Updates on the Ubiquity Conjecture

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The ubiquity question

Potential infinity vs. actual infinity

The Ubiquity Question:

- Fix your favourite connected graph G.
- Suppose have a *host graph* Γ which contains arbitrarily many disjoint copies of G
- Can you find infinitely many disjoint copies of G in Γ?

If yes for all possible host graphs Γ, we say G is ubiquitous.

Small detour: What do we mean by 'copies of G in Γ'?

Embeddings as subgraph, topological minor and minor

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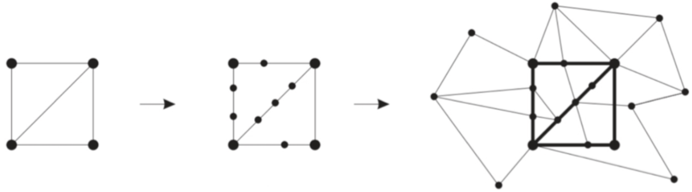

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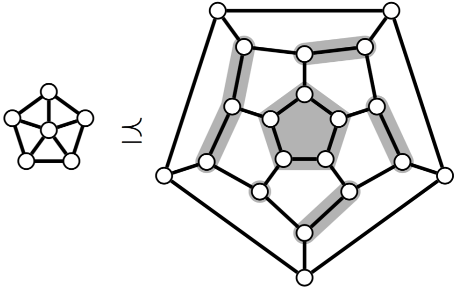
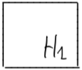


Finite graphs are ubiquitous w.r.t. all relations

We simply pick copies greedily.



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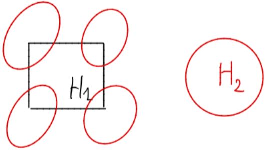

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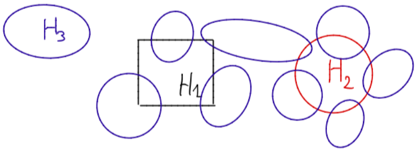

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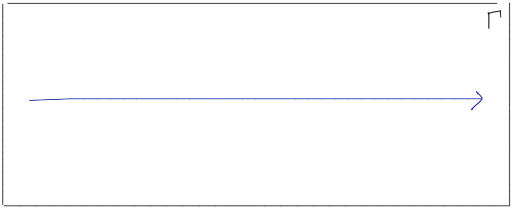
3. Know that Γ contains $|H_1| + |H_2| + 1$ disjoint copies of G....
4. Continue...

Halin (1965): The ray is subgraph-ubiquitous.

Non-trivial, as we can no longer pick copies greedily.



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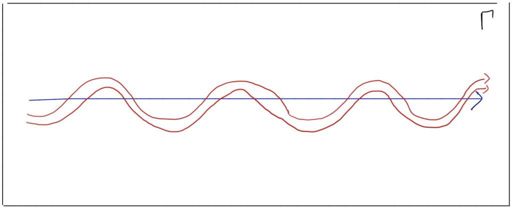

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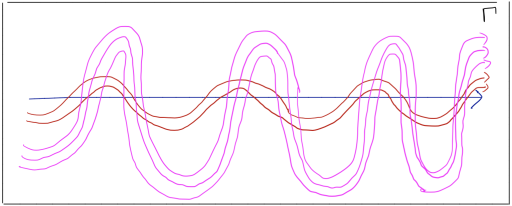

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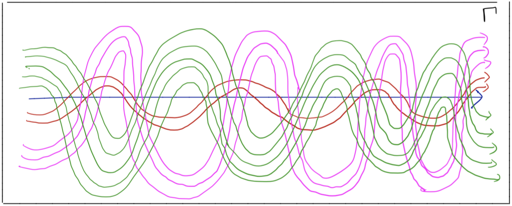

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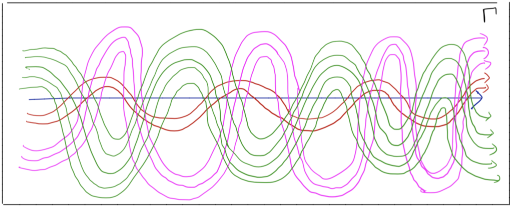

- Any ray from a given *layer* might intersect all rays from all other *layers*.

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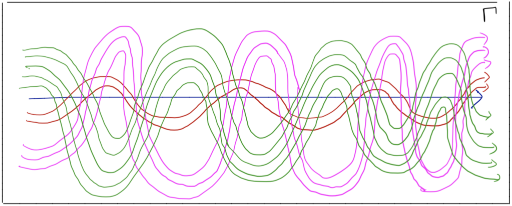

- Any ray from a given *layer* might intersect all rays from all other *layers*.
- Halin's idea:
  If rays don't intersect $\longrightarrow$ pick greedily.
  If rays do intersect $\longrightarrow$ re-route onto the next layer.

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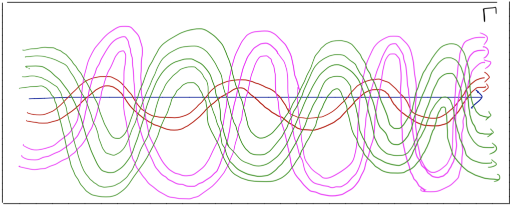

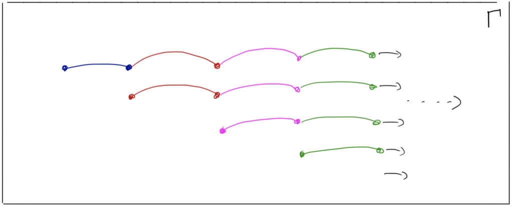

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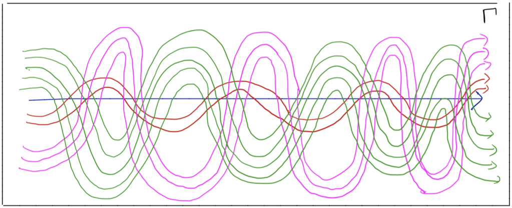
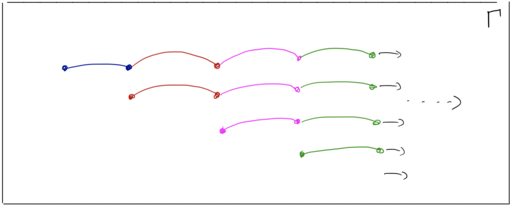




- Our infinitely many rays use finite subpaths from the layers, but otherwise have little in common with our original rays!

Bad news for subgraph and topological minor relation

Counterexamples due to Andreae, Lake and Woodall.

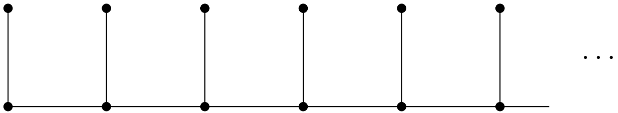
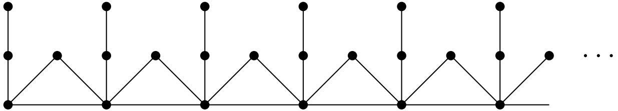


Figure: A graph which is not $\subseteq$-ubiquitous.



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Overview of known ubiquity results

$\subseteq$-Ubiquity: ✓ Finite graphs ✓ Ray / Double ray (Halin, $'65/'70$)
✗ Infinite comb

$\leqslant$-Ubiquity: ✓ Finite graphs ✓ Trees with $\Delta \leq 3$ (Halin, $'75$)
✓ Locally finite trees (Andreae, $'79$)
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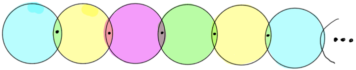

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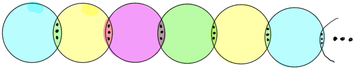

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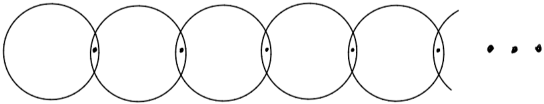
Plan: Show ubiquity ideas in a simple class of examples



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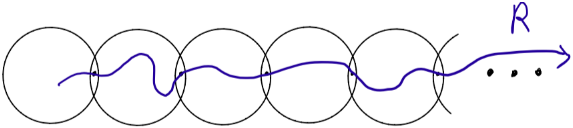

- Let's take an infinite graph G which is glued together from a sequence of finite connected graphs $(G_n)_{n \in \mathbb{N}}$ along 1-separators.
- We may also fix a representative ray $R \subset G$ for later use. Note that R passes through each 1-separator precisely once.

Concentrated families

A simple yet crucial new idea:

- Your task is to hide copies of G in Γ such that
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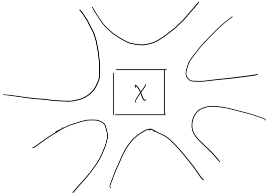

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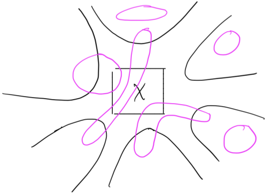

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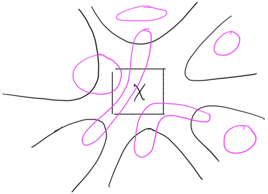

- For every finite vertex set $X \subseteq V(\Gamma)$, at most $|X|$ graphs from each layer can meet X.
- Still $n \cdot G \lhd \Gamma - X$ for all $n \in \mathbb{N}$.
- If $\exists_\infty$ components C of $\Gamma - X$ with $G \lhd C$ then gameover.
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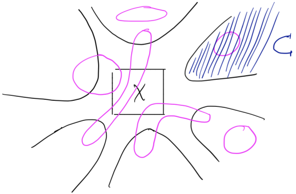

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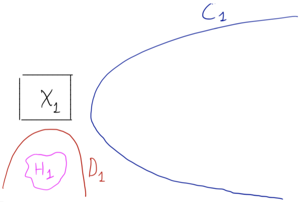

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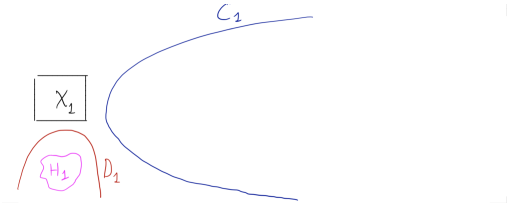

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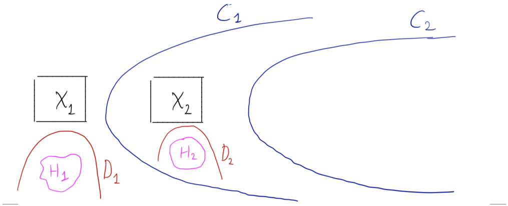

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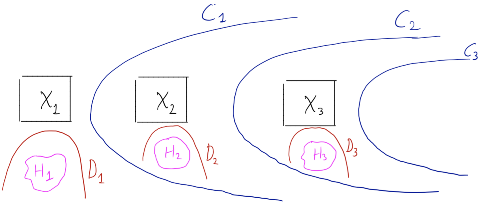

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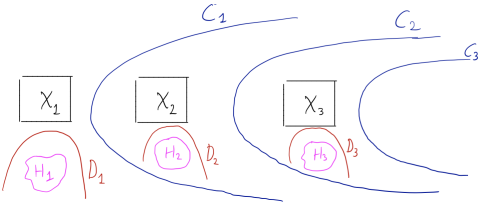

- If this process doesn't stop, then $\{H_n : n \in \mathbb{N}\} \longrightarrow$ gameover.

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Lesson: Place G-copies in Γ s.t. $\forall X \subseteq V(\Gamma)$ finite, $\exists!$ component C_X of $\Gamma - X$ such that 'almost all' copies of G are contained in C_X.

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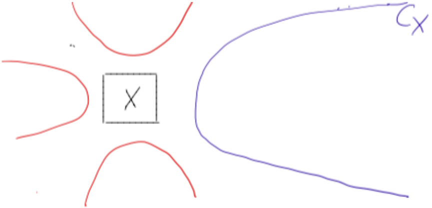

Observation: The family $(C_X)_X$ satisfies

$$X \subseteq X' \to C_X \supseteq C_{X'}.$$

Such a choice of components $(C_X)_X$ is called a *direction in* Γ.

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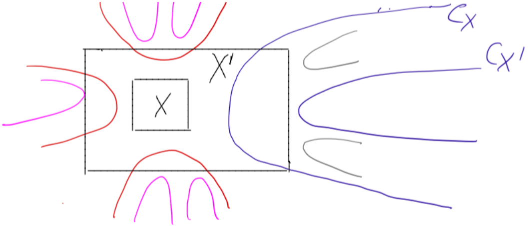

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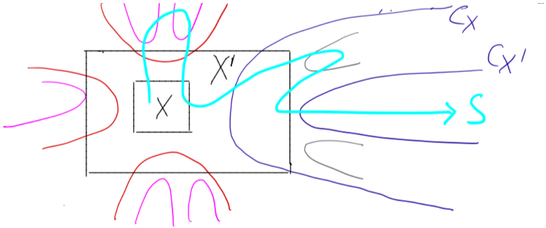

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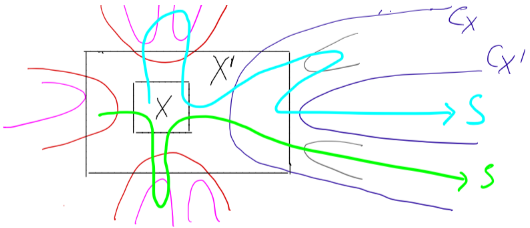

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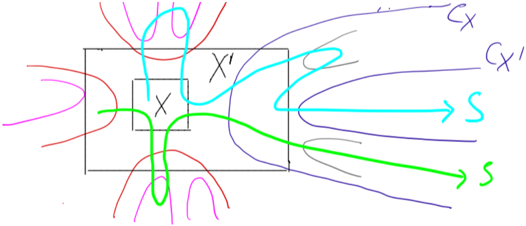
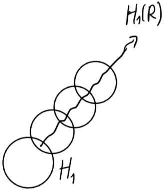
A simple yet crucial new idea:

Lesson: Place G-copies in Γ s.t. $\forall X \subseteq V(\Gamma)$ finite, $\exists!$ component C_X of $\Gamma - X$ such that 'almost all' copies of G are contained in C_X.



- A ray $S \subset \Gamma$ *agrees with* $(C_X)_X$ if S has a tail in every C_X.
- Fix a ray R in our graph G.
- For every G-copy H in Γ, the lifted ray $H(R)$ either agrees with $(C_X)_X$ or not.
- Pigeon hole: May assume that $H(R)$ either agrees with $(C_X)_X$ always or never, uniformly for all G-copies H in Γ.

In the never-agree case, can again pick copies greedily



1. Pick first copy $H_1 \subset \Gamma$ of G.

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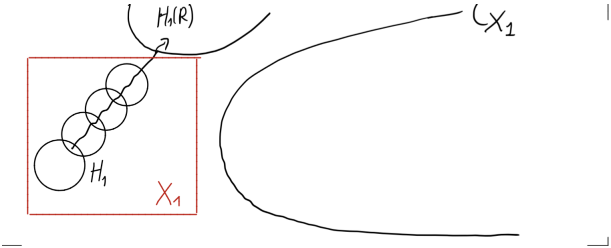

1. Pick first copy $H_1 \subset \Gamma$ of G.
2. Find X_1 where $H_1(R)$ disagrees with C_{X_1}.

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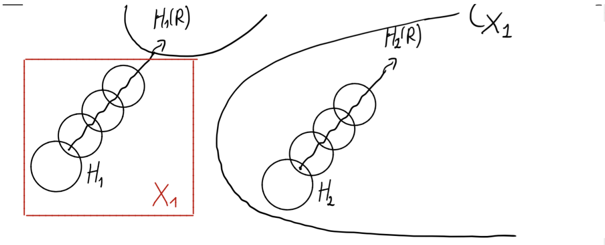

1. Pick first copy $H_1 \subset \Gamma$ of G.
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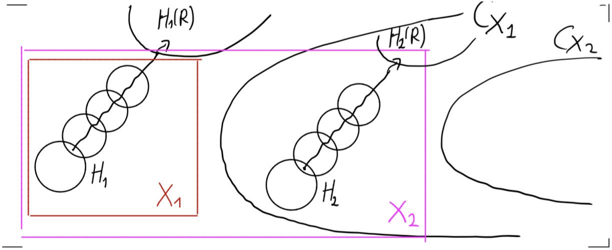

1. Pick first copy $H_1 \subset \Gamma$ of G.
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3. Pick second copy $H_2 \subset C_{X_1}$ of G.
4. Find $X_2 \supset X_1$ where $H_2(R)$ disagrees with C_{X_2}.

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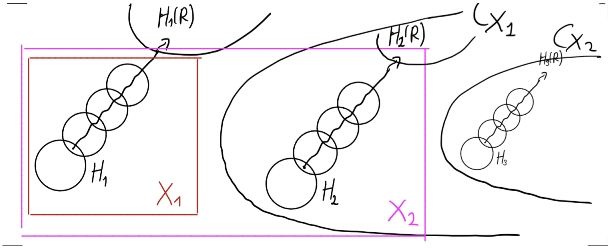

1. Pick first copy $H_1 \subset \Gamma$ of G.
2. Find X_1 where $H_1(R)$ disagrees with C_{X_1}.
3. Pick second copy $H_2 \subset C_{X_1}$ of G.
4. Find $X_2 \supset X_1$ where $H_2(R)$ disagrees with C_{X_2}.
5. Pick third copy $H_3 \subset C_{X_2}$ of G.

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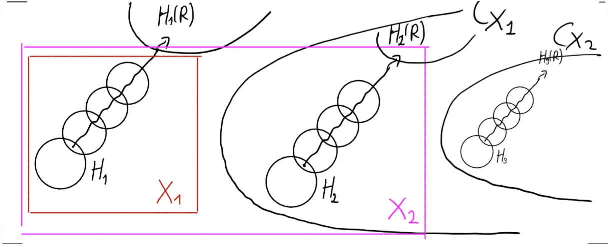

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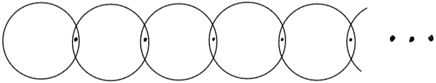
4. Find $X_2 \supset X_1$ where $H_2(R)$ disagrees with C_{X_2}.
5. Pick third copy $H_3 \subset C_{X_2}$ of G.
6. Continue....

In the always-agree case, use well-quasi-ordering theory

Using the Robertson-Seymour result on wqo of finite graphs

- Colour the left cut-vertex of each G_n with 1 and the right cut-vertex with 2.



- Labelled wqo of finite graphs (Robertson-Seymour): $\exists N \in \mathbb{N}$ s.t. every G_n for $n > N$ embeds into infinitely many G_i.

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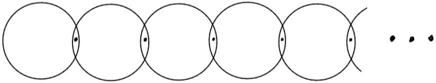

- Labelled wqo of finite graphs (Robertson-Seymour): $\exists N \in \mathbb{N}$ s.t. every G_n for $n > N$ embeds into infinitely many G_i.
- May assume $N = 1$, i.e. can find every blob but the first again and again.

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The construction – a picture proof for a one-ended example

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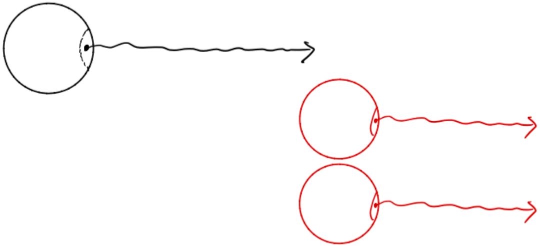

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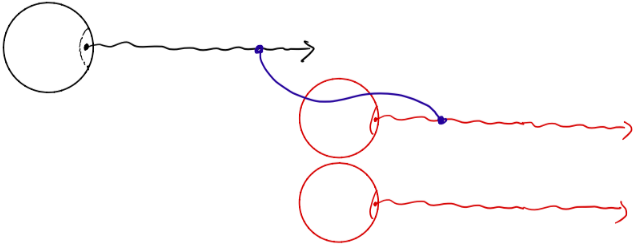

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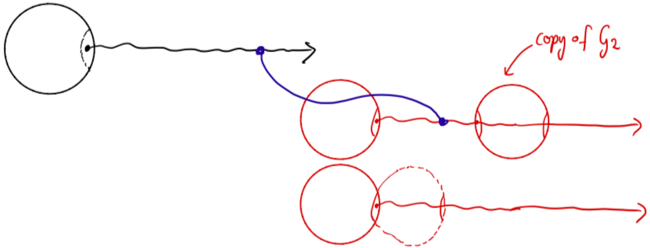

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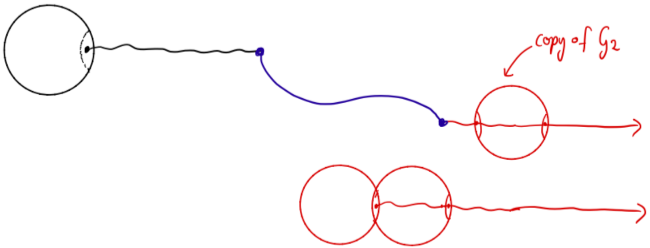

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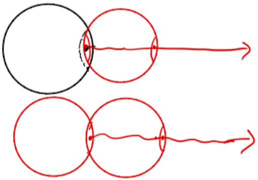

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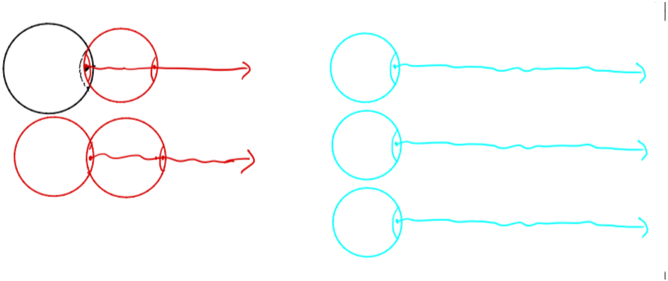

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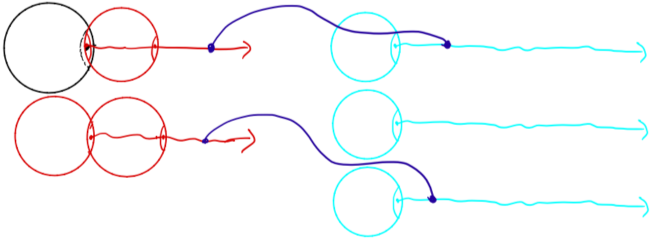

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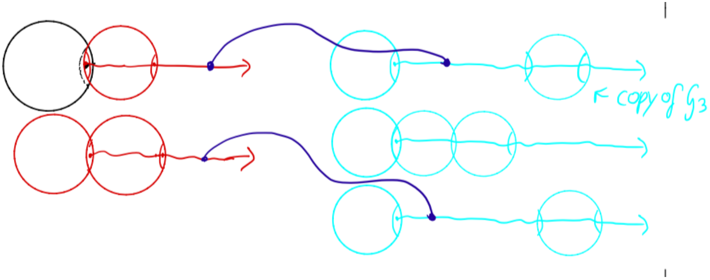

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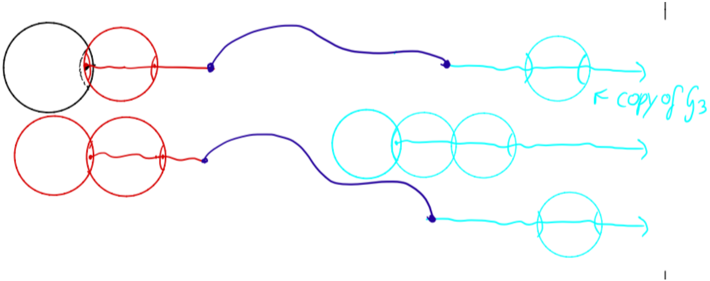

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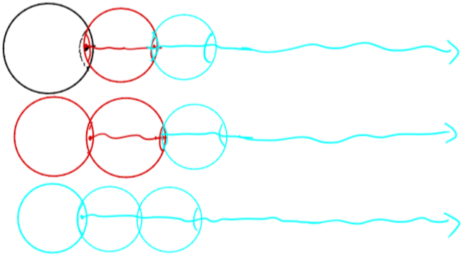

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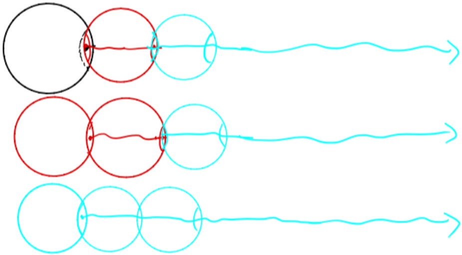

Lesson 1: As with Halin's rays, our G-copies use finite blobs from the layers, but otherwise have little in common with original copies!

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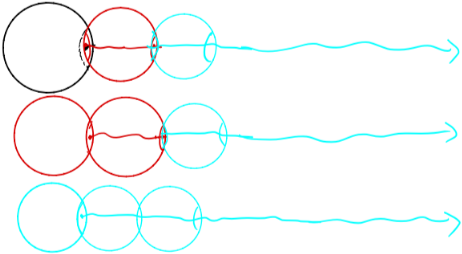

Lesson 1: As with Halin's rays, our G-copies use finite blobs from the layers, but otherwise have little in common with original copies!

Lesson 2: If you place your G-copies all over the host graph Γ, then easy for me to win. And if you place them so that they are concentrated, you will inadvertently create lots of new G-copies due to wqo which I may exploit.

For the details see....

Bowler, Elbracht, Erde, Gollin, Heuer, Pitz, Teegen:

- Ubiquity in graphs I: Topological ubiquity of trees, submitted.
- Ubiquity in graphs II: Ubiquity of graphs with non-linear end structure, submitted.
- Ubiquity in graphs III: Ubiquity of a class of locally finite graphs, preprint available soon.
- Ubiquity in graphs IV: Ubiquity of graphs of bounded tree-width, at some point.