 \longrightarrow G_j \xrightarrow{f_j} M_j \xrightarrow{g_j} G_{j-1} \longrightarrow 0$$

Then obviously $u_j = f_{j-1} \circ g_j$ and $H_{j-1} = \mathrm{Cone}(f_j)$ by the same argument as before and hence

$$Rp_*G_{j-1} \sim \mathrm{Cone}(Rp_*G_j \xrightarrow{Rp_*f_j} Rp_*M_j) \quad \forall j.$$

In the last step we inductively construct complexes R_s with the following two properties:

- $R^k_s = \bigoplus_{j-i=s-k} (Rp_*M_j)^i = \bigoplus_{j-i=s-k} \mathrm{H}^i(\mathcal{F}(-j)) \otimes \Omega^j_X(j)$ which is our desired resolution, and
- $R_s \sim Rp_*G_{s-1}$ in $D^b(\mathrm{Coh}(X))$ relating the new complexes to the old ones.

Starting from $s = n$, inductively going down, there is the natural choice

$$R_n := Rp_*M_n = Rp_*G_{n-1}$$

which has the second property by definition and the first by the properties of the mapping cone in $K^b(\mathrm{Coh}(X))$. Suppose now that we have already constructed R_{s+1} for some s, then the equivalence $Rp_*G_{s-1} \sim Cone(Rp_*G_s \xrightarrow{Rp_*f_s} Rp_*M_s)$ and the induction hypothesis $R_{s+1} \sim Rp_*G_s$ imply that there is a morphism

$$R_{s+1} \dashrightarrow Rp_*M_s$$

in $D^b(X)$, which can again be lifted to a morphism

$$\nu_s\colon\ R_{s+1} \longrightarrow Rp_*M_s$$

in $K^b(X)$. We simply define R_s to be the mapping cone of ν_s in $K^b(X)$, i.e.

$$R_s := \mathrm{Cone}(\nu_s) \sim Rp_*G_{s-1}.$$

So the two properties follow by construction and we finish the proof by remarking that if we change the roles of $\Omega^j_X(j)$ and $\mathcal{O}_X(-j)$ we obtain the second resolution without changing the proof. $\square$

4.3 Beilinson's theorem on Weighted Projective Spaces

In 2000 Alberto Canonaco showed that the Beilinson theorem can be extended to weighted projective spaces. The idea was the following: Take a coherent sheaf $\mathcal{F}$ on $\mathbb{P}(Q)$, then use the finite map π we introduced in the last chapter to lift this sheaf to $\mathbb{P}^n$. Now apply Beilinson to $\pi^*\mathcal{F}$ to obtain a resolution, which then can be pushed down again by the exact direct image functor π_*^0 for a resolution of $\mathcal{F}$. The last step involves some technical problems.
In order to state the theorem of Beilinson on Weighted Projective Spaces we need to introduce a new class of sheaves, which are called sheaves of logarithmic differentials. They are only needed in this section and can be replaced by less complicated sheaves later on, but nevertheless they occur when pushing down the differential sheaves on $\mathbb{P}^n$ by the functor π_*^χ and we need them for the building blocks on $\mathbb{P}(Q)$.

Logarithmic Differential Forms

Definition 4.29. Let $I \subset \{0, \ldots, n\}$ and $\mathbf{1}_I\colon \{0, \ldots, n\} \to \{0, 1\}$ the characteristic function of I. Define for every $j \in \mathbb{N}$ the free S'-module $\Omega^j_{S'}(\log y^I)$ generated by

$$\left\{ \frac{dy_{i_1}}{y_{i_1}^{\mathbf{1}_I(i_1)}} \wedge \ldots \wedge \frac{dy_{i_j}}{y_{i_j}^{\mathbf{1}_I(i_j)}} \middle| 0 \leq i_1 < \ldots i_j \leq n \right\}$$

and naturally graded by

$$\deg\left(\frac{dy_{i_1}}{y_{i_1}^{\mathbf{1}_I(i_1)}} \wedge \ldots \wedge \frac{dy_{i_j}}{y_{i_j}^{\mathbf{1}_I(i_j)}} \right) = q_{i_1}^{1-\mathbf{1}_I(i_1)} + \ldots + q_{i_j}^{1-\mathbf{1}_I(i_j)}.$$

We call this the **module of logarithmic differentials** with respect to I.

Similarly to the case of regular differentials we can now define degree zero morphisms

$$\Delta^j_I\colon \Omega^j_{S'}(\log y^I) \to \Omega^{j-1}_{S'}(\log y^I)$$

for any fixed index set I by the formula

$$\begin{aligned} &\Delta^j_I\left(\frac{dy_{i_1}}{y_{i_1}^{\mathbf{1}_I(i_1)}} \wedge \ldots \wedge \frac{dy_{i_j}}{y_{i_j}^{\mathbf{1}_I(i_j)}} \right) \\ &= \sum_{l=1}^{j} (-1)^{l-1} q_{i_l} y_{i_l}^{1-\mathbf{1}_I(i_l)} \frac{dy_{i_1}}{y_{i_1}^{\mathbf{1}_I(i_1)}} \wedge \ldots \wedge \widehat{\frac{dy_{i_l}}{y_{i_l}^{\mathbf{1}_I(i_l)}}} \wedge \ldots \wedge \frac{dy_{i_j}}{y_{i_j}^{\mathbf{1}_I(i_j)}} \end{aligned}$$

and we set $\overline{\Omega}^j_{S'}(\log y^I) := \mathrm{im}\Delta^j_I$, as before with the induced graduation.

Remark. Note that in the case of $I = \varnothing$ the definition of logarithmic differentials coincides with the regular differentials defined in 3.14, i.e.

$$\Omega^j_{S'} = \Omega^j_{S'}(\log y^\varnothing)$$

and $\Delta^j = \Delta^j_\varnothing$ for the morphisms, so the above definition is just an extension of the old one.

Moreover, by the same arguments and the convention $S' := \Omega^0_{S'}(\log y^I)$ we obtain that the sequence

$$0 \to \Omega^{n+1}_{S'}(\log y^I) \xrightarrow{\Delta_I^{n+1}} \ldots \xrightarrow{\Delta_I^2} \Omega^1_{S'}(\log y^I) \xrightarrow{\Delta_I^1} S' \to 0$$

is the Koszul sequence of $(q_0 y_0^{1-\mathbf{1}_I(0)}, \ldots, q_n y_n^{1-\mathbf{1}_I(n)})$.

Definition 4.30. For every $I \subset \{0, \ldots, n\}$ we define the **sheaf of logarithmic differentials** $\overline{\Omega}^j_{\mathbb{P}(Q)}(\log y^I)$ with respect to I to be the sheaf associated to $\overline{\Omega}^j_{S'}(\log y^I)$. Analogously we define for the twist by $l \in \mathbb{Z}$

$$\overline{\Omega}^j_{\mathbb{P}(Q)}(\log y^I)(l) := (\overline{\Omega}^j_{S'}(\log y^I)(l))^\sim.$$

Recall that we can identify every character $\chi \in \mu_Q^*$ as $\chi = (p_0, \dots, p_n)$ with uniquely determined $0 \leq p_i < q_i$. Hence we can define

$$I(\chi) := \{i \in \{0, \dots, n\} | p_i \neq 0\}$$

to be the index set where χ differs from the zero character. With this notation we finally have the following

Lemma 4.31. *For all $\chi \in \mu_Q^*$ and all integers l there is an $S(Q)$-module isomorphism*

$$(\overline{\Omega}^j_S(l))^\chi \cong \overline{\Omega}^j_{S'}(l)(\log y^{I(\chi)})(l - |\chi|).$$

Proof. [Can00, Lemma 2.6] □

Corollary 4.32. *With the same assumptions as in the lemma there is an isomorphism of coherent sheaves on $\mathbb{P}(Q)$*

$$\pi_*^\chi \Omega^j_{\mathbb{P}^n}(l) \cong (\overline{\Omega}^j_{\mathbb{P}(Q)})(\log y^{I(\chi)})(l - |\chi|)$$

Proof. We have $\pi_*^\chi \Omega^j_{\mathbb{P}^n}(l) \cong \pi_*^\chi(\Omega^j_S(l))^\sim \cong (\overline{\Omega}^j_{S'}(l)(\log y^{I(\chi)})(l-|\chi|))^\sim$, so the proof follows directly from the last lemma. □

Now we know everything important for the direct images of the building blocks, so we are ready to proof Beilinson's theorem on $\mathbb{P}(Q)$.

The result of Canonaco

Recall from the last chapter that we denote by μ_Q-Coh($\mathbb{P}^n$) the category of coherent sheaves on $\mathbb{P}^n$ on which the group μ_Q acts on and morphism are those preserving this action. Then the main ingredient for the proof of Canonaco's result can be formulated as follows.

Lemma 4.33. *Let $\mathcal{F} \in \mu_Q$-Coh($\mathbb{P}^n$), then there are two bounded μ_Q-complexes E_S and E_Λ of vector bundles, such that*

$$E^i_S = \bigoplus_{j=0}^n \mathcal{O}_{\mathbb{P}^n}(-j) \otimes_k H^{j+i}(\mathbb{P}^n, \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \Omega^j_{\mathbb{P}^n}(j))$$

$$E^i_\Lambda = \bigoplus_{j=0}^n \Omega^j_{\mathbb{P}^n}(j) \otimes_k H^{j+i}(\mathbb{P}^n, \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(-j))$$

and

$$H^i(E_S) \cong H^i(E_\Lambda) \cong \begin{cases} \mathcal{F} & , \text{if } i = 0 \\ 0 & , \text{else} \end{cases}.$$

Remark. Note that this lemma is very similar to Beilinson's theorem on $\mathbb{P}^n$, with the only difference that we are working in the category of μ_Q-sheaves. So the lemma essentially states that starting with an arbitrary $\mathcal{F}$ in $\mu_Q - Coh(\mathbb{P}^n)$ all sheaves in the classical Beilinson resolutions can be chosen to be μ_Q-sheaves and all morphisms to be μ_Q-equivariant.

Proof. The idea is again rather simple. We repeat the proof of the theorem on $\mathbb{P}^n$ and check at every step that the objects and morphisms belong to μ_Q-Coh($\mathbb{P}^n$). For the sheaves this is quite easy, but for the morphisms there are technical difficulties involved we don't want to stress too much. See [Can00, Proposition 3.2, lemma 3.3] for details. □

Theorem 4.34 (Theorem of Beilinson on Weighted Projective Spaces)**.** *Let $\mathcal{F} \in Coh(\mathbb{P}(Q))$. Then there are two bounded complexes $B_{Q,I}$ and $B_{Q,II}$ of coherent sheaves on $\mathbb{P}(Q)$ with the following properties:*

$$H^i(B_{Q,I}) = H^i(B_{Q,II}) = \begin{cases} \mathcal{F} & , i = 0 \\ 0 & , i \neq 0 \end{cases}$$

and $B^i_{Q,I} = \bigoplus_{j=0}^n V_j^{j+k}$, $B^i_{Q,II} = \bigoplus_{j=0}^n W_j^{j+k}$, *where*

$$V_j^i = \bigoplus_{\chi\in\mu_Q^*} \mathcal{O}_{\mathbb{P}(Q)}(-j-|\chi|)\otimes_k H^i(\mathbb{P}(Q),\mathcal{F}\otimes\overline{\Omega}^j_{\mathbb{P}(Q)}(log\ y^{I(-\chi)})(j-|-\chi|))$$
$$W_j^i = \bigoplus_{\chi\in\mu_Q^*} \overline{\Omega}^j_{\mathbb{P}(Q)}(log\ y^{I(\chi)})(j-|\chi|)\otimes_k H^i(\mathbb{P}(Q),\mathcal{F}\otimes\mathcal{O}_{\mathbb{P}(Q)}(-j-|-\chi|)).$$

Proof. We know that $\pi^*\mathcal{F}$ is in μ_Q-Coh$(\mathbb{P}^n)$, so we can apply lemma 4.33 to obtain the complex $E_S := E_S(\pi^*\mathcal{F})$, respectively $E_\Lambda := E_\Lambda(\pi^*\mathcal{F})$ where

$$E_S^i = \bigoplus_{j=0}^n P_j^{j+k} := \bigoplus_{j=0}^n \mathcal{O}_{\mathbb{P}^n}(-j)\otimes_k \mathrm{H}^{j+i}(\mathbb{P}^n,\pi^*\mathcal{F}\otimes_{\mathcal{O}_{\mathbb{P}^n}}\Omega^j(j)),$$
$$E_\Lambda^i = \bigoplus_{j=0}^n R_j^{j+k} := \bigoplus_{j=0}^n \Omega^j_{\mathbb{P}^n}(j)\otimes_k \mathrm{H}^{j+i}(\mathbb{P}^n,\pi^*\mathcal{F}\otimes_{\mathcal{O}_{\mathbb{P}^n}}\mathcal{O}_{\mathbb{P}^n}(-j))$$

We put $B_{Q,I} := \pi_*^{\mu_Q}(E_S), B_{Q,II} := \pi_*^{\mu_Q}(E_\Lambda)$and see that, since $\pi_*^{\mu_Q}$ is exact,

$$\mathrm{H}^i(B_{Q,I})\cong\mathrm{H}^i(B_{Q,II})\cong\mathrm{H}^i(\pi_*^{\mu_Q}(E_S))\cong\begin{cases}\pi_*^{\mu_Q}\pi^*\mathcal{F}\cong\mathcal{F} & i=0\\ 0 & k\neq 0\end{cases}.$$

Now we only have to calculate the direct image functor by using the results from the last chapter, to obtain the above resolution:

$$V_j^i := \pi_*^{\mu_Q}P_j^i \cong \bigoplus_{\chi\in\mu_Q^*}\pi_*^\chi\mathcal{O}_{\mathbb{P}^n}(-j)\otimes_k\mathrm{H}^i(\mathbb{P}^n,\pi^*\mathcal{F}\otimes\Omega^j_{\mathbb{P}^n}(j))^{-\chi},$$
$$W_j^i := \pi_*^{\mu_Q}R_j^i \cong \bigoplus_{\chi\in\mu_Q^*}\pi_*^\chi\Omega^j_{\mathbb{P}^n}(j)\otimes_k\mathrm{H}^i(\mathbb{P}^n,\pi^*\mathcal{F}\otimes\mathcal{O}_{\mathbb{P}^n}(-j))^{-\chi}$$

Note since we have to apply $\pi_*^{\mu_Q}$ to a tensor product, the null-character splits via this product and we have to take the sum over all characters. Moreover, we know that

$$\pi_*^\chi\mathcal{O}_{\mathbb{P}^n}(-j)\cong\mathcal{O}_{\mathbb{P}(Q)}(-j-|\chi|)$$
$$\pi_*^\chi\Omega^j_{\mathbb{P}^n}(j)\cong\overline{\Omega}^j_{\mathbb{P}(Q)}(\log\ y^{I(\chi)})(j-|\chi|),$$

by corollary 4.32 and lemma 3.10. Moreover

$$\mathrm{H}^i(\mathbb{P}^n,\pi^*\mathcal{F}\otimes\Omega^j_{\mathbb{P}^n}(j))^{-\chi}\cong\mathrm{H}^i(\mathbb{P}(Q)\mathcal{F}\otimes\overline{\Omega}^j_{\mathbb{P}(Q)}(\log\ y^{I(-\chi)})(j-|-\chi|)),$$
$$\mathrm{H}^i(\mathbb{P}^n,\pi^*\mathcal{F}\otimes\mathcal{O}_{\mathbb{P}^n}(-j))^{-\chi}\cong\mathrm{H}^i(\mathbb{P}(Q),\mathcal{F}\otimes\mathcal{O}_{\mathbb{P}(Q)}(-j-|-\chi|))$$

by lemma 3.9, which finishes the proof. □

Remark.
- Note that the structure of the vector spaces $\mathrm{H}^i(\mathbb{P}(Q),\_)$ is very important in the proof. It would be nice to replace the spaces by their dimension before pushing down, but this does not necessarily produce a Beilinson sequence, because the group μ_Q might decompose the cohomology groups into Eigenspaces. This means that after fixing discrete invariants in order to obtain moduli of sheaves on $\mathbb{P}(Q)$, we would only obtain a subclass of possible $\mathcal{F}$ with those prescribed invariants.
- Of course the two building blocks of $\mathbb{P}(Q)$,

$$\mathcal{O}_{\mathbb{P}(Q)}(-j-|\chi|)\quad\text{and}$$
$$\overline{\Omega}^j_{\mathbb{P}(Q)}(\log y^{I(-\chi)})(j-|-\chi|)$$

generate $D^b(Coh(\mathbb{P}(Q)$ as a triangulated category, but it is a priori not clear how to formulate a corresponding abstract version of the theorem for several reasons. For example, although we still have

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}(Q)}(-i), \mathcal{O}_{\mathbb{P}(Q)}(-j)) \cong S'_{i-j}$$

on $\mathbb{P}(Q)$, we loose the analogous description for the morphisms between the differential sheaves, since the vector space V does not have a meaning anymore. Moreover

$$\mathrm{Ext}^l(\mathcal{O}_{\mathbb{P}(Q)}(-i), \mathcal{O}_{\mathbb{P}(Q)}(-j)) \neq 0$$

in general, but this was essential for the proof of the theorem on $\mathbb{P}^n$. Hence both building blocks come with problems. In [Can06] Canonaco managed to solve these problems by introducing a graded scheme structure on weighted projective spaces, but this is far beyond the scope of this thesis.

Lemma 4.35. *For every $\varnothing \neq I \subset \{0, \ldots, n\}$ we have isomorphisms*

$$\overline{\Omega}^j_{\mathbb{P}(Q)}(log\ y^{I(\chi)})(j - |\chi|) \cong \bigoplus_{J \cap I \neq \varnothing} \mathcal{O}_{\mathbb{P}(Q)}(j - |\chi| - |Q_J|)^{\binom{\# I(\chi)-1}{j-\# J}},$$

where $|Q_J| := \sum_{i \in J} q_i$ and the power by the binomial coefficient means direct sums.

Proof. The proof is found in a remark in [Can06, p.51]. □

The consequence of the last lemma is that all terms in the Beilinson resolution involving sheaves of logarithmic differentials can be decomposed into direct sums of structure sheaves, so every $\mathcal{F}$ can be resolved by $\mathcal{O}_{\mathbb{P}(Q)}(-j)$ and $\overline{\Omega}^j_{\mathbb{P}(Q)}(j)$ for $j = 0, \ldots, n$.

4.4 Examples

Example 4.36. On $\mathbb{P}^2$ we can compute the first Beilinson complex explicitly:

$$\begin{aligned}
C^{-2} &= H^0(\mathcal{F}(-2)) \otimes \Omega^2_{\mathbb{P}^2}(2) \\
C^{-1} &= H^0(\mathcal{F}(-1)) \otimes \Omega^1_{\mathbb{P}^2}(1) \oplus H^1(\mathcal{F}(-2)) \otimes \Omega^2_{\mathbb{P}^2}(2) \\
C^{1} &= H^1(\mathcal{F}) \otimes \mathcal{O}_{\mathbb{P}^2} \oplus H^2(\mathcal{F}(-1)) \otimes \Omega^1_{\mathbb{P}^2}(1) \\
C^{0} &= H^0(\mathcal{F}) \otimes \mathcal{O}_{\mathbb{P}^2} \oplus H^1(\mathcal{F}(-1)) \otimes \Omega^1_{\mathbb{P}^2}(1) \oplus H^2(\mathcal{F}(-2)) \otimes \Omega^2_{\mathbb{P}^2}(2) \\
C^{2} &= H^2(\mathcal{F}) \otimes \mathcal{O}_{\mathbb{P}^2}
\end{aligned}$$

This sequence can be nicely reduces in order to obtain some classification results, for example in the following setting. (The reader who is not familiar with stable sheaves can find the necessary definitions in [Har80] or look them up in the next chapter.)

Lemma 4.37. *Let $\mathcal{F}$ be a stable vector bundle of rank 2 on $\mathbb{P}^2$ with fixed Chern classes $c_1 := c_1(\mathcal{F}) = 0$ and $c_2 := c_2(\mathcal{F}) = n \in \mathbb{Z}$. Then we have a Beilinson resolution of $\mathcal{F}$, given by*

$$0 \longrightarrow n\Omega^2_{\mathbb{P}^2}(2) \xrightarrow{A} n\Omega^1_{\mathbb{P}^2}(1) \xrightarrow{B} (n-2)\mathcal{O}_{\mathbb{P}^2} \longrightarrow 0,$$

where $A \in Mat(n \times n, V)$, $B \in Mat(n \times (n-2), V)$ and $\mathcal{F} \cong ker(B)/im(A)$. We will sometimes call this sequences a ***Beilinson monad****.*

Remark. This resolution is a complex in the sense that $A \wedge B = 0$, i.e.

$$(a_{ij}) \wedge (b_{jk}) = \sum_{j=1}^{n} a_{ij} \wedge b_{jk} = 0 \in \Lambda^2 V \qquad \forall i, k.$$

Proof. The proof of this statement can also be found in [CO80]. First of all we remark that on $\mathbb{P}^2$ $\mathcal{F}$ being stable is equivalent to $H^0(F) = 0$, see [Har80]. Tensoring the Euler sequence by $\mathcal{F}$, we obtain the sequence

$$0 \longrightarrow \Omega^1_{\mathbb{P}^2} \otimes \mathcal{F} \longrightarrow 3\mathcal{F}(-1) \longrightarrow \mathcal{F} \longrightarrow 0$$

and therefore $3H^0(\mathcal{F}(-1)) \subset H^0(\mathcal{F})$, which inductively implies that $H^0(F(-i)) = 0$ for all i. So we can already cancel out the three H^0-terms.
Next, by Serre duality, we get

$$\begin{aligned}
H^p(\mathcal{F})^* &\cong H^{2-p}(\mathcal{F}^* \otimes \Omega^2_{\mathbb{P}^2}) \\
&\cong H^{2-p}(\mathcal{F}^* \otimes \mathcal{O}_{\mathbb{P}^2}(-3)) \\
&\cong H^{2-p}(\mathcal{F}^*(-3)),
\end{aligned}$$

where the second isomorphism follows from the fact that $\Omega^2_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$, which can directly be read of from the Koszul sequence. Moreover, it is known that

$$0 = c_1(\mathcal{F}) = c_1(\Lambda^2(\mathcal{F}))$$

which implies for the line bundle $\Lambda^2(\mathcal{F})$ that it has to be isomorphic to $\mathcal{O}_{\mathbb{P}^2}$, since on $\mathbb{P}^2$ we have $\mathrm{Pic}(\mathbb{P}^2) \cong \mathbb{Z}$ via the first Chern class. Now we can consider the canonical pairing

$$\begin{tikzcd}
\mathcal{F} \times \mathcal{F} \arrow[r, "\phi"] \arrow[d] & \Lambda^2(\mathcal{F}) \cong \mathcal{O}_{\mathbb{P}^2} \; . \\
\mathcal{F} \otimes \mathcal{F} \arrow[ur] &
\end{tikzcd}$$

This pairing is non-degenerated, since $\mathcal{O}_{\mathbb{P}^2}$ has no zero divisors. Thus we in fact have an isomorphism $\mathcal{F} \cong \mathcal{F}^*$. Coming back to what we know from Serre duality, this actually means

$$H^p(\mathcal{F})^* \cong H^{2-p}(\mathcal{F}(-3)).$$

In particular

$$
\begin{aligned}
H^2(\mathcal{F})^* &\cong H^0(\mathcal{F}(-3)) \\
H^2(\mathcal{F}(-1))^* &\cong H^0(\mathcal{F}(-2)) \\
H^2(\mathcal{F}(-2))^* &\cong H^0(\mathcal{F}(-1)),
\end{aligned}
$$

hence we can also cancel all H^2-terms and the Beilinson sequence reduces to:

$$
0 \longrightarrow H^1(\mathcal{F}(-2)) \otimes \Omega^2_{\mathbb{P}^2}(2) \longrightarrow H^1(\mathcal{F}(-1)) \otimes \Omega^1_{\mathbb{P}^2}(1) \xrightarrow{\psi} H^1(\mathcal{F}) \otimes \mathcal{O}_{\mathbb{P}^2} \longrightarrow 0
$$

Since, again by Serre duality, we have an identification

$$
H^1(\mathcal{F}(-2)) \cong H^1(\mathcal{F}(-1))
$$

we only need to calculate the dimensions of the two finite dimensional vector spaces $H^1(\mathcal{F}(-1))$ and $H^1(\mathcal{F})$ to finish the proof. This can be done with the help of the Hirzebruch-Riemann-Roch theorem, which states that the Euler characteristic of $\mathcal{F}$ is given by a polynomial expression in the Chern classes. Consider the following diagram:

$$
\begin{tikzcd}
0 \arrow[r] & H^1(\mathcal{F}(-2))\otimes\Omega^2_{\mathbb{P}^2}(2) \arrow[r] \arrow[dr] & H^1(\mathcal{F}(-1))\otimes\Omega^1_{\mathbb{P}^2}(1) \arrow[r, "\psi"] & H^1(\mathcal{F})\otimes\mathcal{O}_{\mathbb{P}^2} \arrow[r] & 0 \\
 & & \mathcal{K}:=\mathrm{Ker}(\psi) \arrow[u] \arrow[r] & \mathcal{F} \arrow[r] & 0 \\
 & & 0 \arrow[u] & &
\end{tikzcd}
$$

We of course know the ranks of the sheaves involved, namely

$$
\mathrm{rk}(\Omega^2_{\mathbb{P}^2}(2)) = 1 \qquad \mathrm{rk}(\Omega^1_{\mathbb{P}^2}(1)) = 2 \qquad \mathrm{rk}(\mathcal{O}_{\mathbb{P}^2}) = 1.
$$

Thus we have a first relation for the dimensions of the cohomology groups, given by

$$
\begin{aligned}
2 = \mathrm{rk}(\mathcal{F}) &= -h^1(\mathcal{F}(-2)) + \mathrm{rk}(\mathcal{K}) \\
&= 2h^1(\mathcal{F}(-1)) - h^1(\mathcal{F}) - h^1(\mathcal{F}(-2)) \\
&= h^1(\mathcal{F}(-1)) - h^1(\mathcal{F}),
\end{aligned}
$$

or equivalently, abbreviating by $q := h^1(\mathcal{F}(-1))$ and $p := h^1(\mathcal{F})$, $q = p + 2$. Thus it suffices to show that $q = n$. Picking a class $h \in A^1(\mathbb{P}^2)$ which is under the isomorphism $A^1(\mathbb{P}^2) \cong \mathbb{Z}$ mapped to 1, by assumption of the lemma we know that the total Chern class of $\mathcal{F}$ is given by

$$
C(\mathcal{F}) = 1 + nh^2 = (1 + h^2)^n = (1 - h^2)^{-n} = (1 + h)^n \cdot (1 - h)^{-n},
$$

but on the other hand it is also determined by the resolution using the kernel $\mathcal{K}$ and the fact that the total Chern class is multiplicative, i.e.

$$
\begin{aligned}
c(\mathcal{F}) &= c(\mathcal{K}) \cdot c(q\Omega^2_{\mathbb{P}^2}(2))^{-1} \\
&= c(\mathcal{K}) \cdot c(q\mathcal{O}_{\mathbb{P}^2}(-1))^{-1} \\
&= c(\mathcal{K}) \cdot (1 - h)^{-q}
\end{aligned}
$$

Furthermore, by the Euler sequence we obtain

$$
c(\Omega^1_{\mathbb{P}^2}(1)) = c(3\mathcal{O}_{\mathbb{P}^2}) \cdot c(\mathcal{O}_{\mathbb{P}^2}(1))^{-1} = (1 + h)^{-1}
$$

and hence, using $c(r \cdot \mathcal{O}_{\mathbb{P}^2}) = 1$ for all r, we get

$$
C(\mathcal{K}) = c(q \cdot \Omega^1_{\mathbb{P}^2}(1))^{-1} = (1 + h)^q.
$$

Finally, combining these computations yields

$$(1+h)^q \cdot (1-h)^{-q} = c(\mathcal{K}) \cdot (1-h)^{-q} = C(\mathcal{F}) = (1+h)^n \cdot (1-h)^{-n}$$

and so $n = q$. $\square$

Corollary 4.38. *Every $\mathcal{F}$ of the lemma admits another resolution with $\mathcal{F}$ as cohomology, which we call the "Beilinson IIA" resolution, explicitly given by the following sequence*

$$0 \longrightarrow n\mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{M} (2n+2)\mathcal{O}_{\mathbb{P}^2} \xrightarrow{N} n\mathcal{O}_{\mathbb{P}^2}(1) \longrightarrow 0,$$

where this time M and N have entries in $S^1(V^*) \cong S_1$, i.e. the matrices consist of linear forms in the corresponding Cox-coordinates.

Proof. Starting from the sequence of the lemma, we can vertically add the Euler sequence

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & n\Omega^2_{\mathbb{P}^2}(2) \arrow[r] & n\Omega^1_{\mathbb{P}^2}(1) \arrow[r] \arrow[d] & (n-2)\mathcal{O}_{\mathbb{P}^2} \arrow[r] & 0 \\
 & & n(V^* \otimes \mathcal{O}_{\mathbb{P}^2}) \arrow[d] & & \\
 & & n\mathcal{O}_{\mathbb{P}^2}(1) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

Using again the isomorphism $\Omega^2_{\mathbb{P}^2}(2) \cong \mathcal{O}_{\mathbb{P}^2}(-1)$ we can naturally complete this diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & n\Omega^2_{\mathbb{P}^2}(2) \arrow[r] \arrow[d, "\cong"'] & n\Omega^1_{\mathbb{P}^2}(1) \arrow[r] \arrow[dd] & (n-2)\mathcal{O}_{\mathbb{P}^2} \arrow[r] \arrow[dd, equal] & 0 \\
 & n\mathcal{O}_{\mathbb{P}^2}(-1) \arrow[d, dotted, "M"', "\exists"] & & & \\
0 \arrow[r] & (2n+2)\mathcal{O}_{\mathbb{P}^2} \arrow[r] \arrow[d, dotted, "N"', "\exists"] & n(V^* \otimes \mathcal{O}_{\mathbb{P}^2}) \arrow[r] \arrow[d] & (n-2)\mathcal{O}_{\mathbb{P}^2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & n\mathcal{O}_{\mathbb{P}^2}(1) \arrow[r, equal] \arrow[d] & n\mathcal{O}_{\mathbb{P}^2}(1) \arrow[r] \arrow[d] & 0, & \\
 & 0 & 0 & &
\end{tikzcd}$$

where $(2n+2)\mathcal{O}_{\mathbb{P}^2}$ is by linear algebra the kernel of the middle row. Because of the two equalities, it is easy to check that the "outer" short exact sequences in fact have the same cohomology, which finishes the proof. $\square$

Corollary 4.39. *Completely analogous to the last corollary we obtain a third equivalent resolution, which we will call the "Beilinson IIB" resolution, given by*

$$0 \longrightarrow n\mathcal{O}_{\mathbb{P}^2}(-2) \xrightarrow{M} 2n\mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{N} (n-2)\mathcal{O}_{\mathbb{P}^2} \longrightarrow 0.$$

Proof. This time we simply have to extend the sequence of the lemma from above by the Koszul complex

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & n\Lambda^3 V^* \otimes \mathcal{O}_{\mathbb{P}^2}(-2) \arrow[d] & & \\
 & & n\Lambda^2 V^* \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \arrow[d] & & \\
0 \arrow[r] & n\Omega^2_{\mathbb{P}^2}(2) \arrow[r] & n\Omega^1_{\mathbb{P}^2}(1) \arrow[r] \arrow[d] & (n-2)\mathcal{O}_{\mathbb{P}^2} \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

and use that for the three dimensional vector space V we have an isomorphism $n \wedge^3 V^* \cong k$, which allows us to fill the diagram in a similar way:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & n\Lambda^3 V^* \otimes \mathcal{O}_{\mathbb{P}^2}(-2) \arrow[r, "\cong"] \arrow[d] & n\mathcal{O}_{\mathbb{P}^2}(-2) \arrow[r] \arrow[d, dotted, "M"', "\exists"] & 0 \\
0 \arrow[r] & n\mathcal{O}_{\mathbb{P}^2}(-1) \arrow[r] \arrow[d, "\cong"'] & n\Lambda^2 V^* \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \arrow[r] \arrow[d] & 2n\mathcal{O}_{\mathbb{P}^2}(-1) \arrow[r] \arrow[d, dotted, "N"', "\exists"] & 0 \\
0 \arrow[r] & n\Omega^2_{\mathbb{P}^2}(2) \arrow[r] \arrow[d] & n\Omega^1_{\mathbb{P}^2}(1) \arrow[r] \arrow[d] & (n-2)\mathcal{O}_{\mathbb{P}^2} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

□

5 On Moduli problems for toric sheaves

Since we want to use the theorem of Beilinson to obtain toric sheaves we have to know more about the T-equivariances of the building blocks and - even more important - the toric morphisms between them. After giving some criteria for the computation of the morphisms we will talk about the existence of ”toric monads” and give a short introduction to known moduli spaces on $\mathbb{P}^2$. As an application we close with two explicit examples. The first filters toric bundles and sheaves from the space $\overline{M}_{\mathbb{P}^2}(2; c_1, c_2)$ in special cases and in the second we construct some toric sheaves on $\mathbb{P}(1,1,2)$, using all the theoretical tools from the last chapters.

5.1 More about Toric Sheaves

In this section we will investigate necessary and sufficient conditions for a matrix to be toric as morphism between structure and differential sheaves on a toric variety.

Toric Structure Sheaves

We begin with structure sheaves on an arbitrary toric variety.

Lemma 5.1. *Let X be a toric variety and S its homogeneous coordinate ring, then for all twists $\alpha, \beta \in A_{n-1}(X)$ we have*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(\alpha), \mathcal{O}_X(\beta)) \cong \mathcal{O}_X(\beta - \alpha).$$

Proof. For the rings we clearly have $\mathrm{Hom}_S(S(\alpha), S(\beta)) \cong S(\beta - \alpha)$ and since $S(\beta - \alpha)$ is a finitely generated S-module we obtain by [Gro61, proposition 2.5.13] for the corresponding sheaves

$$\begin{aligned}\mathcal{O}_X(\beta - \alpha) \cong \widetilde{S(\beta - \alpha)} &\cong (\mathrm{Hom}_S(S(\alpha), S(\beta)))^\sim \\ &\cong \mathcal{H}om_{\mathcal{O}_X}(\widetilde{S(\alpha)}, \widetilde{S(\beta)}) \\ &\cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(\alpha), \mathcal{O}_X(\beta)).\end{aligned}$$

□

Lemma 5.2. *Let $X = X(\Delta)$ and $a, b \in \mathbb{Z}^{\Delta(1)}$ be toric divisors. Then there is an isomorphism*

$$Hom_{\mathcal{O}_X}(\mathcal{O}_X(a), \mathcal{O}_X(b))^T \cong S_{b-a}.$$

Proof. Taking global sections of the sheaves from the last lemma and using proposition 2.47 we obtain an isomorphism

$$\mathrm{Hom}(\mathcal{O}_X(\alpha), \mathcal{O}_X(\beta)) \cong \Gamma(\mathcal{O}_X(\beta - \alpha)) \cong S_{\beta-\alpha}.$$

Then it is easy to verify that there is also an isomorphism for the T-invariant sections, i.e.

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(a), \mathcal{O}_X(b))^T \cong \Gamma(\mathcal{O}_X(b - a))^T.$$

So we have to consider T-stable sections $s\colon \mathcal{O}_X \to \mathcal{O}_X(b - a)$. From the theory of the last chapter s has to make the following diagram commutative

$$\begin{tikzcd} \sigma^*\mathcal{O}_X \arrow[r, "\sigma^* s"] \arrow[d, "\Psi_0"'] & \sigma^*\mathcal{O}_X(b-a) \arrow[d, "\Psi_{b-a}"] \\ p_2^*\mathcal{O}_X \arrow[r, "p_2^* s"'] & p_2^*\mathcal{O}_X(b-a), \end{tikzcd}$$

where Ψ_0 and Ψ_a denote the given equivariances of S and $S(b-a)$. We saw that those equivariances where determined by the fine gradings of the rings and so the commutativatity of the diagram is equivalent to say that the induced map

$$s\colon S \to S(b - a)$$

respects the fine gradings, i.e. is a fine graded ring homomorphism of degree 0. Hence s has to be in $S(b-a)_0 = S_{b-a}$. This can also formally be checked by on the level of the diagrams of definition 2.49, but this gives rise to a lengthy computation that does not give more insight. $\square$

Thus the T-invariant morphisms consist of multiples of a single monomial, but might also be zero. In the case of a nonzero morphism, the divisor $b-a$ has to be effective.

Example 5.3. Let $X = \mathbb{P}^n$, then all coordinates $x_0, \ldots, x_n$ have $\mathbb{Z}$-degree one, which in particular means that S is generated by S_1 as S_0-algebra. Thus the above result reduces to

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}(a), \mathcal{O}(b))^T \cong \begin{cases} k \cdot x^{b-a} & \text{, if } b_i \geq a_i \quad \forall i \in \Delta(1) \\ 0 & \text{, else} \end{cases}$$

We can now extend this to morphisms between finite sums of structure sheaves, i.e. to matrices, which is exactly what we need for applications of the theorem of Beilinson. We have seen that it is necessary that all entries have to be monomials and the surprising result is that it is sufficient that all minors are monomials. So let's make the following

Definition 5.4. We call a matrix $F \in \mathrm{Mat}(p \times q, k[x_0, \ldots, x_n])$ **minor-monomial**, if all of its minors are monomials. In particular all entries of F have to be monomials.

Using this definition, we can now formulate the following result, which allows easy computations.

Theorem 5.5. *Let $F = (f_{ij})$ be a $p \times q$-matrix consisting of monomials $f_{ij} = \alpha_{ij} x^{a_{ij}}$ where $a_{ij} \in \mathbb{Z}^{n+1}$ in the case of $\alpha_{ij} \neq 0$. Then the following conditions are equivalent:*

1. *F is a T-equivariant morphism between structure sheaves on X.*
2. *There are $a_i, b_j \in \mathbb{Z}^{n+1}$ for $i = 1, \ldots, p$ and $j = 1, \ldots q$ with the property that if $f_{ij} \neq 0$, then $a_{ij} = b_j - a_i$.*
3. *F is minor-monomial.*

Proof. The equivalence of the first two statements follows from the observation that F is in equivariant if and only there are toric divisors a_i and b_j identified with elements of $\mathbb{Z}^{\Delta(1)}$ such that

$$\bigoplus_{i=1}^{p} \mathcal{O}_X(a_i) \xrightarrow{F} \bigoplus_{j=1}^{q} \mathcal{O}_X(b_j)$$

is a homomorphism of toric sheaves. We know that

$$\mathrm{Hom}(\mathcal{O}_X(a_i), \mathcal{O}_X(b_j))^T = k \cdot x^{b_j - a_i},$$

so the elements of F have the property $a_{ij} = b_j - a_i$ for all i and j.
For "1. $\Rightarrow$ 3." we use the more general fact that a toric morphism F between two sheaves $\mathcal{E}$ and $\mathcal{F}$, given by a commutative diagram

$$\begin{tikzcd} \sigma^*\mathcal{E} \arrow[r, "\phi"] \arrow[d, "\sigma^* F"'] & pr_2^*\mathcal{E} \arrow[d, "pr_2^* F"] \\ \sigma^*\mathcal{F} \arrow[r, "\psi"'] & pr_2^*\mathcal{F}, \end{tikzcd}$$

remains toric if we apply Λ^m for any natural number m, since we obtain an induced equivariance by

$$\begin{tikzcd} \sigma^*\Lambda^m(\mathcal{E}) \arrow[r, "\Lambda^m(\phi)"] \arrow[d, "\sigma^*\Lambda^m(F)"'] & pr_2^*\Lambda^m(\mathcal{E}) \arrow[d, "pr_2^*\Lambda^m(F)"] \\ \sigma^*\Lambda^m(\mathcal{F}) \arrow[r, "\Lambda^m(\psi)"'] & pr_2^*\Lambda^m(\mathcal{F}). \end{tikzcd}$$

This means that all minors have to be monomials again.
The proof of "3. $\Leftarrow$ 2." can be found in [MP] for the case of $X = \mathbb{P}^n$, but it applies to the general case without any change. $\square$

Toric Differential Sheaves

Next we will describe morphisms of differential sheaves. It turns out that a description similar to the case of structure sheaves above seems to be only possible in the case of $X = \mathbb{P}^n$. The reason for this mainly is that for projective space we can always choose an $(n+1)$-dimensional vector space V such that $\mathbb{P}^n = \mathbb{P}(V)$, which gives us an additional structure to work with. For weighted projective spaces this already does not work anymore, the case of an arbitrary toric variety X is even harder.

So let's restrict ourselves for the moment to the case of $X = \mathbb{P}^n$. We find the following maps for every $p \in \mathbb{Z}$:

$$\begin{tikzcd}
\Lambda^{p+1}V^* \otimes \mathcal{O}_{\mathbb{P}^n}(-p-1) \arrow[r, dotted, "\neg"] \arrow[d, shift right, "\phi"'] & \Lambda^p V^* \otimes \mathcal{O}_{\mathbb{P}^n}(-p) \\
(\Lambda^p V^* \otimes V^*) \otimes \mathcal{O}_{\mathbb{P}^n}(-p-1) \arrow[r, "\cong"'] \arrow[u, shift right, "\wedge"'] & \Lambda^p V^* \otimes (V^* \otimes \mathcal{O}_{\mathbb{P}^n}(-p-1)), \arrow[u]
\end{tikzcd}$$

where the right arrow is induced by the twisted Euler sequence and the map ϕ is given by

$$\phi(z_1 \wedge \ldots \wedge z_{p+1}) := \frac{1}{p+1} \sum_{i=1}^{p+1} (-1)^{p+1} z_i \otimes z_1 \wedge \ldots \wedge \widehat{z_i} \wedge \ldots \wedge z_{p+1},$$

for every $z_1 \wedge \ldots \wedge z_{p+1} \in \Lambda^p V^*$ and determined by linear extension. We call this Koszul-like map the **de-antisymmetric map** for every p and it is easy to check that it in fact splits with $\wedge$.

Definition 5.6. The morphism $\neg$ given by composition is called the **contraction map** or simply **contraction**.

We will see that we can relate the contraction maps with the morphisms between differential sheaves. To do so recall that the sheaves $\Omega^p_{\mathbb{P}^n}$ occur as kernels of the morphisms in the Koszul complex

$$0 \to \Lambda^n V^* \otimes \mathcal{O}_{\mathbb{P}^n}(-n) \to \ldots \to \Lambda^2 V^* \otimes \mathcal{O}_{\mathbb{P}^n}(-2) \to \Lambda^1 V^* \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n} \to 0,$$

and we therefore have the following short exact sequence

$$0 \to \Omega^p_{\mathbb{P}^n} \to \Lambda^p V^* \otimes \mathcal{O}_{\mathbb{P}^n}(-p) \to \Omega^{p-1}_{\mathbb{P}^n} \to 0$$

for every p. Moreover there is always a morphism

$$\begin{aligned} \text{ev}\colon \Lambda^q V \otimes \Omega^{p+q}_{\mathbb{P}^n}(p+q) &\to \Omega^p_{\mathbb{P}^n}(p) \\ \xi \otimes \omega &\longmapsto \omega(\xi) \end{aligned}$$

which we call the **evaluation map**. Fixing a single $\xi \in \Lambda^q V$ then induces a morphism

$$\Omega^{p+q}_{\mathbb{P}^n}(p+q) \xrightarrow{\xi} \Omega^p_{\mathbb{P}^n}(p) \qquad \omega \mapsto \omega(\xi)$$

and the next theorem ensures us that this is already the general situation.

Lemma 5.7. *For all $p, q \geq 0$ there is an isomorphism*

$$\mathit{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\Omega^{p+q}(p+q), \Omega^p(p)) \cong \Lambda^q V$$

for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.

Proof. The proof of this lemma is well known, see for example [Bei78], but since we need some important steps from it ,we will partially reprove this statement in the next proposition. □

Lemma 5.8. *The sheaves $\Omega^i_{\mathbb{P}^n}(j)$ are toric.*

Proof. For any $j = 0, \ldots, n$ the T-equivariance of the sheaves is canonically induced by the Koszul complex, explicitly by

$$0 \longrightarrow \Omega^j_{\mathbb{P}^n} \longrightarrow \Lambda^j V^* \otimes \mathcal{O}_{\mathbb{P}^n}(-j) \xrightarrow{Z_j} \Lambda^{j-1} V^* \otimes \mathcal{O}_{\mathbb{P}^n}(-j+1) \xrightarrow{Z_{j-1}} \cdots \xrightarrow{Z_0} \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0.$$

We can inductively describe the Koszul matrices Z_p by setting

$$Z_0 = \begin{pmatrix} Z_0' \\ x_n \end{pmatrix} \qquad \text{and } Z_p = \left(\begin{array}{ccc|c} & Z_p' & & 0 \\ \hline (-1)^q x_n & \ldots & 0 & \\ \vdots & \ddots & \vdots & Z_{p-1}' \\ 0 & \ldots & (-1)^q x_n & \end{array}\right),$$

but is not clear by this description that those matrices are minor-monomial. Howeverg we have a more theoretical argument to show this. Indeed, the matrix Z_0 is minor-monomial and we can apply the Λ-functor without destroying this property, as we have shown in the proof of theorem 5.5. Note that this is exactly what we need for the Koszul complex. By choosing an arbitrary toric divisor D of $\mathbb{Z}$-degree i, we obtain an equivariance of the sheaf $\Omega^j_{\mathbb{P}^n}(i)$ by the following sequence

$$\begin{tikzcd}
0 \arrow[r] & \Omega^j_{\mathbb{P}^n}(i) \arrow[r] \arrow[d, dotted] & \Lambda^j V^* \otimes \mathcal{O}_{\mathbb{P}^n}(i-j) \arrow[r, "Z_j"] \arrow[d, dotted] & \cdots . \\
0 \arrow[r] & \Omega^j_{\mathbb{P}^n}(D) \arrow[r] & \sum_{0 \le k_1 < \ldots < k_{i-j} \le 0} \mathcal{O}_{\mathbb{P}^n}(D - E_{k_1} - \ldots - E_{k_{i-j}}) \arrow[r, "Z_j"'] & \cdots
\end{tikzcd}$$

Thus $\Omega^j_{\mathbb{P}^n}(i)$ is the kernel of a toric morphism between toric sheaves, which therefore canonically inherits the induced toric structure of the Koszul sequence. □

Corollary 5.9. *The sheaves $\overline{\Omega}^i_{\mathbb{P}(Q)}(j)$ are toric.*

Proof. This can be done using the same arguments as above and the generalized Euler sequence

$$0 \longrightarrow \overline{\Omega}^1_{\mathbb{P}(Q)} \longrightarrow \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}(Q)}(-D_i) \xrightarrow{\begin{pmatrix} q_0 y_0 \\ \vdots \\ q_n y_n \end{pmatrix}} \mathcal{O}_{\mathbb{P}(Q)} \longrightarrow 0.$$

from theorem 3.16 and the corresponding Koszul sequence introduced there. Note that this sequence as well as the one from the last lemma is induced by a fine-graded sequence of modules. □

Proposition 5.10. *Let V be an $(n+1)$-dimensional vector space with basis $(e_0, \ldots, e_n)$ such that $\mathbb{P}^n = \mathbb{P}(V)$. Moreover for every $J := (j_1, \ldots, j_q)$ with $0 \le j_1 < \ldots < j_q \le n$ we abbreviate a basis element $\alpha_{j_1,\ldots,j_q} e_{j_1} \wedge \ldots \wedge e_{j_q} \in \Lambda^q V$ by $\xi = \alpha_J e^{\Lambda(J)}$ and call it a* ***Λ-monomial*** *with respect to $\Lambda^q V$. Then a morphism*

$$\Lambda^q V \ni \xi \colon \Omega^{p+q}_{\mathbb{P}^n}(p+q) \to \Omega^p_{\mathbb{P}^n}(p)$$

is a toric if and only if ξ is a Λ-monomial, i.e.

$$\xi = \alpha_J e^{\Lambda(J)} = \alpha_{j_1,\ldots,j_q} e_{j_1} \wedge \ldots \wedge e_{j_q}$$

for some J and $\alpha_{j_1,\ldots,j_q} \in k$.

Proof. From the previous lemma we know that both $\Omega^{p+q}_{\mathbb{P}^n}(p+q)$ and $\Omega^p_{\mathbb{P}^n}(p)$ are toric. In the proof of the lemma we have seen that equivariance is induced by the Koszul sequence, so we simply compare the morphism ξ with the induced morphisms between the Koszul complexes. Clearly, by composition we obtain a morphism $\overline{\xi}$. Embedded into the short exact sequences discussed before, this gives us

$$\begin{tikzcd} 0 \arrow[r] & \Omega^{p+q}_{\mathbb{P}^n}(p+q) \arrow[r] \arrow[d, "\xi"'] \arrow[dr, dotted, "\overline{\xi}"] & \Lambda^{p+q}V^* \otimes \mathcal{O}_{\mathbb{P}^n} \arrow[r] \arrow[d, dotted, "\neg\xi"] & \Omega^{p+q-1}_{\mathbb{P}^n}(p+q) \arrow[r] & 0 \\ 0 \arrow[r] & \Omega^{p}_{\mathbb{P}^n}(p) \arrow[r] & \Lambda^{p}V^* \otimes \mathcal{O}_{\mathbb{P}^n} \arrow[r] & \Omega^{p-1}_{\mathbb{P}^n}(p) \arrow[r] & 0 \end{tikzcd}$$

and we claim that this morphism uniquely extends to a morphism $\neg\xi$. We prove this claim by showing

$$\mathrm{Hom}(\Lambda^{p+q}V^* \otimes \mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}) \cong \mathrm{Hom}(\Omega^{p+q}_{\mathbb{P}^n}(p+q), \mathcal{O}_{\mathbb{P}^n}).$$

This can be done by directly counting the dimension of both spaces, or by considering the long exact Ext-sequence

$$\begin{tikzcd} \mathrm{Hom}(\Lambda^{p+q}V^*\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}) \arrow[r] & \mathrm{Hom}(\Omega^{p+q}_{\mathbb{P}^n}(p+q), \mathcal{O}_{\mathbb{P}^n}) \arrow[d] \\ \mathrm{Hom}(\Omega^{p+q-1}_{\mathbb{P}^n}(p+q), \mathcal{O}_{\mathbb{P}^n}) \arrow[u] & \mathrm{Ext}^1(\Omega^{p+q-1}_{\mathbb{P}^n}(p+q), \mathcal{O}_{\mathbb{P}^n}) \arrow[d] \\ \vdots \arrow[u] & \vdots \end{tikzcd} .$$

We prove that $\mathrm{Hom}(\Omega^{p+q-1}_{\mathbb{P}^n}(p+q), \mathcal{O}_{\mathbb{P}^n}) = 0 = \mathrm{Ext}^1(\Omega^{p+q-1}_{\mathbb{P}^n}(p+q), \mathcal{O}_{\mathbb{P}^n})$. For the first part we simply use the fact that

$$\mathrm{Hom}(\Omega^{p+q-1}_{\mathbb{P}^n}(p+q), \mathcal{O}_{\mathbb{P}^n}) \cong \mathrm{H}^0(\Omega^{p+q-1}_{\mathbb{P}^n}(p+q)) = 0$$

on projective spaces, for example by theorem 3.19, but the second part needs a more sophisticated argument. Recall that in the more general situation, where $\mathcal{E}$ is locally free and $\mathcal{F}$ an arbitrary coherent sheaf on $\mathbb{P}^n$, we have the following identification:

$$\mathrm{Ext}^i(\mathcal{E}, \mathcal{F}) \cong \mathrm{H}^i(\mathcal{E}^* \otimes \mathcal{F})$$

for any $i, j \in \mathbb{Z}$. This applies to our situation and we can conclude

$$\begin{aligned} \mathrm{Ext}^1(\Omega^{p+q-1}_{\mathbb{P}^n}(p+q), \mathcal{O}) &\cong \mathrm{H}^1((\Omega^{p+q-1}_{\mathbb{P}^n}(p+q))^* \otimes \mathcal{O}_{\mathbb{P}^n}) \\ &\cong \mathrm{H}^1(\Omega^{n-(p+q-1)}_{\mathbb{P}^n}(n+1-(p+q))) \\ &\cong \mathrm{H}^1(\Omega^{n+1-p-q}_{\mathbb{P}^n}(n+1-p-q)) = 0. \end{aligned}$$

This has two major consequences. First of all we can now inductively go down to the case of $p = 0$ and study the morphism

$$\xi\colon \Omega^q_{\mathbb{P}^n}(q) \to \mathcal{O}_{\mathbb{P}^n},$$

which makes live much easier, and second we only have to care about the maps $\neg\xi$, since we can extend through the Koszul sequence

$$\begin{tikzcd} 0 \arrow[r] & \Omega^{p+q}_{\mathbb{P}^n}(p+q) \arrow[r] \arrow[d, "\xi"'] & \Lambda^{p+q}V^* \otimes \mathcal{O}_{\mathbb{P}^n} \arrow[r, "Z_{p+q}"] \arrow[d, dotted, "\neg\xi"] & \Lambda^{p+q-1}V^* \otimes \mathcal{O}_{\mathbb{P}^n}(1) \arrow[r, "Z_{p+q-1}"] \arrow[d, dotted, "\neg\xi"] & \cdots , \\ 0 \arrow[r] & \Omega^{p}_{\mathbb{P}^n}(p) \arrow[r] & \Lambda^{p}V^* \otimes \mathcal{O}_{\mathbb{P}^n} \arrow[r, "Z_p"] & \Lambda^{p-1}V^* \otimes \mathcal{O}_{\mathbb{P}^n} \arrow[r, "Z_{p-1}"] & \cdots \end{tikzcd}$$

where we write by abuse of notation $\neg\xi$ at every stage. It is now easy to see that ξ is equivariant if and only if all of the $\neg\xi$ are equivariant. Moreover, choosing toric divisors A and B of $\mathbb{Z}$-degree $p+q$ and p respectively, we get induced matrices M_j^ξ, where j denotes the power of the wedge product in the lower row, from

$$\begin{tikzcd}
\Omega^{p+q}_{\mathbb{P}^n}(A) \arrow[r] \arrow[ddd, "\xi"'] & \Lambda^{p+q}V^* \otimes \mathcal{O}_{\mathbb{P}^n} \arrow[r] \arrow[d, "\approx"] & \cdots . \\
& \sum_{0\le i_1,\dots,i_{p+q}} \mathcal{O}_{\mathbb{P}^n}(A - E_{i_1} - \ldots - E_{i_{p+q}}) \arrow[d, "M_p^\xi"'] & \\
& \sum_{0\le j_1,\dots,i_p} \mathcal{O}_{\mathbb{P}^n}(B - E_{j_1} - \ldots - E_{j_p}) \arrow[d, "\approx"] & \\
\Omega^{p}_{\mathbb{P}^n}(B) \arrow[r] & \Lambda^{p}V^* \otimes \mathcal{O}_{\mathbb{P}^n} \arrow[r] & \cdots
\end{tikzcd}$$

Since all morphisms and sheaves in the rows of the diagram are toric, we see that ξ is a toric morphism if and only if all the M_j^ξ are, by what we know so far.
As said before we now consider the case $p = 0$, where the statement reduces to

$$\xi \text{ is toric } \Leftrightarrow M_0^\xi \text{ is toric}$$

in the following diagram (where we assume without loss of generality that $B = 0$)

$$\begin{tikzcd}
0 \arrow[r] & \Omega^q_{\mathbb{P}^n}(A) \arrow[r] \arrow[d, "\xi"'] & \sum_{0\le i_1<\ldots<i_q\le n} \mathcal{O}_{\mathbb{P}^n}(A-E_{i_1}-\ldots-E_{i_q}) \arrow[r] \arrow[d, "M_0^\xi"'] & \cdots . \\
0 \arrow[r] & \mathcal{O} \arrow[r] & \mathcal{O}_{\mathbb{P}^n} \arrow[r] & 0
\end{tikzcd}$$

Let us assume that the basis elements of $\Lambda^q V^*$ are in lexicographic ordering with respect to the q-tupel index of each such element in order to write

$$\xi = \alpha_{0,\dots,q} e_0 \wedge \ldots \wedge e_q + \ldots + \alpha_{n-q,\dots,n} e_{n-q} \wedge \ldots \wedge e_n = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix},$$

where N denotes the dimension of $\Lambda^q V^*$. The structure of M_0^ξ is determined by $\neg\xi$. Let $\xi = \alpha_I e^{\Lambda(I)}$ for some index I, then $\neg\xi$ is given by

$$\begin{aligned}
\Lambda^q V^* &\longrightarrow k \\
x_{j_1} \wedge \ldots \wedge x_{j_q} &\longmapsto \neg\xi(x_{i_1} \wedge \ldots \wedge x_{i_q}) = \alpha_I \det \begin{pmatrix} x_{j_1}(e_{i_1}) & \ldots & x_{j_1}(e_{i_q}) \\ \vdots & & \vdots \\ x_{j_q}(e_{i_1}) & \ldots & x_{j_q}(e_{i_q}) \end{pmatrix} = \alpha_I .
\end{aligned}$$

Because of the multilinearity of the determinant, this extends linearly to general ξ and so we see that the matrix M_0^ξ, which does in fact nothing else but separating the basis vectors, has to have coefficients α_I at the I-th position.
Finally we only have to mention what toric morphisms between structure sheaves of the same degree are. For two divisors a and b in $\mathbb{Z}^{n+1}$ of $\mathbb{Z}$-degree 0 we have

$$\begin{aligned}
\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}(a), \mathcal{O}_{\mathbb{P}^n}(b))^T \cong \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n}(b-a)^T &= \begin{cases} S_{b-a} & , \text{ if } \beta_i \ge \alpha_i \quad \forall i \\ 0 & , \text{ else} \end{cases} \\
&= \begin{cases} S_0 \cong k & , \text{ if } \beta_i = \alpha_i \quad \forall i \\ 0 & , \text{ else} \end{cases}
\end{aligned}$$

Hence such a morphism is toric iff $a = b$, since in all other cases we would need fractions of the coordinates x_i to get a morphism of sheaves, which then would not be toric. In our case this means that

$$\begin{aligned}
\xi \text{ is toric } &\Leftrightarrow M_0^\xi \text{ is toric} \\
&\Leftrightarrow \text{ all coordinate functions of } M_0^\xi \text{ are toric} \\
&\Leftrightarrow \exists I = (i_1, \ldots, i_q)\colon\ A - E_{i_1} - \ldots - E_{i_q} = 0 \\
&\Leftrightarrow \exists I = (i_1, \ldots, i_q)\colon\ A = E_{i_1} + \ldots + E_{i_q} \\
&\Leftrightarrow \text{ Exactly one of the coordinates of } M_0^\xi \text{ at the } I\text{-th position is nonzero and equals } \alpha_I \\
&\Leftrightarrow \xi = \alpha_I e^{\wedge(I)} \text{ is a basis element.}
\end{aligned}$$

This finishes the proof. □

Remark. Note that it is necessary for the equivariance of ξ that M_0^ξ consists of only one nonzero entry, which is in fact a scalar, but it is also necessary. This also holds true when we consider maps with $p > 0$, since by the inductive definition of the Koszul matrices it then turns out that all other M_j^ξ are automatically minor-monomial (they only consist of a scalars). In this case we have the same relations for divisors A and B of $\mathbb{Z}$-degree $p + q$ and p as in the theorem, where we chose $B = 0$. It follows that

$$\xi = \alpha_I e^{\wedge(I)} \text{ is toric } \Leftrightarrow A - B = E_{i_1} + \ldots + E_{i_q}$$

or, if we identify the divisors directly with elements in $\mathbb{Z}^{n+1}$, we can write

$$A - B = \begin{pmatrix} 0 & \ldots & 1 & \ldots & 1 & \ldots & 0 \end{pmatrix}$$

with an entry 1 on each of the positions $i_1, \ldots, i_q$ and 0 elsewhere.

Now we extend the result to matrices, as we did before. The skew-symmetric structure of $\Lambda^q V$ does not allow us to use linear algebra and the concept of determinants, but with the following definition we can at least partially keep things together.

Definition 5.11. Let $F = (f_{ij})$ with $i = 1, \ldots, p$, $j = 1, \ldots, q$, be a matrix consisting of Λ-monomials, i.e. $f_{ij} = \alpha_{ij} e^{\Lambda(a_{ij})}$ where $\alpha_{ij} \in k$ and $a_{ij} \in [0,1]^{n+1}$. Then we call F **Λ-minor-monomial** if for any l-minor M with row indices $i_1, \ldots, i_l$ the following condition holds. If we replace F by the matrix $\hat{F} = \hat{f}_{ij}$ with $\hat{f}_{ij} := \alpha_{ij} x^{a_{ij}}$, where $x = (x_0, \ldots, x_n)$ denotes the Cox-coordinates, then all terms occurring in the calculation of the determinant of the induced matrix $\hat{M}$ have the same $\mathbb{Z}^{n+1}$-degree, i.e. for all permutations $\sigma\colon \{1, \ldots, l\} \to \{1, \ldots, l\}$ the terms

$$x^{a_{i_1, i_{\sigma(1)}}} \cdot \ldots \cdot x^{a_{i_l, i_{\sigma(l)}}}$$

are in the same fine-graded part of $S = S(\mathbb{P}^n)$.

Note that this condition is stronger then the minor-monomiality defined before, since if all terms of the determinant of a minor $\hat{M}$ have the same $\mathbb{Z}^{n+1}$-degree, then the determinant is a monomial too. With this definition at hand we can formulate the following theorem much easier.

Theorem 5.12. *Let $F = (f_{ij})$ be a matrix as given in the above definition, i.e. $f_{ij} = \alpha_{ij} e^{\Lambda(a_{ij})}$ for the basis $(e_0, \ldots, e_n)$ of the vector space V with $\mathbb{P}^n = \mathbb{P}(V)$. Then the following are equivalent:*

1. *F is toric as a morphism between differential sheaves on $\mathbb{P}^n$.*
2. *There are toric divisors $D_i, E_j \in \mathbb{Z}^{n+1}$ such that $D_i - E_j = a_{ij}$.*
3. *F is $\wedge$-minor-monomial.*

Proof. A matrix F is toric as morphism between differential sheaves, if and only if there are divisors $D_i, E_j \in \mathbb{Z}^{n+1}$ and indices $k_1, \ldots, k_p$ and $l_1, \ldots, l_q$ such that

$$\sum_{i=1}^{p} \Omega^{k_i}_{\mathbb{P}^n}(D_i) \xrightarrow{F} \sum_{j=1}^{q} \Omega^{l_j}_{\mathbb{P}^n}(E_i)$$

is toric in each component in the sense of the last proposition. As a byproduct of the proof of this proposition we saw that the toric divisors chosen and the degree of the entries are directly related to each other. Therefore F is toric if and only if

$$G := \begin{pmatrix} D_1 - E_1 & \ldots & D_1 - E_q \\ \vdots & & \vdots \\ D_p - E_1 & \ldots & D_p - E_q \end{pmatrix} = \begin{pmatrix} a_{11} & \ldots & a_{1q} \\ \vdots & & \vdots \\ a_{p1} & \ldots & a_{pq} \end{pmatrix}.$$

Thus the first two conditions are equivalent. The equivalence of the first and the last condition can immediately be read of from the structure of the matrix G, since each term of an l-minor with rows $i_1, \ldots, i_l$ has the same $\mathbb{Z}^{n+1}$-degree, namely

$$\sum_{k=1}^{l} D_{i_k} - \sum_{k=1}^{l} E_{i_k}$$

which is independent of the chosen permutation. □

Remark. The proof technique of the corresponding theorem for structure sheaves does not apply in this case for several reasons. First of all, the monomials do not have "higher degrees" by definition of the exterior algebra and so we cannot use elementary row transformations, since they might disturb the structure. Moreover we do not have a well developed theory of determinants, so we loose the elegant description using the minor-monomiality property of the matrices.

However in the following special case, which is particularly interesting for the Beilinson monads, we get the following result.

Corollary 5.13. *If the matrix F contains only Λ-monomials of the vector space V, then if we replace F by $\hat{F}$, that is $f_{ij} = \alpha_{ij} e^{\Lambda(a_{ij})}$ by $\hat{f}_{ij} = \alpha_{ij} x^{a_{ij}}$ we get the same result as for the structure sheaves, i.e. F is toric if and only if $\hat{F}$ is minor-monomial.*

Proof. Since here we don't have to care about the skew-symmetric structure the proof of theorem 5.5 applies to the matrix $\hat{F}$. □

Example 5.14. The formulation of the theorem might seem cryptic, but let's check on an example what can go wrong and why we have to do like we did: Let $X = \mathbb{P}^3$ with coordinates x_0, x_1, x_2, x_3 and consider the following map

$$F = \begin{pmatrix} x_0 \wedge x_1 & 2x_0 \wedge x_2 \\ x_1 & x_2 \end{pmatrix}.$$

This matrix does in fact define a toric morphism. To see this we have to find toric divisors D_1, D_2, E_1 and E_2 such that

$$\Omega^3_{\mathbb{P}^3}(D_1) \oplus \Omega^2_{\mathbb{P}^3}(D_2) \xrightarrow{\left(\begin{smallmatrix} x_0 \wedge x_1 & x_0 \wedge x_2 \\ x_1 & x_2 \end{smallmatrix}\right)} \Omega^1_{\mathbb{P}^3}(E_1) \oplus \Omega^1_{\mathbb{P}^3}(E_2)$$

is toric, i.e. if the following relations are satisfied

$$D_1 - E_1 = (1,1,0,0) \qquad D_1 - E_2 = (0,1,1,0)$$
$$D_2 - E_1 = (1,0,0,0) \qquad D_2 - E_2 = (0,0,1,0),$$

so a possible choice might be

$$
\begin{aligned}
D_1 &= (1,1,1,0) & \qquad E_1 &= (0,0,1,0) \\
D_2 &= (1,0,1,0) & \qquad E_2 &= (1,0,0,0).
\end{aligned}
$$

Note however that there is no reasonable choice to define the determinant of this matrix in order to obtain a monomial. For example

$$
\begin{aligned}
&(x_0 \wedge x_1) \wedge x_2 \pm x_1 \wedge (x_0 \wedge x_2) \\
\text{and } &(x_0 \wedge x_1) x_2 \pm x_1 (x_0 \wedge x_2)
\end{aligned}
$$

cannot be seen as monomials. So this example shows already that we have to make "everything" commutative or in other words interpret the terms as monomials in the usual coordinates. This reflects the idea that the commutative structure of the torus should not be disturbed by the antisymmetric structure of the exterior product when we talk about questions of T-equivariance.

Now that we have seen how the toric structure of the differential sheaves is related to the structure of the structure sheaves, let us check an example to get a deeper understanding of this relation.

Example 5.15. Recall that for a stable rank 2 bundle $\mathcal{F}$ on $\mathbb{P}^2 = \mathbb{P}(V)$ with $c_1(\mathcal{F}) = 0$ and $c_2(\mathcal{F}) = n$ we had the Beilinson I and IIA resolutions, see section 4.4 for the details. In the case of $n = 2$ these resolutions can be written as follows:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 2\Omega^2_{\mathbb{P}^2}(2) \arrow[r, "A=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)"] \arrow[d, "\cong"'] & 2\Omega^1_{\mathbb{P}^2}(1) \arrow[r] \arrow[dd, "\left(\begin{smallmatrix} e_0 & e_1 & e_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_0 & e_1 & e_2 \end{smallmatrix}\right)"] & 0 \\
 & 2\mathcal{O}_{\mathbb{P}^2}(-1) \arrow[d, "\left(\begin{smallmatrix} a_0 & a_1 & a_2 & b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 & d_0 & d_1 & d_2 \end{smallmatrix}\right)=\tilde{A}"'] & & \\
0 \arrow[r] & 2V^* \otimes \mathcal{O}_{\mathbb{P}^2} \arrow[r, equal] \arrow[d, "\left(\begin{smallmatrix} x_0 & 0 \\ x_1 & 0 \\ x_2 & 0 \\ 0 & x_0 \\ 0 & x_1 \\ 0 & x_2 \end{smallmatrix}\right)"'] & 2V^* \otimes \mathcal{O}_{\mathbb{P}^2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 2\mathcal{O}_{\mathbb{P}^2}(1) \arrow[r, equal] \arrow[d] & 2\mathcal{O}_{\mathbb{P}^2}(1) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}
$$

Here the entries of A are elements of the vector space V and the entries of $\tilde{A}$ are linear forms in the Cox coordinates. It is now easy to see that, neglecting the structure of the vector spaces, for any given A we obtain an induced map $\tilde{A}$ by the sequence

$$
\begin{tikzcd}
k^2 \arrow[r, "A"] \arrow[ddr, dotted, "\tilde{A}"'] & k^2 \otimes V \arrow[d, "\left(\begin{smallmatrix} e_0 & e_1 & e_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_0 & e_1 & e_2 \end{smallmatrix}\right)"] \\
 & k^6 \otimes \Lambda^2 V \arrow[d, "\approx"] \\
 & k^6 \otimes V^*
\end{tikzcd}
$$

where the isomorphism $\Lambda^2 V \to V^*$ is given by

$$\begin{aligned} e_0 \wedge e_1 &\longleftrightarrow x_2 =: \overline{e_0 \wedge e_1} \\ e_0 \wedge e_2 &\longleftrightarrow -x_1 =: \overline{e_0 \wedge e_2} \\ e_1 \wedge e_2 &\longleftrightarrow x_0 =: \overline{e_1 \wedge e_2}. \end{aligned}$$

This means that we have the following relation

$$\begin{aligned} \tilde{A} &= \begin{pmatrix} a_0 & a_1 & a_2 & b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 & d_0 & d_1 & d_2 \end{pmatrix} \\ &= \begin{pmatrix} \overline{a \wedge e_0} & \overline{a \wedge e_1} & \overline{a \wedge e_2} & \overline{b \wedge e_0} & \overline{b \wedge e_1} & \overline{b \wedge e_2} \\ \overline{c \wedge e_0} & \overline{c \wedge e_1} & \overline{c \wedge e_2} & \overline{d \wedge e_0} & \overline{d \wedge e_1} & \overline{d \wedge e_2} \end{pmatrix}, \end{aligned}$$

which implies that $\tilde{A}$ also determines A and moreover that we can compare the coefficients of the single "blocks", e.g. a and (a_o, a_1, a_2), directly.
We claim that $\tilde{A}$ is toric if and only if A consists of $\wedge$-monomials and w.l.o.g. the entry c is zero (which is in this case equivalent to the last property of the theorem, since if A has a minor as determinant, this simply means that c is a multiple of a and can be deleted by row transformations).
First, if A consists only of $\wedge$-monomials and $c = 0$, then by the above relation we see that every entry of $\tilde{A}$ again has to be a monomial and that the block (c_0, c_1, c_2) is zero. Thus A is in any case minor-monomial, since every other nonzero block consists of exactly two nonzero entries and so the only possibility for a nontrivial 2-minor is the case when $(b_0, b_1, b_2) = (d_0, d_1, d_2)$, but then is clearly has to be a minor.
Second, if $\tilde{A}$ is minor-monomial we use that the Beilinson IIb resolution is a complex, so we have for every block the following conditions:

$$\sum_{i=0}^{2} a_i x_i = 0, \ldots, \sum_{i=0}^{2} d_i x_i = 0.$$

This leaves us with two possibilities. Either a block is completely zero, or it is of the form

$$(\lambda x_1, -\lambda x_0, 0) \qquad (\lambda x_2, 0, -\lambda x_0) \qquad (0, \lambda x_2, -\lambda x_1),$$

for some $\lambda \in k$. Such a block then can be lifted to a $\wedge$-monomial in A, which is explicitly given by

$$\lambda e_2 \qquad -\lambda e_1 \qquad \lambda e_0$$

respectively. Moreover, since $\tilde{A}$ is minor-monomial and has the above determined shape, we can add a suitable multiple of the first row to the second such that w.l.o.g. the block (c_0, c_1, c_2) is zero and thus lifted to the element 0 in A, which proves the claim.

5.2 Theoretical Preparations for Moduli problems

To finally apply the tools developed in the last chapters we need to recall some facts from the theory of moduli spaces and have a closer at the results from this theory in order correctly treat the subclass of toric sheaves. We begin with properties of

The Moduli Spaces $M_{\mathbb{P}^2}(2; c_1, c_2)$ and $\overline{M}_{\mathbb{P}^2}(2; c_1, c_2)$

Most of what we say in this section can for example be found in [Bar77]. We will mainly refer to the book [CO80] Before stating the necessary results we recall what moduli spaces are.

Definition 5.16. Let S be a scheme and fix numbers $c_1, \ldots, c_r$. A **family of stable bundles** on $\mathbb{P}^n$ of rank r and prescribed Chern classes $c_1, \ldots, c_r$ is a rank r vector bundle $\mathcal{E}$ over $S \times \mathbb{P}^n$ such that for all $s \in S$ the restricted bundle

$$E(s) := \mathcal{E}|_{\{s\} \times \mathbb{P}^n}$$

is a vector bundle over $\{s\} \times \mathbb{P}^n \cong \mathbb{P}^n$ with $c_i(E(s)) = c_i$. We say that the family $\mathcal{E}$ is parametrized by S.

There is a natural equivalence relation on such a parametrization. Denote by $p_1\colon S \times \mathbb{P}^n \to S$ the projection onto the first factor and let $\mathcal{E}$ and $\mathcal{F}$ be two families parametrized by S. We say that $\mathcal{E}$ is equivalent to $\mathcal{F}$ iff there is a line bundle $\mathcal{L}$ over S such that

$$\mathcal{F} \cong \mathcal{E} \otimes p_1^* \mathcal{L}.$$

Notation. We denote by $Cl_{\mathbb{P}^n}(r; c_1, \ldots, c_r)(S)$ the set of equivalence classes of families of stable vector bundles of rank r and Chern classes $c_1, \ldots, c_r$ on $\mathbb{P}^n$ parametrized by S. $Cl_{\mathbb{P}^n}(r; c_1, \ldots, c_r)(\_)$ is a contravariant functor from the category of schemes to the category of sets.

Definition 5.17. A pair $(M := M_{\mathbb{P}^n}(r; c_1, \ldots, c_r), \mathcal{U})$ where M is a scheme and $\mathcal{U}$ a bundle over $M \times \mathbb{P}^n$ is called a **fine moduli space** for stable rank r bundles with Chern classes c_i on $\mathbb{P}^n$ if it represents the functor $Cl_{\mathbb{P}^n}(r; c_1, \ldots, c_r)(\_)$. We will usually write only $M_{\mathbb{P}^n}(r; c_1, \ldots, c_r)$ or M for $(M, \mathcal{U})$.

Definition 5.18. We say that a scheme $M := M_{\mathbb{P}^n}(r; c_1, \ldots, c_r)$ is a **coarse moduli space** for $Cl_{\mathbb{P}^n}(r; c_1, \ldots, c_r)(\_)$ if

- there is a natural transformation

  $$Cl_{\mathbb{P}^n}(r; c_1, \ldots, c_r)(\_) \to \mathrm{Hom}(\_, M)$$

  which is bijective for every point x and

- any other transformation factors uniquely through the given one, i.e. for every scheme N and every natural transformation

  $$Cl_{\mathbb{P}^n}(r; c_1, \ldots, c_r)(\_) \to \mathrm{Hom}(\_, N)$$

  there is a unique morphism $\pi\colon M \to N$ such that for the induced morphism $\tilde{\pi}$ the following diagram commutes:

$$\begin{tikzcd}
Cl_{\mathbb{P}^n}(r; c_1, \ldots, c_r)(_) \arrow[rr] \arrow[dr] & & \mathrm{Hom}(_, M) \arrow[dl, dotted, "\tilde{\pi}"] \\
& \mathrm{Hom}(_, N). &
\end{tikzcd}$$

It is known that coarse moduli spaces $M_{\mathbb{P}^n}(r; c_1, \ldots, c_r)$ always exist, but it is in general not clear which geometric properties those spaces have. We restrict ourselves to the special case of $r = 2$ and $n = 2$, where in fact a lot of geometric aspects are known. It can be shown that by twisting every vector bundle of rank 2 on $\mathbb{P}^2$ can be normalized such that either $c_1 = 0$ or $c_1 = -1$, see for example [Har80].

Theorem 5.19.

- *$M_{\mathbb{P}^2}(2; 0, n)$ is a coarse moduli space which is always irreducible and it is fine iff n is odd.*
- *$M_{\mathbb{P}^2}(2; -1, n)$ is a fine moduli space for all n.*

Proof. This is a summary of [CO80, Theorem 4.1.12, Theorem 4.1.14, Theorem 4.1.17 and Theorem 4.2.1] □

In the last section of the chapter about Beilinson's theorem we calculated a monad for the space $M_{\mathbb{P}^2}(2; 0, n)$. As said before we are only interested in the toric bundles within this space, i.e. we want to investigate properties of $M_{\mathbb{P}^2}(2; 0, n)^T$. Moreover we can also consider the closure $\overline{M}_{\mathbb{P}^2}(2; 0, n)$ of the moduli space, which allows us to study the toric sheaves in the boundary of this space.
On weighted projective spaces we do not know about the existence of moduli spaces, but nevertheless we can use Beilinson monads pushed down from $\mathbb{P}^2$ to filter the toric sheaves on weighted projective planes in some special cases. Before doing so we need some theoretical considerations:

Toric Monads

In this section let $\mathcal{F}$ denote a stable sheaf on $\mathbb{P}^2$ of rank 2 with Chern classes $c_1 = c_1(\mathcal{F}) = 0$ and $c_2 = c_2(\mathcal{F}) = n$ for some natural number n.
Recall that by lemma 4.37 if $\mathcal{F}$ is a vector bundle it has a Beilinson monad of the form

$$0 \longrightarrow H^1(\mathcal{F}(-2)) \otimes \Omega^2_{\mathbb{P}^2}(2) \longrightarrow H^1(\mathcal{F}(-1)) \otimes \Omega^1_{\mathbb{P}^2}(1) \longrightarrow H^1(\mathcal{F}) \otimes \mathcal{O}_{\mathbb{P}^2} \longrightarrow 0.$$

Since we know from our preparations that in the case of a toric $\mathcal{F}$ the torus canonically acts on the cohomology groups, we can replace these finite-dimensional vector spaces by their dimension we computed in the last chapter:

$$0 \longrightarrow n\Omega^2_{\mathbb{P}^2}(2) \xrightarrow{M} n\Omega^1_{\mathbb{P}^2}(1) \xrightarrow{N} (n-2)\mathcal{O}_{\mathbb{P}^2} \longrightarrow 0$$

Proposition 5.20. *$\mathcal{F}$ is toric $\Leftrightarrow$ There is a monad given by T-equivariant matrices M and N*

Proof. Since the cohomology given by two equivariant morphisms is equivariant, one direction is clear. Let $\mathcal{F}$ be toric. Then the torus acts on the cohomology groups $\mathrm{H}^1(\mathcal{F}(-i))$, $i = 0, 1, 2$ occurring in the Beilinson monad as well as on the dual vector space V^*.This gives clearly gives us T-equivariant morphisms

$$\mu\colon \mathrm{H}^1(\mathcal{F}(-2)) \otimes V^* \to \mathrm{H}^1(\mathcal{F}(-1)) \quad \text{and} \quad \nu\colon \mathrm{H}^1(\mathcal{F}(-1)) \otimes V^* \to \mathrm{H}^1(\mathcal{F})$$

respecting this action and dually corresponding to T-equivariant matrices

$$M\colon \mathrm{H}^1(\mathcal{F}(-2)) \to \mathrm{H}^1(\mathcal{F}(-1)) \otimes V \quad \text{and} \quad N\colon \mathrm{H}^1(\mathcal{F}(-1)) \to \mathrm{H}^1(\mathcal{F}) \otimes V$$

which give us our desired monad. This finishes the proof. □

Remark. Note that if we are given two T-equivariant matrices M and N, then there always are by definition minor-monomial matrices M' and N' such that the following diagram commutes

$$\begin{tikzcd}
 & n\Omega^2_{\mathbb{P}^2}(2) \arrow[r, "M'"] \arrow[d, "\cong"] & n\Omega^1_{\mathbb{P}^2}(1) \arrow[d, "\cong"] & & \\
0 \arrow[r] & n\Omega^2_{\mathbb{P}^2}(2) \arrow[r, "M"] & n\Omega^1_{\mathbb{P}^2}(1) \arrow[r, "N"] & (n-2)\mathcal{O}_{\mathbb{P}^2} \arrow[r] & 0 \\
 & & n\Omega^1_{\mathbb{P}^2}(1) \arrow[u, "\cong"] \arrow[r, "N'"] & (n-2)\mathcal{O}_{\mathbb{P}^2}. \arrow[u, "\cong"] &
\end{tikzcd}$$

The problem lies in the fact that M and N may be chosen with respect to different T-linearizations of the involved sheaves. In particular the sheaf $n\Omega^1(1)$ could have a toric structure coming from toric divisors $A_1, \ldots, A_n$ inducing M and completely different divisors $B_1, \ldots, B_n$ inducing the matrix N. This can equivalently be expressed by saying that the composition of the isomorphisms in the middle column of the diagram might not be the identity and we have to plug in an additional isomorphism to make the sequence

$$0 \longrightarrow n\Omega^2_{\mathbb{P}^2}(2) \xrightarrow{M} n\Omega^1_{\mathbb{P}^2}(1) \xrightarrow{\cong} n\Omega^1_{\mathbb{P}^2}(1) \xrightarrow{N} (n-2)\mathcal{O}_{\mathbb{P}^2} \longrightarrow 0$$

respect the given T-linearizations. Another way to express this is that M and N can not be chosen simultaneously with respect to a fixed toric structure of the differential sheaves.

Example 5.21. A minor-monomial matrix M does not automatically imply that N is also minor-monomial. Let for example

$$M = \begin{pmatrix} x_2 & x_2 & x_1 \\ -x_1 & x_1 & 0 \\ 0 & 0 & x_2 \end{pmatrix}$$

then N could for example be

$$N = \begin{pmatrix} x_1 + x_2 \\ x_1 + x_2 \\ 2x_2 \end{pmatrix}.$$

However there might be the possibility that other choices of N are minor-monomial, e.g.

$$N = \begin{pmatrix} x_2 \\ x_2 \\ 0 \end{pmatrix}$$

in our case.

Since we want to compute vector bundles from the Beilinson monads we need to know when the matrix M is injective, respectively N surjective. The following lemma gives an easy criterion

Lemma 5.22. *1. N is surjective if and only if the columns in N do not generate a column that consists only of linearly dependent entries.*

2. M is injective iff the same holds true for the rows of M, i.e. every row in M and every row generated by these rows consists of at least two linearly independent entries.

Proof. First of all we know that that M is injective iff the dual map

$$n(\Omega^1_{\mathbb{P}^2}(1))^* \xrightarrow{M^T} n(\Omega^2_{\mathbb{P}^2}(2))^*$$

is surjective. This is obviously the same as to say that

$$n\Omega^1_{\mathbb{P}^2}(2) \xrightarrow{M^T} n\mathcal{O}_{\mathbb{P}^2}(1).$$

This means that after twisting by -1 we see that have the same condition for N and M^T. Thus it suffices to prove the statement for N. Recall that for all $v \in V$ we have

$$\Omega^1_{\mathbb{P}^2}(1)(\langle v \rangle) = (V/\langle v \rangle)^*,$$

which implies that $N\colon k^n \otimes (V/\langle v \rangle)^* \to k^{n-2}$ is surjective iff $N^T\colon k^{n-2} \to k^n \otimes (V/\langle v \rangle)^*$ is injective iff $(c_1, \ldots, c_{n-2}) \mapsto ((c_1, \ldots, c_{n-2}) \cdot N^T)/\langle v \rangle$ is nonzero for all $(c_1, \ldots, c_{n-2})$. This is after picking a basis equivalent to say that

$$0 \neq w := (\overline{w_1}, \ldots, \overline{w_n}) \in k^n \otimes V/\langle v \rangle$$

for all w in the image of the map, which means that there exist at least one i such that $w_i \neq \lambda_i v$ and this is true for every $v \in V$. Thus N^T is injective if and only if there are at least two linear independent rows in N^T. This shows the analogous statement for N and finishes the proof. $\square$

5.3 Applications

In this last section we apply the theory developed so far to some examples. After having mastered all the technical details of the last chapters, the whole problem of classifying toric sheaves using the Beilinson monads comes down to rather trivial arguments from linear algebra.
As a first example we find, all toric vector bundles on $\mathbb{P}^2$ with $c_2 = 2$ and $c_2 = 3$ and all toric sheaves in the boundary of the corresponding compactified moduli spaces $\overline{\mathrm{M}}_{\mathbb{P}^2}(2;0,n)$. In the second example we compute some toric stable sheaves of rank 2 with $c_1 = 0$ and $c_2 = 1$ on the weighted projective plane $\mathbb{P}(1,1,2)$.

The space $\overline{\mathbf{M}}_{\mathbb{P}^2}(2;0,2)^T$

Proposition 5.23. *Within the moduli space $\overline{M}_{\mathbb{P}^2}(2;0,2)$ there are 18 toric sheaves in the boundary of the space and no toric vector bundles.*

Proof. We start with the case of We know that in this case any toric bundle of this type is determined as the cokernel of the monad

$$0 \to 2\Omega^2_{\mathbb{P}^2}(2) \xrightarrow{M} 2\Omega^1_{\mathbb{P}^2}(1) \to \mathcal{F} \to 0,$$

where M is minor-monomial and its rows have at least two linearly independent entries. This means that M has no zero entry and is therefore because of the minor-monomiality of the form

$$M = \begin{pmatrix} \lambda_{00}e_i & \lambda_{01}e_j \\ \lambda_{10}e_i & \lambda_{11}x_j \end{pmatrix}$$

for $i \neq j \in \{0,1,2\}$. Thus by elementary row transformations we can assume that w.l.o.g. $\lambda_{01} = 0$, but this contradicts the assumption for M. So we see that there are no toric bundles in this case, but we can still consider the toric sheaves in the boundary. There are essentially two possibilities:
$\underline{\boldsymbol{\lambda_{10} = 0}:}$ Since we can multiply M from both sides with matrices from $GL_2(k)$ which do not destroy minor-monomiality, no matter how we choose the coefficients λ_{00} and λ_{11} we can alway normalize them:

$$\begin{pmatrix} \lambda_{00}^{-1} & 0 \\ 0 & \lambda_{11}^{-1} \end{pmatrix} \begin{pmatrix} \lambda_{00}e_i & 0 \\ 0 & \lambda_{11}e_j \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e_i & 0 \\ 0 & e_j \end{pmatrix}$$

Hence we are left with six possibilities for the choice of i and j, namely the following:

$$\begin{pmatrix} e_0 & 0 \\ 0 & e_0 \end{pmatrix} \quad \begin{pmatrix} e_1 & 0 \\ 0 & e_1 \end{pmatrix} \quad \begin{pmatrix} e_2 & 0 \\ 0 & e_2 \end{pmatrix}$$
$$\begin{pmatrix} e_0 & 0 \\ 0 & e_1 \end{pmatrix} \quad \begin{pmatrix} e_0 & 0 \\ 0 & e_2 \end{pmatrix} \quad \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$$

Note that the corresponding sheaves have fitting support on the set of points where the determinant $\det(M) \in S^2(V)$ vanishes, so in our case this corresponds to the following degenerated quadrics:

$$\{e_0^2 = 0\} \qquad \{e_1^2 = 0\} \qquad \{e_2^2 = 0\}$$
$$\{e_0e_1 = 0\} \qquad \{e_0e_2 = 0\} \qquad \{e_1e_2 = 0\}.$$

$\underline{\boldsymbol{\lambda_{10} \neq 0}:}$ In this case the matrix M has, after similar transformations as above, to be of the form

$$M = \begin{pmatrix} e_i & 0 \\ e_l & e_j \end{pmatrix}$$

where $i \neq l \neq j$ since otherwise we would be in the first case by linear transformations. Thus we get twelve new sheaves from the matrices

$$\begin{pmatrix} e_0 & 0 \\ e_1 & e_0 \end{pmatrix} \quad \begin{pmatrix} e_0 & 0 \\ e_2 & e_0 \end{pmatrix} \quad \begin{pmatrix} e_1 & 0 \\ e_0 & e_1 \end{pmatrix}$$
$$\begin{pmatrix} e_1 & 0 \\ e_2 & e_1 \end{pmatrix} \quad \begin{pmatrix} e_2 & 0 \\ e_0 & e_2 \end{pmatrix} \quad \begin{pmatrix} e_2 & 0 \\ e_1 & e_2 \end{pmatrix}$$
$$\begin{pmatrix} e_0 & 0 \\ e_2 & e_1 \end{pmatrix} \quad \begin{pmatrix} e_1 & 0 \\ e_2 & e_0 \end{pmatrix} \quad \begin{pmatrix} e_0 & 0 \\ e_1 & e_2 \end{pmatrix}$$
$$\begin{pmatrix} e_2 & 0 \\ e_1 & e_0 \end{pmatrix} \quad \begin{pmatrix} e_1 & 0 \\ e_0 & e_2 \end{pmatrix} \quad \begin{pmatrix} e_2 & 0 \\ e_0 & e_1 \end{pmatrix}$$

since this time the order of the entries on the diagonal is important. To summarize this we see that for every type of determinant from above there are exactly three sheaves, so altogether there are 18 of them. □

The space $\overline{\mathbf{M}}_{\mathbb{P}^2}(2;0,3)^T$

Proposition 5.24. *There are exactly 6 toric vector bundles in $M_{\mathbb{P}^2}(2;0,3)$ given by the Koszul-like monad*

$$M = \begin{pmatrix} 0 & e_j & e_i \\ e_j & 0 & e_l \\ e_i & e_l & 0 \end{pmatrix} \quad N = \begin{pmatrix} e_l \\ e_i \\ e_j \end{pmatrix},$$

with $i \neq j \neq l \neq i$. Moreover there are 70 toric sheaves in the boundary of this space.

Proof. We first consider toric vector bundles. The condition for M is that every row and column has at least two linearly independent entries and M is monomial. In this case M does not have a full 2×2-minor. Assume this would be true, then w.l.o.g. M would be of the form

$$M = \begin{pmatrix} \lambda_{00}e_i & \lambda_{01}e_i & * \\ \lambda_{10}e_j & \lambda_{11}e_j & * \\ * & * & * \end{pmatrix}.$$

Since all rows have to have two independent entries it follows that the two upper right stars also have to be nonzero. But then the minor-monomiality of M implies that those entries also have to be of the form $\left(\begin{smallmatrix} \lambda_{02}e_i \\ \lambda_{12}e_j \end{smallmatrix}\right)$ contradicting the row-property of M. Therefore we can assume that every row and column has exactly two nonzero entries. Thus M has to be of the form

$$M = \begin{pmatrix} 0 & \lambda_{01}e_j & \lambda_{02}e_i \\ \lambda_{10}e_j & 0 & \lambda_{12}e_l \\ \lambda_{20}e_i & \lambda_{21}e_l & 0 \end{pmatrix},$$

where $i \neq j \neq l \neq i$. This symmetric matrix is minor-monomial and it is easy to check that the only possible nonzero choice producing a complex is

$$N = \begin{pmatrix} \lambda e_l \\ \mu e_i \\ \nu e_j \end{pmatrix}$$

for some nonzero scalars $\lambda, \mu, \nu \in k$ with the condition

$$
\begin{aligned}
M \wedge N &= \begin{pmatrix} (\lambda_{01}\mu - \lambda_{02}\nu)e_i \wedge e_j \\ (\lambda_{10}\lambda - \lambda_{12}\nu)e_j \wedge e_l \\ (\lambda_{20}\lambda - \lambda_{21}\mu)e_i \wedge e_l \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
&\Leftrightarrow \nu = \frac{\lambda_{01}}{\lambda_{02}}\mu, \quad \nu = \frac{\lambda_{10}}{\lambda_{12}}\lambda, \quad \mu = \frac{\lambda_{20}}{\lambda_{21}}\lambda \\
&\Leftrightarrow \lambda \text{ is arbitrary, } N = \lambda \cdot \begin{pmatrix} e_l \\ \frac{\lambda_{20}}{\lambda_{21}} e_i \\ \frac{\lambda_{10}}{\lambda_{12}} e_j \end{pmatrix} \text{ and } \lambda_{10} \cdot \lambda_{02} \cdot \lambda_{21} = \lambda_{01} \cdot \lambda_{20} \cdot \lambda_{12}
\end{aligned}
$$

Thus we see that the monad gives us a vector bundle if and only if M is of the form

$$
M = \begin{pmatrix} 0 & \lambda_{01}e_j & \lambda_{02}e_i \\ \lambda_{10}e_j & 0 & \lambda_{12}e_l \\ \lambda_{20}e_i & \lambda_{21}e_l & 0 \end{pmatrix}
$$

and $\lambda_{10} \cdot \lambda_{02} \cdot \lambda_{21} = \lambda_{01} \cdot \lambda_{20} \cdot \lambda_{12}$. We show that this M can also be normalized. For this we can use matrices $g_1, g_2 \in GL_3(k)$ that preserve the minor-monomiality of M. It is easy to see that in our special case because of the structure if M only diagonal matrices are allowed, i.e.

$$
g_1 = \begin{pmatrix} \alpha_0 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_2 \end{pmatrix} \quad g_2 = \begin{pmatrix} \beta_0 & 0 & 0 \\ 0 & \beta_1 & 0 \\ 0 & 0 & \beta_2 \end{pmatrix}.
$$

Thus we obtain

$$
g_1 M g_2 = \begin{pmatrix} 0 & \beta_1^{-1}\alpha_0\lambda_{01}e_j & \beta_2^{-1}\alpha_0\lambda_{02}e_i \\ \beta_0^{-1}\alpha_1\lambda_{10}e_j & 0 & \beta_2^{-1}\alpha_1\lambda_{12}e_l \\ \beta_0^{-1}\alpha_2\lambda_{20}e_i & \beta_1^{-1}\alpha_2\lambda_{21}e_l & 0 \end{pmatrix}
$$

and we have to check if those six equations

$$
\begin{gathered}
\beta_1 = \alpha_0\lambda_{01}, \quad \beta_2 = \alpha_0\lambda_{02}, \quad \beta_0 = \alpha_1\lambda_{10}, \\
\beta_2 = \alpha_1\lambda_{12}, \quad \beta_0 = \alpha_2\lambda_{20}, \quad \beta_1 = \alpha_2\lambda_{21}
\end{gathered}
$$

are compatible with the relation of the λ_{ij}. This is clear since if we fix α_0 then every other coefficient is determined and:

$$
\begin{aligned}
\beta_1 = \alpha_0\lambda_{01} \Rightarrow \beta_1 = \alpha_2\lambda_{21} &\Leftrightarrow \alpha_2 = \alpha_0 \cdot \frac{\lambda_{01}}{\lambda_{21}} \\
\Rightarrow \beta_0 = \alpha_2\lambda_{20} &= \lambda_{20}\alpha_0 \cdot \frac{\lambda_{01}}{\lambda_{21}} \\
\Rightarrow \beta_0 = \alpha_1\lambda_{10} &\Leftrightarrow \alpha_1 = \frac{\beta_0}{\lambda_{10}} = \alpha_0 \cdot \frac{\lambda_{20} \cdot \lambda_{01}}{\lambda_{21} \cdot \lambda_{10}} \\
\Rightarrow \alpha_0\lambda_{02} = \beta_2 &= \alpha_1\lambda_{12} = \alpha_0 \frac{\lambda_{12} \cdot \lambda_{20} \cdot \lambda_{01}}{\lambda_{21} \cdot \lambda_{10}},
\end{aligned}
$$

but this is exactly the relation of the λ_{ij}. Hence by permutation of the coordinates we get exactly 6 toric vector bundles from the monads

$$
M = \begin{pmatrix} 0 & e_j & e_i \\ e_j & 0 & e_l \\ e_i & e_l & 0 \end{pmatrix} \quad N = \begin{pmatrix} e_l \\ e_i \\ e_j \end{pmatrix}.
$$

Note that every such toric sheaf has support on the degenerated cubic $\{e_0e_1e_2 = 0\}$ corresponding to the three coordinate lines.

Let us now look at the stable sheaves in the boundary. We therefore drop the condition on the rows of M, but we are still looking for surjective matrices N, i.e. matrices with at least two independent vectors. Thus M has to have at least (which means exactly) one row with two zero entries and can be seen as an extension of the case $n = 2$, i.e.

$$M = \left(\begin{array}{c|cc} \lambda_{00}e_k & 0 & 0 \\ \hline * & \lambda_{11}e_i & 0 \\ * & \lambda_{21}e_l & \lambda_{22}e_j \end{array}\right)$$

and we have to distinguish several cases depending on what the two stars are. It is clear by the last case that every full 2×2-matrix can either be replaced by a matrix with at least one zero or is not minor-monomial. Moreover a 2×2-matrix with three nonzero entries has to be one of the twelve types above. Thus we are essentially left with the following types (as before we can normalize the matrix, but this can be done in the same way as before, so we don't repeat the rather boring procedure):

1. Since permutation of the coordinates can be realized by invertible matrices we only have one M and four choices for N.

$$M = \begin{pmatrix} e_0 & 0 & 0 \\ 0 & e_1 & 0 \\ 0 & 0 & e_2 \end{pmatrix}, \qquad N = \begin{pmatrix} e_0 \\ e_1 \\ 0 \end{pmatrix}, \begin{pmatrix} e_0 \\ 0 \\ e_2 \end{pmatrix}, \begin{pmatrix} 0 \\ e_1 \\ e_2 \end{pmatrix} \text{ or } \begin{pmatrix} e_0 \\ e_1 \\ e_2 \end{pmatrix}.$$

2. In the next case permutation plays a role, since two of the vectors are the same, so we obtain six possible matrices M and have for each of them three choices for N, thus 18 new sheaves:

$$M = \begin{pmatrix} e_0 & 0 & 0 \\ 0 & e_1 & 0 \\ 0 & 0 & e_1 \end{pmatrix}, \qquad N = \begin{pmatrix} e_0 \\ e_1 \\ 0 \end{pmatrix}, \begin{pmatrix} e_0 \\ 0 \\ e_1 \end{pmatrix} \text{ or } \begin{pmatrix} e_0 \\ e_1 \\ e_1 \end{pmatrix}.$$

3. There are twelve possible matrices M of the following type

$$M = \begin{pmatrix} e_0 & 0 & 0 \\ 0 & e_1 & 0 \\ 0 & e_{0/2} & e_1 \end{pmatrix}, \qquad N = \begin{pmatrix} e_0 \\ e_1 \\ e_{0/2} \end{pmatrix}, \begin{pmatrix} 0 \\ e_1 \\ e_{0/2} \end{pmatrix} \text{ or } \begin{pmatrix} e_0 \\ 0 \\ e_1 \end{pmatrix}$$

   inducing three possible choices for N and hence gives us 36 new toric sheaves.

4. The fourth possible type for M, inducing a unique surjective N, is the following

$$M = \begin{pmatrix} e_0 & 0 & 0 \\ e_1 & e_0 & 0 \\ 0 & e_1 & e_0 \end{pmatrix}, \qquad N = \begin{pmatrix} e_0 \\ e_1 \\ e_0 \end{pmatrix}$$

   and so we can add 6 sheaves for each choice of M obtained from permutations.

5. The last possible M not equivalent to any of the types listed above looks like this

$$M = \begin{pmatrix} e_0 & 0 & 0 \\ e_1 & e_0 & 0 \\ e_2 & 0 & e_0 \end{pmatrix}, \qquad N = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \end{pmatrix}$$

   with N again determined by M and thus we get the last six sheaves.

So all in all we have 70 toric sheaves in the boundary of $M_{\mathbb{P}^2}(2; 0, 3)$. □

This finishes our first examples and we now pass over to to the weighted projective plane $\mathbb{P}(1, 1, 2)$ and construct some toric sheaves from Beilinson sequences.

Some toric sheaves on $\mathbb{P}(1,1,2)$

Proposition 5.25. *Let $\mathcal{F}$ be a vector bundle on $\mathbb{P}(Q) = \mathbb{P}(1,1,2)$ of rank 2 with Chern classes $c_1(\mathcal{F}) = 0$ and $c_2(\mathcal{F}) = 1$ and having no global sections, i.e. $H^0(\mathcal{F}) = 0$. Then all possible $\mathcal{F}$ are obtained by one of the following sequences:*

1. $0 \longrightarrow 2\mathcal{O}_{\mathbb{P}(Q)}(-2) \longrightarrow 4\mathcal{O}_{\mathbb{P}(Q)}(-1) \longrightarrow \mathcal{F} \longrightarrow 0$
2. $0 \longrightarrow \mathcal{O}_{\mathbb{P}(Q)}(-2) \oplus \mathcal{O}_{\mathbb{P}(Q)}(-1) \longrightarrow \overline{\Omega}^1_{\mathbb{P}(Q)}(1) \oplus 2 \cdot \mathcal{O}_{\mathbb{P}(Q)}(-1) \longrightarrow \mathcal{F} \longrightarrow 0$
3. $0 \longrightarrow 2\mathcal{O}_{\mathbb{P}(Q)}(-1) \longrightarrow 2\overline{\Omega}^1_{\mathbb{P}(Q)}(1) \longrightarrow \mathcal{F} \longrightarrow 0$

Proof. We use the morphism $\pi\colon \mathbb{P}^2 \to \mathbb{P}(1,1,2)$ to pull this sheaf back to $\mathbb{P}^2$. Note that $\pi^*\mathcal{F}$ is of course a rank 2 vector bundle. Since the induced morphism of the Chow rings

$$\pi^*\colon \mathbb{Z} \cong A^*(\mathbb{P}(1,1,2) \to A^*(\mathbb{P}^2) \cong \mathbb{Z}$$

is just multiplication by 2 by example 3.27 the Chern classes of $\pi^*\mathcal{F}$ are $c_1(\pi^*\mathcal{F}) = 0$ and $c_2(\pi^*\mathcal{F}) = 2 \cdot n$. Moreover, since $\mathrm{H}^0(\mathcal{F}) = 0$ the sheaf $\pi^*\mathcal{F}$ also has no sections. Assume this would be wrong, then since $\pi_*^{\mu_Q}$ is exact $\pi_*^{\mu_Q}\pi^*\mathcal{F} \cong \mathcal{F}$ would also have sections, which is a contradiction. By lemma 3.1 from [Har80] we know that $\mathrm{H}^0(\pi^*\mathcal{F}) = 0$ this is equivalent to $\pi^*\mathcal{F}$ being stable. Thus we can apply Lemma 4.37 to get the Beilinson I resolution we used before. We only consider the first and easiest case $n = 1$ and we will see that this is combinatorial already hard enough. We have the following resolution for $\pi^*\mathcal{F}$

$$0 \longrightarrow \mathrm{H}^1(\pi^*\mathcal{F}(-2)) \otimes \Omega^2_{\mathbb{P}^2}(2) \longrightarrow \mathrm{H}^1(\pi^*\mathcal{F}(-1)) \otimes \Omega^1_{\mathbb{P}^2}(1) \longrightarrow \pi^*\mathcal{F} \longrightarrow 0$$

where both cohomology groups have the dimension $2 \cdot n = 2$. Now we push this Beilinson sequence with $\pi_*^0 = \pi_*^{\mu_Q}$ down to $\mathbb{P}(1,1,2)$. We abbreviate the second character by $\pi_*^1 := \pi_*^{(0,0,1)}$ and obtain

$$0 \to \begin{matrix} [\mathrm{H}^1(\pi^*\mathcal{F}(-2))]^0 \otimes \pi_*^0\Omega^2_{\mathbb{P}^2}(2) \\ \oplus \\ [\mathrm{H}^1(\pi^*\mathcal{F}(-2))]^1 \otimes \pi_*^1\Omega^2_{\mathbb{P}^2}(2) \end{matrix} \to \begin{matrix} [\mathrm{H}^1(\pi^*\mathcal{F}(-1))]^0 \otimes \pi_*^0\Omega^1_{\mathbb{P}^2}(1) \\ \oplus \\ [\mathrm{H}^1(\pi^*\mathcal{F}(-1))]^1 \otimes \pi_*^1\Omega^1_{\mathbb{P}^2}(1) \end{matrix} \to \pi_*^0\pi^*\mathcal{F} \to 0.$$

By lemma 3.9 we know how the Eigenspace decomposition of the cohomology groups looks like, namely

$$0 \to \begin{matrix} \mathrm{H}^1(\pi_*^0\pi^*\mathcal{F} \otimes \pi_*^0\mathcal{O}_{\mathbb{P}^2}(-2)) \otimes \pi_*^0\Omega^2_{\mathbb{P}^2}(2) \\ \oplus \\ \mathrm{H}^1(\pi_*^1\pi^*\mathcal{F} \otimes \pi_*^1\mathcal{O}_{\mathbb{P}^2}(-2)) \otimes \pi_*^1\Omega^2_{\mathbb{P}^2}(2) \end{matrix} \to \begin{matrix} \mathrm{H}^1(\pi_*^0\pi^*\mathcal{F} \otimes \pi_*^0\mathcal{O}_{\mathbb{P}^2}(-1)) \otimes \pi_*^0\Omega^1_{\mathbb{P}^2}(1) \\ \oplus \\ \mathrm{H}^1(\pi_*^1\pi^*\mathcal{F} \otimes \pi_*^1\mathcal{O}_{\mathbb{P}^2}(-1)) \otimes \pi_*^1\Omega^1_{\mathbb{P}^2}(1) \end{matrix} \to \pi_*^0\pi^*\mathcal{F} \to 0$$

Using $\pi_*^0\pi^*\mathcal{F} \cong \mathcal{F}$ and $\pi_*^\chi\mathcal{O}_{\mathbb{P}^2}(j) \cong \mathcal{O}_{\mathbb{P}(Q)}(j - |\chi|)$ from Lemma 3.10 we obtain

$$0 \to \begin{matrix} \mathrm{H}^1(\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(Q)}(-2)) \otimes \pi_*^0\Omega^2_{\mathbb{P}^2}(2) \\ \oplus \\ \mathrm{H}^1(\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(Q)}(-3)) \otimes \pi_*^1\Omega^2_{\mathbb{P}^2}(2) \end{matrix} \to \begin{matrix} \mathrm{H}^1(\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(Q)}(-1)) \otimes \pi_*^0\Omega^1_{\mathbb{P}^2}(1) \\ \oplus \\ \mathrm{H}^1(\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(Q)}(-2)) \otimes \pi_*^1\Omega^1_{\mathbb{P}^2}(1) \end{matrix} \to \mathcal{F} \to 0$$

Moreover, since $\pi_*^\chi\Omega^j_{\mathbb{P}^n}(l) \cong (\overline{\Omega}^j_{\mathbb{P}(Q)})(\log y^{I(\chi)})(l - |\chi|)$ by corollary 4.32, the sequence is isomorphic to

$$0 \to \begin{matrix} \mathrm{H}^1(\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(Q)}(-2)) \otimes \overline{\Omega}^2_{\mathbb{P}(Q)}(2) \\ \oplus \\ \mathrm{H}^1(\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(Q)}(-3)) \otimes \overline{\Omega}^2_{\mathbb{P}(Q)}(\log y^{I(1)})(2-1) \end{matrix} \to \begin{matrix} \mathrm{H}^1(\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(Q)}(-1)) \otimes \pi_*^0\Omega^1(1) \\ \oplus \\ \mathrm{H}^1(\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(Q)}(-2)) \otimes \overline{\Omega}^1_{\mathbb{P}(Q)}(\log y^{I(1)})(1-1) \end{matrix} \to \mathcal{F} \to 0$$

We can replace the nontrivial logarithmic differential sheaves by the rule

$$\overline{\Omega}^j_{\mathbb{P}(Q)}(\log y^{I(-\chi)})(j - |-\chi|) \cong \bigoplus_{J \cap I \neq \varnothing} \mathcal{O}_{\mathbb{P}(Q)}(j - |\chi| - |Q_J|)^{\binom{\#I(\chi)-1}{j-\#J}},$$

where $|Q_J| := \sum_{i \in J} q_i$. Thus, for $\overline{\Omega}^1_{\mathbb{P}(Q)}(\log y^{I(1)})$ we have exactly two choices for J, namely $J = (1,0,0)$ and $J = (0,1,0)$ which shows that

$$\overline{\Omega}^1_{\mathbb{P}(Q)}(\log y^{I(1)}) \cong 2 \cdot \mathcal{O}_{\mathbb{P}(Q)}(-1).$$

In the case of $\overline{\Omega}^2_{\mathbb{P}(Q)}(\log y^{I(1)})(1)$ the only possibility is $J = (1,1,0)$ and we therefore get an isomorphism

$$\overline{\Omega}^1_{\mathbb{P}(Q)}(\log y^{I(1)}) \cong \mathcal{O}_{\mathbb{P}(Q)}(-1).$$

Furthermore, there is an isomorphism

$$\overline{\Omega}^2_{\mathbb{P}(Q)}(2) \cong S'\widetilde{(2-|Q|)} \cong \mathcal{O}_{\mathbb{P}(Q)}(-2)$$

by remark 3.2. Therefore we can write the above sequence as

$$0 \to \begin{array}{c} \mathrm{H}^1(\mathcal{F}\otimes\mathcal{O}_{\mathbb{P}(Q)}(-2))\otimes\mathcal{O}_{\mathbb{P}(Q)}(-2) \\ \oplus \\ \mathrm{H}^1(\mathcal{F}\otimes\mathcal{O}_{\mathbb{P}(Q)}(-3))\otimes\mathcal{O}_{\mathbb{P}(Q)}(-1) \end{array} \to \begin{array}{c} \mathrm{H}^1(\mathcal{F}\otimes\mathcal{O}_{\mathbb{P}(Q)}(-1))\otimes\overline{\Omega}^1(1) \\ \oplus \\ \mathrm{H}^1(\mathcal{F}\otimes\mathcal{O}_{\mathbb{P}(Q)}(-2))\otimes 2\cdot\mathcal{O}_{\mathbb{P}(Q)}(-1) \end{array} \to \mathcal{F} \to 0.$$

Now we have to distinguish several cases, since it is a priori not clear how the cohomology groups are decomposed into Eigenspaces under the action of the group μ_Q. Since the group $\mathrm{H}^1(\mathcal{F}\otimes\mathcal{O}_{\mathbb{P}(Q)}(-2))$, which can only have dimension $0,1$ or 2, occurs in both sides of the monad and all other dimensions are determined by such a choice (since the dimension has to sum up to 2), we are left with the following three cases:

1. $0 \longrightarrow 2\mathcal{O}_{\mathbb{P}(Q)}(-2) \longrightarrow 4\mathcal{O}_{\mathbb{P}(Q)}(-1) \longrightarrow \mathcal{F} \longrightarrow 0$
2. $0 \longrightarrow \mathcal{O}_{\mathbb{P}(Q)}(-2) \oplus \mathcal{O}_{\mathbb{P}(Q)}(-1) \longrightarrow \overline{\Omega}^1_{\mathbb{P}(Q)}(1) \oplus 2\cdot\mathcal{O}_{\mathbb{P}(Q)}(-1) \longrightarrow \mathcal{F} \longrightarrow 0$
3. $0 \longrightarrow 2\mathcal{O}_{\mathbb{P}(Q)}(-1) \longrightarrow 2\overline{\Omega}^1_{\mathbb{P}(Q)}(1) \longrightarrow \mathcal{F} \longrightarrow 0,$

which finishes the proof. □

Warning. Since we don't know how μ_Q acts on the cohomology groups, we also don't know whether the three cases listed above are nonempty. In what follows we simply assume that they are and check all the possibilities to obtain a toric sheaf.
Since except for $\mathcal{O}_{\mathbb{P}(Q)}(-2)$ all the sheaves in our three resolutions are not locally free, we cannot use Whitney's sum formula to compute the Chern classes "by hand", so the sheaves $\mathcal{F}$ have the right Chern classes, although we cannot check this via the given resolutions.

Remark. To avoid the technically difficult computations with the logarithmic differential sheaves, one might have the idea to use the well known isomorphism $\Omega^2_{\mathbb{P}^2}(2) \cong \mathcal{O}_{\mathbb{P}^2}(-1)$. Then by the calculations of the above proof we clearly have to have

$$\pi_*\Omega^2_{\mathbb{P}^2}(2) \cong \mathcal{O}_{\mathbb{P}(Q)}(-1) \oplus \mathcal{O}_{\mathbb{P}(Q)}(-2) \cong \pi_*\mathcal{O}_{\mathbb{P}^2}(-1),$$

using the morphism π_* to push the isomorphic sheaves. However, for single characters this might be wrong. We see by a direct calculation

$$\pi^0_*\mathcal{O}_{\mathbb{P}^2}(-1) \cong \mathcal{O}_{\mathbb{P}(Q)}(-1) \qquad \pi^1_*\mathcal{O}_{\mathbb{P}^2}(-1) \cong \mathcal{O}_{\mathbb{P}(Q)}(-2)$$

that the pushed down sheaves occur in a different order. The reason for this is that the isomorphism $\Omega^2_{\mathbb{P}^2}(2) \cong \mathcal{O}_{\mathbb{P}^2}(-1)$ is not μ_Q-invariant and therefore changes the action of the characters. Thus we always have to be careful using such isomorphisms.

Note that there are no toric bundles $\mathcal{F}$ with the properties of the proposition, since pulling back via π respects the T-equivariance of a toric sheaf on $\mathbb{P}(Q)$ and have seen before that there are no toric bundles $\pi^*\mathcal{F}$ on $\mathbb{P}^2$ in this setting. Although we don't know whether the cases are empty, we can still filter the possible toric sheaves from those sequences. To do so we first of all have to know all morphisms involved, i.e. we only have to find out what

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}(Q)}(-j), \overline{\Omega}^1_{\mathbb{P}(Q)}(1))$$

for $j = 1$ and 2 is, since all other morphisms are known. Recall from the chapter about weighted projective spaces that there is the generalized Euler short exact sequence

$$0 \to \overline{\Omega}^1_{\mathbb{P}(Q)}(1) \to \bigoplus_{i=0}^{2} \mathcal{O}_{\mathbb{P}(Q)}(1-q_i) \xrightarrow{\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}} \mathcal{O}_{\mathbb{P}(Q)} \to 0$$

and so applying the covariant functor $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}(Q)}(-j), \_)$ gives us the induced long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}(\mathcal{O}_{\mathbb{P}(Q)}(-j), \overline{\Omega}^1_{\mathbb{P}(Q)}(1)) \to \mathrm{Hom}(\mathcal{O}_{\mathbb{P}(Q)}(-j), \bigoplus_{i=0}^{2} \mathcal{O}_{\mathbb{P}(Q)}(1-q_i)) \\ \to \mathrm{Hom}(\mathcal{O}_{\mathbb{P}(Q)}(-j), \mathcal{O}_{\mathbb{P}(Q)}(1)) \to \dots \end{aligned}$$

Hence a morphism from $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}(Q)}(-j), \overline{\Omega}^1_{\mathbb{P}(Q)}(1))$ is isomorphic to a morphism from

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}(Q)}(-j), \bigoplus_{i=0}^{2} \mathcal{O}_{\mathbb{P}(Q)}(1-q_i)) \cong \bigoplus_{i=0}^{2} \mathrm{Hom}(\mathcal{O}_{\mathbb{P}(Q)}(-j), \mathcal{O}_{\mathbb{P}(Q)}(1-q_i)) \cong \bigoplus_{i=0}^{2} S'_{q_i-1-j}$$

which is zero if we compose it with $\left(\begin{smallmatrix} y_0 \\ y_1 \\ y_2 \end{smallmatrix}\right)$.
In the case of $j = 1$ this reduces to

$$\begin{tikzcd} \mathcal{O}_{\mathbb{P}(Q)}(-1) \arrow[r] \arrow[dr, "0"'] & \mathcal{O}_{\mathbb{P}(Q)} \oplus \mathcal{O}_{\mathbb{P}(Q)} \oplus \mathcal{O}_{\mathbb{P}(Q)}(-1) \arrow[d, "\left(\begin{smallmatrix} y_0 \\ y_1 \\ y_2 \end{smallmatrix}\right)"] \\ & \mathcal{O}_{\mathbb{P}(Q)}(1) \end{tikzcd}$$

and therefore it is clear that

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}(Q)}(-j), \overline{\Omega}^1_{\mathbb{P}(Q)}(1)) \cong \{\lambda \cdot h := (\lambda y_1, -\lambda y_0, 0) | \lambda \in k\} \cong k.$$

The case $j = 2$ is a little bit more complicated, since the twist by -2 allows more choices for the morphisms, i.e. we have

$$\begin{tikzcd} \mathcal{O}_{\mathbb{P}(Q)}(-2) \arrow[r, "f"] \arrow[dr, "0"'] & \mathcal{O}_{\mathbb{P}(Q)} \oplus \mathcal{O}_{\mathbb{P}(Q)} \oplus \mathcal{O}_{\mathbb{P}(Q)}(-1) \arrow[d, "\left(\begin{smallmatrix} y_0 \\ y_1 \\ y_2 \end{smallmatrix}\right)"] \\ & \mathcal{O}_{\mathbb{P}(Q)}(1) \end{tikzcd}$$

where $f \in S'_2 \oplus S'_2 \oplus S'_1$. Since S'_1 is generated by the monomials y_0 and y_1 and S_2 is generated by $y_0 y_1$, y_0^2, y_1^2 and y_2 we see that

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}(Q)}(-j), \overline{\Omega}^1_{\mathbb{P}(Q)}(1))$$

is generated by the four one-dimensional spaces

$$\begin{aligned} &\{\lambda \cdot f_1 := \lambda \cdot (y_0 y_1, -y_0^2, 0) | \lambda \in k\} \\ &\{\lambda \cdot f_2 := \lambda \cdot (y_1^2, -y_0 y_1, 0) | \lambda \in k\} \\ &\{\lambda \cdot f_3 := \lambda(y_2, 0, -y_0) | \lambda \in k\} \\ &\{\lambda \cdot f_4 := \lambda(y_2, 0, -\ y_1) | \lambda \in k\}. \end{aligned}$$

So let's check if there are some toric sheaves in the "boundary" given by the monads. We begin with the first resolution:

1.
$$0 \longrightarrow 2\mathcal{O}_{\mathbb{P}(Q)}(-2) \xrightarrow{M_1} 4\mathcal{O}_{\mathbb{P}(Q)}(-1) \longrightarrow \mathcal{F} \longrightarrow 0$$

The 2×4 matrix M_1 consists of monomials of $S_1 = k \cdot y_0 \oplus k \cdot y_1$. Therefore we obtain 11 possible minor-monomial M_1:

$$\begin{pmatrix} y_0 & 0 & y_1 & 0 \\ 0 & y_0 & 0 & y_1 \end{pmatrix} \quad \begin{pmatrix} y_0 & 0 & 0 & 0 \\ 0 & y_0 & y_1 & 0 \end{pmatrix} \quad \begin{pmatrix} y_0 & 0 & y_1 & 0 \\ 0 & y_0 & y_1 & 0 \end{pmatrix}$$
$$\begin{pmatrix} y_0 & 0 & 0 & 0 \\ 0 & y_0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} y_1 & 0 & 0 & 0 \\ 0 & y_1 & y_0 & 0 \end{pmatrix} \quad \begin{pmatrix} y_1 & 0 & y_0 & 0 \\ 0 & y_1 & y_0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} y_1 & 0 & 0 & 0 \\ 0 & y_1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} y_0 & y_1 & 0 & 0 \\ 0 & y_0 & y_1 & 0 \end{pmatrix} \quad \begin{pmatrix} y_0 & y_1 & 0 & 0 \\ 0 & y_0 & 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} y_1 & y_0 & 0 & 0 \\ 0 & y_1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} y_0 & 0 & 0 & 0 \\ 0 & y_1 & 0 & 0 \end{pmatrix}.$$

2. The second case is combinatorially the hardest. The reason for this is that in the sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(Q)}(-2) \oplus \mathcal{O}_{\mathbb{P}(Q)}(-1) \xrightarrow{M_2} \overline{\Omega}^1_{\mathbb{P}(Q)}(1) \oplus 2\mathcal{O}_{\mathbb{P}(Q)}(-1) \longrightarrow \mathcal{F} \longrightarrow 0$$

the matrix defining the sheaf is of the form

$$M_2 = \begin{pmatrix} * & k \cdot y_{0/1} & k \cdot y_{0/1} \\ * & k & k \end{pmatrix}$$

where the upper star is a morphism from

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}(Q)}(-2), \overline{\Omega}^1_{\mathbb{P}(Q)}(1))$$

and the lower a morphism from

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}(Q)}(-1), \overline{\Omega}^1_{\mathbb{P}(Q)}(1)).$$

The problem is that these morphism are not necessarily "compatible" anymore, as in the cases we studied on $\mathbb{P}^2$, i.e. we can not use linear algebra and even worse we don't have a good criterion to decide whether M_2 is toric or not, since minor-monomiality does not make sense in this case. However, the right part of the matrix is rather easy, so we mainly have to care about the stars. We therefore have to go one step back and consider the toric divisors that we have to choose for the T-equivariances of the sheaves.
First consider the case $j = 1$ and check that the one-dimensional space of homomorphism consists of toric morphisms. To show this we have to find toric divisors A, B_0, B_1 and B_2 in $\mathbb{Z}^3$ of $\mathbb{Z}$-degree -1, 0, 0 and -1, respectively, such that

$$S'_{B_0-A} = k \cdot y^{B_0-A} = k \cdot y^{(0,1,0)} = k \cdot y_1$$
$$\text{and } S'_{B_1-A} = k \cdot y^{B_1-A} = k \cdot y^{(1,0,0)} = k \cdot y_0.$$

Clearly, it does not matter what B_2 is and the choice

$$A = (0, -1, 0) \quad B_0 = (0, 0, 0) \quad B_1 = (1, -1, 0)$$

is modulo divisors of degree 0 unique, hence the morphism $(\lambda y_1, -\lambda y_0, 0)$ is toric for every $\lambda \in k$.

For the morphisms in the case of $j = 2$ we have to find toric divisors as above, with the little difference that the degree of A is -2, and consider four different cases:

- In the first case the divisors have to satisfy the following relation

$$S_{B_0-A} = k \cdot x^{(1,1,0)} \qquad S_{B_1-A} = k \cdot x^{(2,0,0)}$$

  to get the morphism f_1, which means that we can again choose B_2 to be arbitrary and

$$A = (-2, 0, 0) \quad B_0 = (-1, 1, 0) \quad B_1 = (0, 0, 0).$$

- The second case works analogously:

$$S'_{B_0-A} = k \cdot y^{(2,0,0)} \qquad S'_{B_1-A} = k \cdot y^{(1,1,0)}$$

  gives us f_2. Thus B_2 can be chosen arbitrary and

$$A = (-2, 0, 0) \quad B_0 = (0, 0, 0) \quad B_1 = (-1, 1, 0).$$

- To obtain f_3 as toric morphism we have to have that

$$S'_{B_0-A} = k \cdot y^{(0,0,1)} \qquad S'_{B_2-A} = k \cdot y^{(1,0,0)}$$

  This time B_1 can be chosen arbitrary and

$$A = (0, 0, -1) \quad B_0 = (0, 0, 0) \quad B_1 = (1, 0, -1).$$

- The last case is analogous to the third, i.e.

$$S'_{B_0-A} = k \cdot y^{(0,0,1)} \qquad S'_{B_2-A} = k \cdot y^{(0,1,0)}$$

  Hence B_0 is arbitrary and

$$A = (0, 0, -1) \quad B_1 = (0, 0, 0) \quad B_2 = (0, 1, -1).$$

So we see that every morphism calculated above inherits its own specific T-equivariant structure and by denoting the 2×2 matrix obtained by deleting the first column of M_2 by M_2' we can deduce several things from this calculation:

- The sum of two elements f_i and f_j for $i \neq j$ is not toric, thus we can think of the f_i as monomials, as in the case of $\mathbb{P}^2$.
- Assume that both stars in M_2 are nonzero. Then we see from the cases above that this is only possible if the lower star is a morphisms consisting of f_3 or f_4, i.e. of the form $(\lambda y_2, 0, -\lambda y_0)$ or $(0, \lambda y_2, -\lambda y_0)$, since otherwise there would be a contradiction to the choice of the toric divisors. Moreover in this case we only have two more free parameters for the choice of the T-equivariant divisors for $2 \cdot \mathcal{O}_{\mathbb{P}(Q)}(-1)$ and so we see that there at most two other nonzero entries in M_2'.
- If one star is zero, then we are left with three free parameters for the choices of the divisors of all other morphisms, which essentially means that we have at most three nonzero choices for the remaining entries of M_2'.

Now we can simply list all possible matrices M_2'

$$\begin{pmatrix} * & y_0 & y_1 \\ * & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} * & y_0 & 0 \\ * & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} * & y_1 & 0 \\ * & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} * & y_0 & 0 \\ * & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} * & y_1 & 0 \\ * & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} * & y_0 & 0 \\ * & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} * & y_1 & 0 \\ * & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} * & 0 & 0 \\ * & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} * & y_0 & y_1 \\ * & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} * & y_1 & y_0 \\ * & 1 & 0 \end{pmatrix}$$

and show which of them are compatible with the possible entries of the stars:

(a) If both stars are nonzero, by what we said above we can only use the first nine matrices for M_2'. Thus we can construct $2 \cdot 1 \cdot 9 = 18$ matrices.

(b) If both are zero, then M_2' has to have two independent rows, which is only true for six of the above matrices.

(c) If only the upper star is zero, then all matrices M_2' with at least one nonzero entry in the lower row are "allowed", i.e. again nine of them. Therefore we get 18 new sheaves from this calculation.

(d) If the lower star is zero we have four choices for the lower star by the above considerations and at least one entry of the lower row of M_2' has to be zero. Thus we can add $4 \cdot 7 = 28$ sheaves to our list, which makes altogether 70 of them.

3. The third sequence

$$0 \longrightarrow 2\mathcal{O}_{\mathbb{P}(Q)}(-1) \xrightarrow{M_3} 2\overline{\Omega}^1_{\mathbb{P}(Q)}(1) \longrightarrow \mathcal{F} \longrightarrow 0$$

is again easier to describe. We calculated that the space

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}(Q)}(-1), \overline{\Omega}^1_{\mathbb{P}(Q)}(1))$$

is isomorphic to k and spanned by the toric morphism h. So the only feasible choice for M_3 is the following

$$M_3 = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}.$$

Conclusions

The last example already shows how difficult it is to obtain computable Beilinson monads on weighted projective spaces for certain subclasses of (toric) sheaves and that we are far from being able to construct moduli spaces as in the case of $\mathbb{P}^n$. For this we would need much more information about these spaces, for example we would need a reasonable notion of stability. Of course there are such notions even in a more general context, but the definition of the Hilbert polynomial depends on the choice of an ample line bundle, which is not canonically given on $\mathbb{P}(Q)$. Moreover the theory lacks of such powerful tools such as Serre duality and the Hirzebruch-Riemann-Roch theorem, which basically enabled us to simplify the Beilinson sequences in special cases on $\mathbb{P}^2$. However there are a lot of things that could be done.

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Hiermit erkläre ich, dass ich die vorliegende Diplomarbeit selbständig und nur unter Verwendung der angegebenen Literatur bearbeitet habe.

Kaiserslautern, 19. Juli 2007