Infinite Eulerian graphs and strongly irreducible images of intervals

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Joint work with Paul Gartside

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Overview

1. What is an Eulerian space?
   Edge-wise Eulerian tours in infinite graphs
   ... vs ...
   irreducible images of I and S^1
2. The Eulerianity conjecture
3. Affirmative results towards the Eulerianity conjecture
4. Proof impressions

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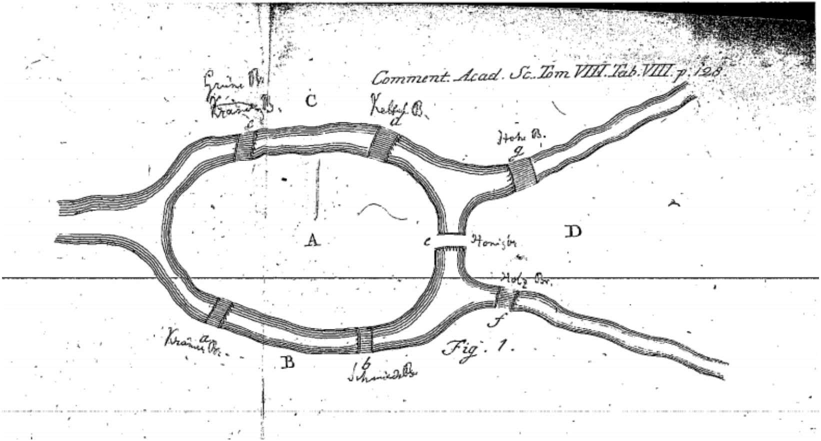
The Koenigsberg Bridge Problem



Schematic drawings of the seven bridges of Koenigsberg, in:

Leonhard Euler (1736): "Solutio problematis ad geometriam situs pertinentis" (Solution of a problem about the geometry of position)

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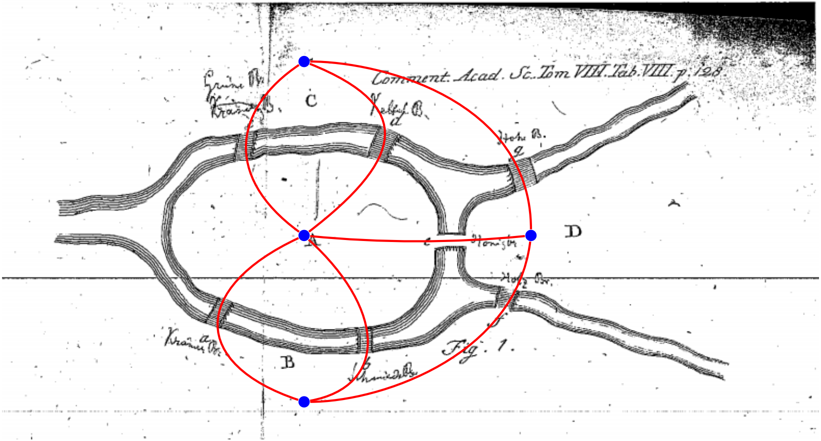

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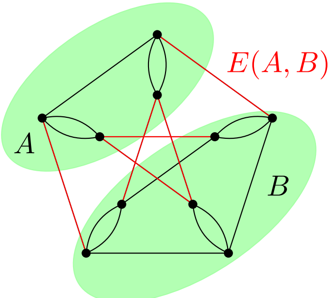
Finite Eulerian graphs

Characterisations in terms of vertex degrees, decomposition results and edge-cuts

Theorem: For a finite connected multi-graph G, tfae:

1. G is Eulerian,
2. all vertices of G have even degree, (Euler)
3. G can be decomposed into edge-disjoint cycles, (Veblen)
4. all edge-cuts of G are even. (Nash-Williams)

An edge-cut of G is a set $E(A, B) \subseteq E(G)$ of edges crossing a bipartition (A, B) of $V(G)$.



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Question: What about infinite (multi-)graphs?

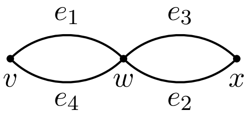
- Erdős, Grünwald, Vàzsonyi (1938)
- Nash-Williams (1960, 1962)
- Sabidussi (1964)
- Rothschild (1965)
- Laviolette (1997)
- Diestel & Kühn (2004)

The topological viewpoint

Combinatorial vs. topological Eulerian tours

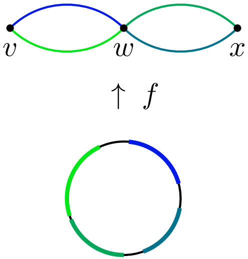
Finite multi-graph G turns naturally into a topological space $|G|$.

A combinatorial Euler tour is a closed walk containing every edge of G precisely once.



$$W = ve_1we_2xe_3we_4v$$

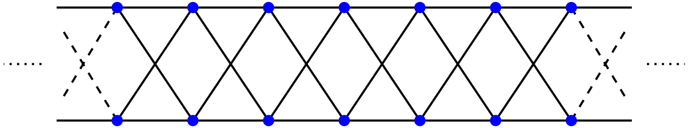
An edge-wise Eulerian map is a continuous surjection $f\colon S^1 \twoheadrightarrow |G|$ which is injective for interior points on edges.



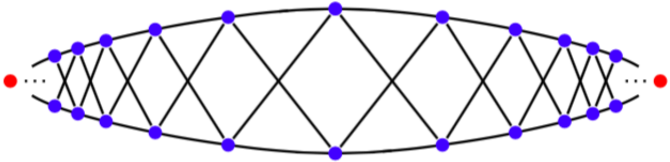
Solution: Add the Ends, and Compactify

The Freudenthal compactification FG

R. Diestel, *Locally finite graphs with ends: a topological approach I-III*, Discrete Math (2010–11).



...turns into...



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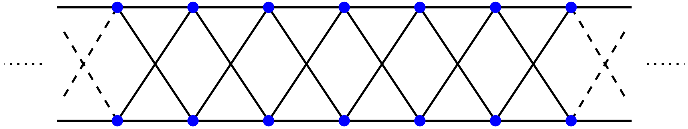
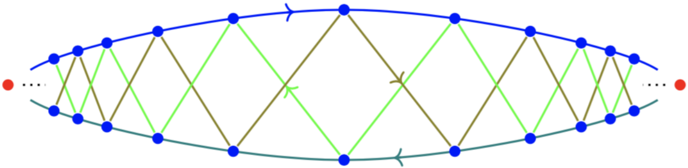


...with edge-wise Eulerian map $f \colon S^1 \twoheadrightarrow FG$



Infinite Eulerian graphs

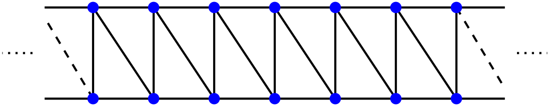
With this definition, the finite characterisation extends best possible

Definition (Diestel & Kühn): G Eulerian $\Leftrightarrow \exists$ edge-wise Eulerian surjection $f\colon S^1 \twoheadrightarrow FG$

Theorem: (DK '04) For a locally finite connected multi-graph G, tfae:

1. G is Eulerian,
2. G can be decomposed into edge-disjoint (finite) cycles
3. all (finite) edge-cuts of G are even.

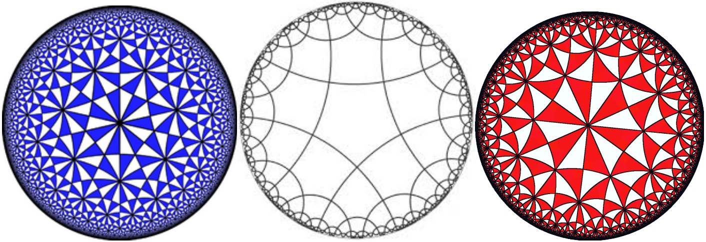
Euler's original even-degree condition is no longer sufficient:



Eulerian problem for infinite topological graphs

What about other naturally occurring compactifications of locally finite graphs?

Do these 'graphs' admit edge-wise Eulerian surjections?



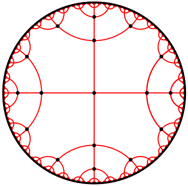
[Credit: https://en.wikipedia.org/wiki/Wythoff_symbol]

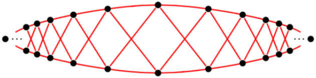
Edge-wise Eulerian maps in topological spaces

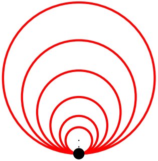
A general definition of edges in topological spaces

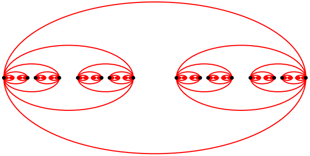
Let X be a metrizable space.

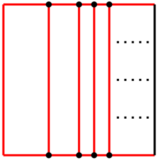
- An edge (i.e. free arc) of X is an inclusion-maximal open subset of X homeomorphic to $(0, 1)$. Let $E(X)$ be the edge set of X. The ground space of X is $\mathcal{G}(X) = X - E(X)$
- X is edge-wise Eulerian if $\exists$ edge-wise Eulerian $f\colon S^1 \twoheadrightarrow X$.

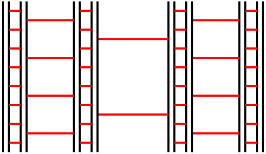












Strongly irreducible maps and Eulerian continua

Hahn-Mazurkiewicz: A space X is the continuous image of I or S^1 if and only if X is a Peano continuum.

Question: What about 'nice' space-filling curves?

- Hilbert (1891)
- Nöbling (1933)
- Harrold (1940, 1942)
- Ward (1977)
- Treybig & Ward (1981)
- Bula, Nikiel & Tymchatyn (1994)

Definition: A continuous surjection $f\colon S^1 \twoheadrightarrow X$ is strongly irreducible if for all closed proper subsets $A \subsetneq S^1$, we have $f(A) \subsetneq X$.

Problem (Treybig & Ward '81): Characterize the strongly irreducible images of S^1.

Exercise: Every strongly irreducible $f\colon S^1 \twoheadrightarrow X$ is edge-wise Eulerian.

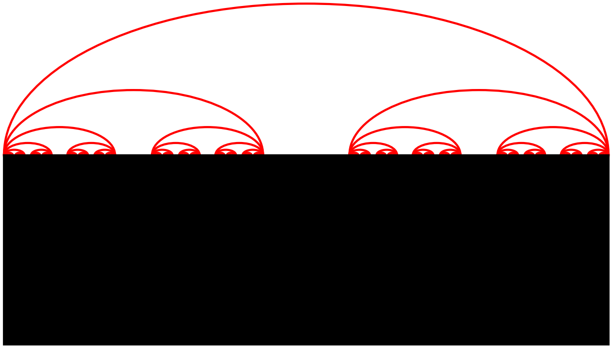
Definition: A space X is Eulerian if there exists a strongly irreducible surjection $f\colon S^1 \twoheadrightarrow X$. Call any such map an Eulerian map.

Understanding Eulerian maps

What makes a map strongly irreducible?

Observation (Harrold): If $E(X) = \emptyset$ and $f\colon S^1 \twoheadrightarrow X$ is Eulerian, then $f \restriction J$ *never* traces out an arc for any time interval $J \subset S^1$.

Proof: Otherwise, $f(S^1 \setminus \mathrm{int}(J))$ contains a dense subset of X, so – since compact – must be onto, contradicting strongly irreducible.



Jekyll-Hyde behaviour

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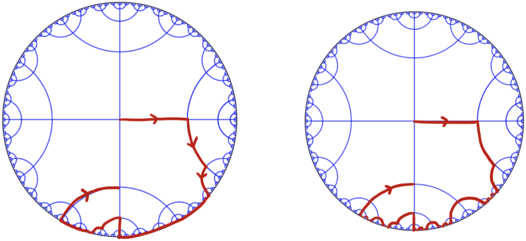
Lemma (GP 19^+): If X has dense edges, then $f\colon S^1 \twoheadrightarrow X$ is Eulerian iff f is edge-wise Eulerian & $f^{-1}(\mathcal{G}(X))$ is zero-dimensional.



Admissible trace of an edge-wise Eulerian map on the left, and an Eulerian map on the right.

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What is known about Eulerian continua?

Problem (Treybig & Ward, '81): Characterize the Eulerian continua!

Answer known in the following special cases:

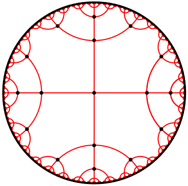
- ✓ Peano continua without free arcs (Harrold '42) $\rightarrow$ *always Eulerian*
- ✓ Finite graphs (Euler) & Freudenthal compactification of locally finite graphs (Diestel, Kühn '04)
- ✓ Continua with zero-dimensional ground space (called completely regular continua or graph-like continua) (Bula, Nikiel, Tymchatyn '94) / (Espinoza, Gartside, Pitz '16)
- ✓ X with dense edges, $\mathcal{G}(X)$ Peano continuum (BNT '94) $\rightarrow$ *always Eulerian*

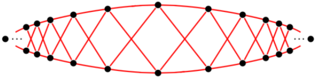
But: So far, no structural condition describing the Eulerian continua was even conjectured.

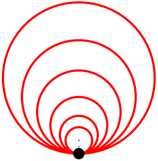
The Eulerianity Conjecture

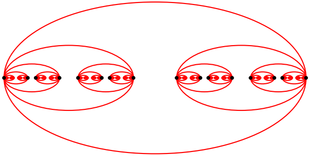
Edges and edge-cuts in Peano continua $X \neq S^1$

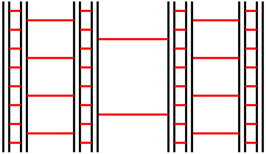
- Let $E(X)$ be the set of edges of X. The ground space of X is $\mathcal{G}(X) = X - E(X)$.
- $E(X)$ is a zero-sequence of disjoint open sets.
- Every edge has at most two boundary points.











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- The ground space of X is $\mathcal{G}(X) = X - E(X)$.
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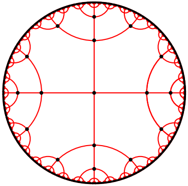
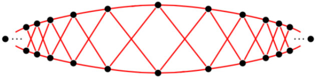
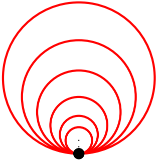
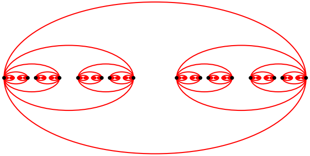
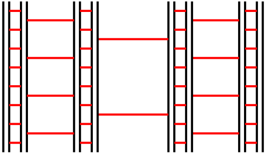
- Edge-cuts in Peano continua are finite.

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Observation: Edge-cuts of edge-wise Eulerian spaces are even.

Eulerianity Conjecture (Gartside & Pitz): A Peano continuum is Eulerian if and only if all its edge-cuts are even.

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Results and Evidence towards the Eulerianity Conjecture

Eulerianity Conjecture (Gartside & Pitz): A Peano continuum is Eulerian if and only if all its edge-cuts are even.

Theorem 1 (GP 19^+):
A space is Eulerian if and only if it is edge-wise Eulerian.

Theorem 2 (GP 19^+): The Eulerianity Conjecture holds for every Peano continuum whose ground space

1. consists of finitely many Peano continua, or
2. is homeomorphic to a product $V \times P$, where V is zero-dimensional and P a Peano continuum, or
3. is at most one-dimensional.

3 *says the conjecture holds for all one-dimensional Peano continua.*
2 *(kind of) answers Problem 3 of Bula, Nikiel, Tymchatyn ('94).*

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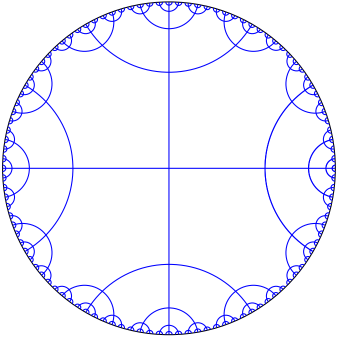
Framework: Approximating by finite Eulerian graphs

Step-by-step approximations towards a cyclic order for the hyperbolic 4-regular tree X



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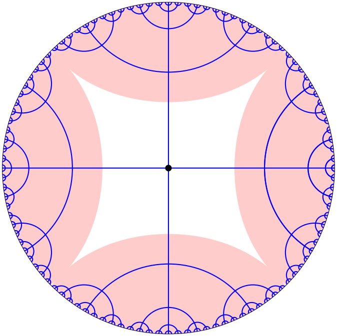
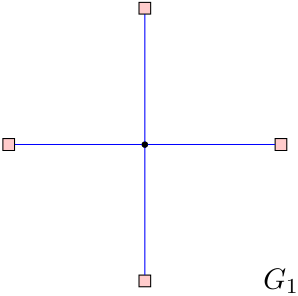
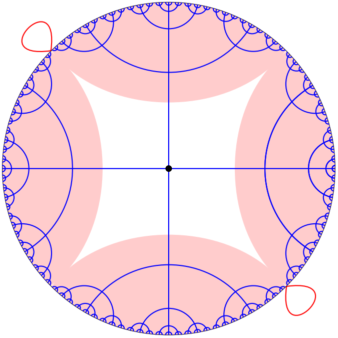


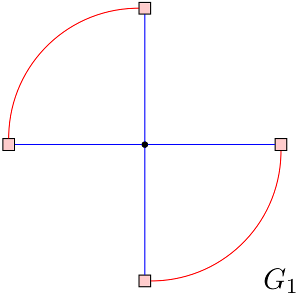


1. Partition into almost Eulerian tiles. (This step uses Bing's and Andersen's Brick Partition Theorem and the combinatorial theory for Freudenthal compactifications by Diestel et al...).
2. Let G_1 be graph on the tiles with edge set all uncovered edges.

Framework: Approximating by finite Eulerian graphs

Step-by-step approximations towards a cyclic order for the hyperbolic 4-regular tree X





3. Carefully add dummy edges to G_1 in order to make it Eulerian, drawing a dummy loop in X for each new dummy edge at the intersection of corresponding tiles.
4. Repeat!

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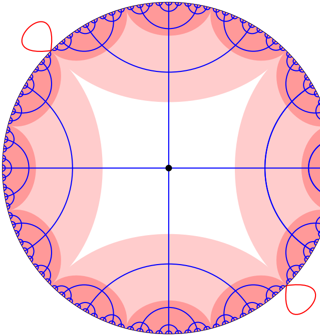
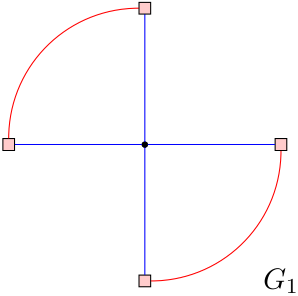
Step-by-step approximations towards a cyclic order for the hyperbolic 4-regular tree X





1' Partition each tile into (smaller) almost Eulerian tiles.

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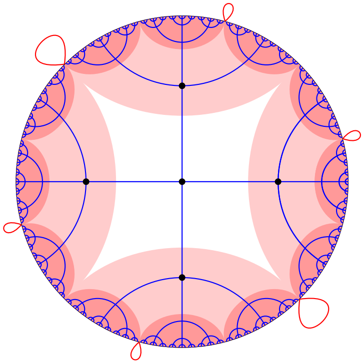
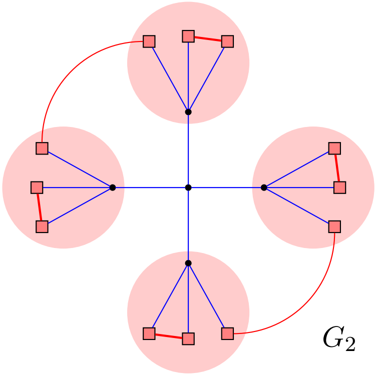
Step-by-step approximations towards a cyclic order for the hyperbolic 4-regular tree X





2' Obtain a "finer" graph G_2 on the new tiles.

3' Add dummy edges to G_2 in order to make it Eulerian–inside the old tiles!–and add dummy loop for each new dummy edge at the intersection of corresponding tiles.

4' Repeat!

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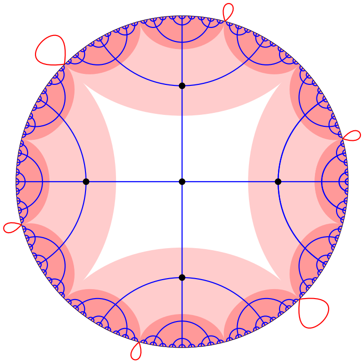
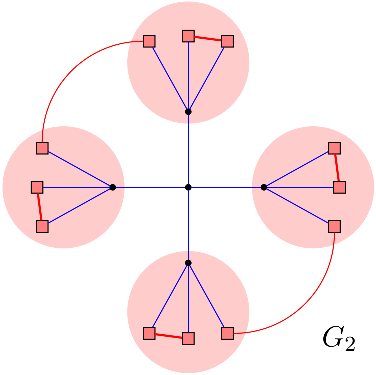
Step-by-step approximations towards a cyclic order for the hyperbolic 4-regular tree X





$\Rightarrow$ Obtain a sequence of finite Eulerian graphs $G_1, G_2, G_3, \ldots$ such that every G_i is an edge-quotient of G_{i+1}.
$\Rightarrow$ Then $\varprojlim G_i$ is Eulerian projecting 'nicely' onto X.
$\Rightarrow$ Every Eulerian map for $\varprojlim G_i$ projects to an edge-wise Eulerian map for X.

Outlook

Eulerianity Conjecture: A Peano continuum is Eulerian if and only if all its edge-cuts are even.

Open problems / next steps:

- Prove the conjecture for all hyperbolic graphs with boundary S^n for $n \geq 2$.
- Can one extend this to deal with n-dimensional spaces?
- Resolve the full conjecture!