Recent developments in reconstruction of infinite graphs

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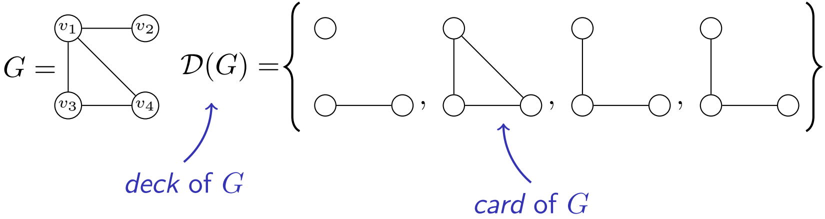
The reconstruction conjecture in graph theory

Examples of decks and cards



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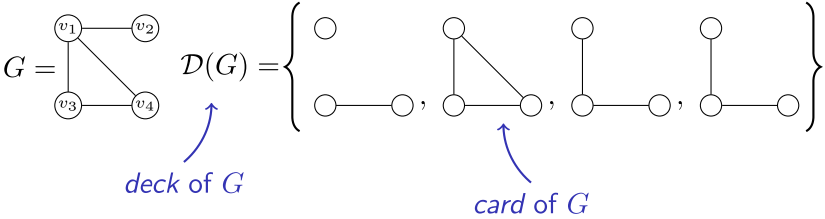

- A graph G is *reconstructible* if $\mathcal{D}(G) = \mathcal{D}(H)$ only if $G \cong H$.

The reconstruction conjecture in graph theory

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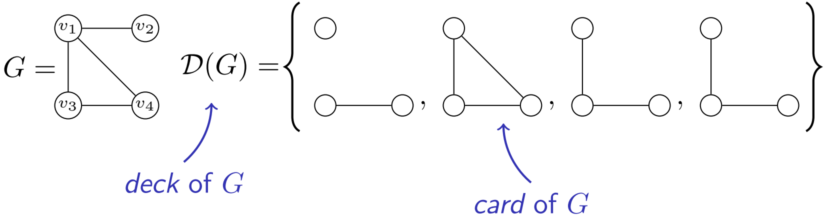

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The Reconstruction Conjecture (Ulam, Kelly, 1941):

Every finite graph with at least 3 vertices is reconstructible.

Why restricting the conjecture is necessary

Infinite graphs are in general not reconstructible

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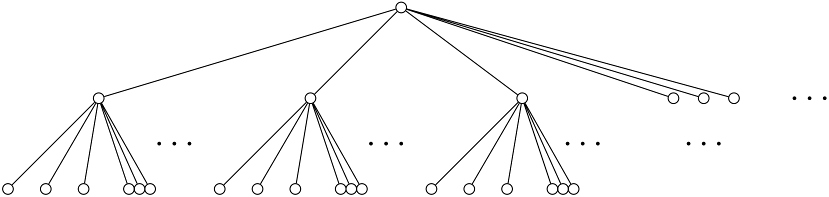
Every finite graph with at least 3 vertices is reconstructible.

Counterexample for infinite graphs: Countably branching tree T_∞.



We have $\mathcal{D}(T_\infty) = \{\infty \cdot T_\infty, \infty \cdot T_\infty, \ldots\} = \mathcal{D}(2 \cdot T_\infty)$.

Reconstruction results for infinite graphs

Due to non-reconstructible T_∞, should restrict to locally finite conn'd graphs.

The Harary-Schwenk-Scott Conjecture (1972):

Every locally finite tree is reconstructible.

Nash-Williams' Problem (1991):

Is every locally finite connected infinite graph reconstructible?

# ends	Locally finite trees	Locally finite graphs
1		
2		
$3, 4, \ldots$		
$\lvert\mathbb{N}\rvert$		
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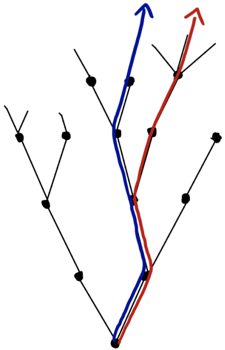
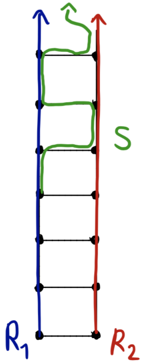
Ends of trees and graphs

A small detour



ends $\overset{1-1}{\longleftrightarrow}$ (infinite) rays starting at the root.



ends $\overset{1-1}{\longleftrightarrow}$ equivalence classes of rays: $R_1 \sim R_2 :\Leftrightarrow \exists S$ s.t. $|S \cap R_i| = \infty$.

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Amalgamating isomorphisms via promise structures

A versatile proof technique to control isomorphisms of locally finite connected graphs

A general set-up. Given:

- Disjoint graphs G, H, N,
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Want:

- $G' \supset G$ and $H' \supset N \dot{\cup} H$ s.t.
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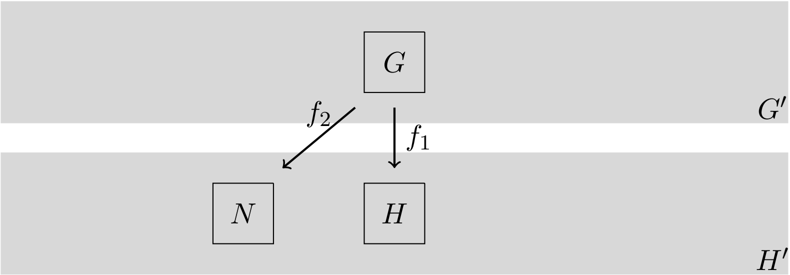

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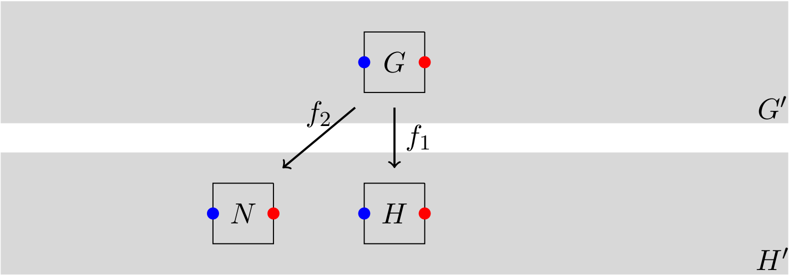

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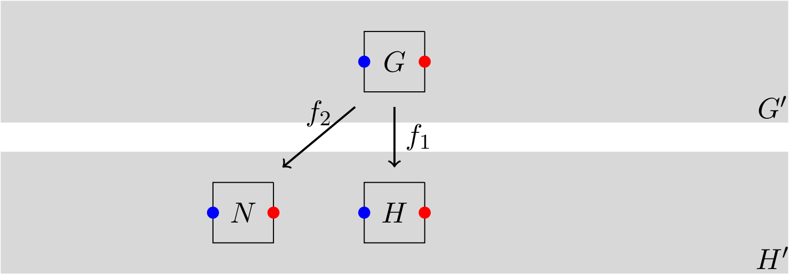

Promise: ① Viewed from G and H: attach same graph behind red and blue leaves respectively. Then old iso $f_1 \colon G \to H$ extends.

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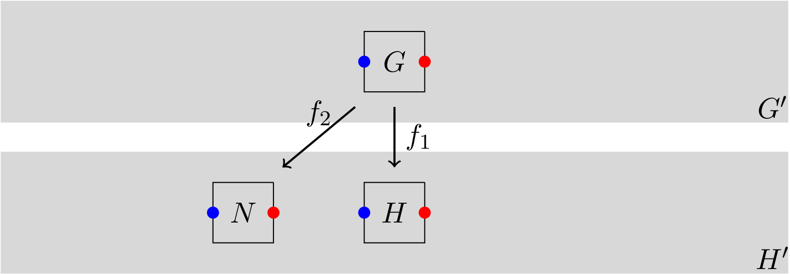

Promise: ② Viewed from G and N: attach same graph behind red and blue leaves respectively. Then new iso $f_2 \colon G \to N$ extends.

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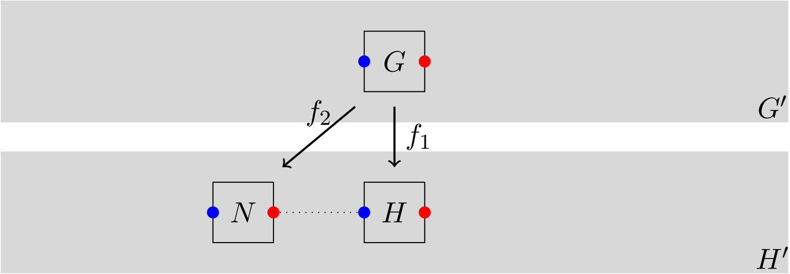

To make H' connected, add an edge between N and H.

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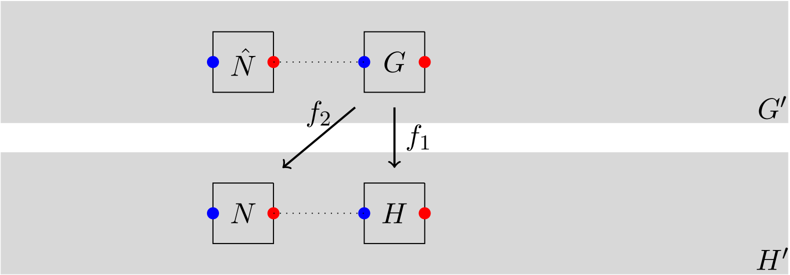

To make f_1 happy: Add copy $\hat{N}$ of N behind blue leaf of G.

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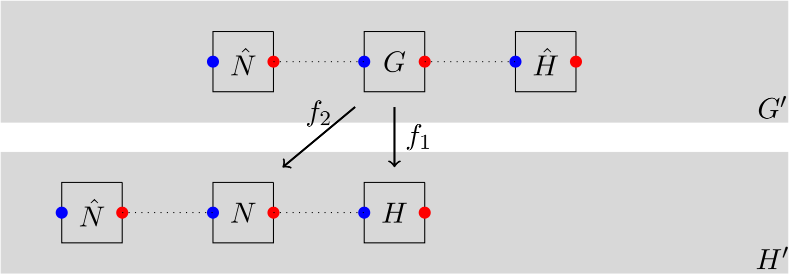

To make f_2 happy: Add copy $\hat{N}$ of N behind blue leaf of N, and copy $\hat{H}$ of H behind red leaf of G.

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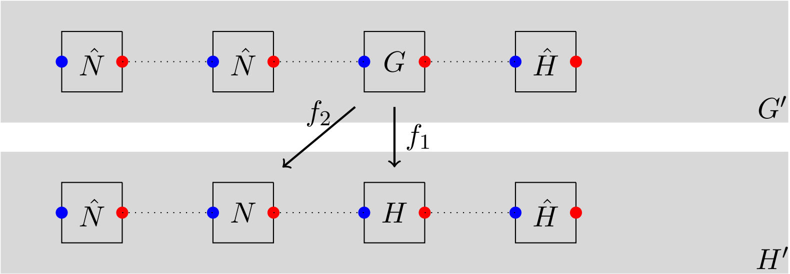

To make f_1 happy: Add copy $\hat{H}$ of H behind red leaf of H, and another copy $\hat{N}$ of N upstairs.

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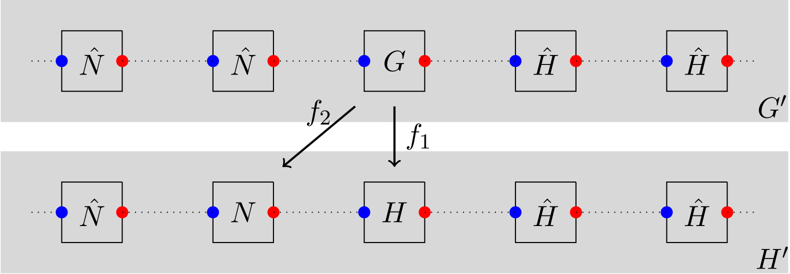

At the end of time, both f_1 and f_2 are simultaneously happy.

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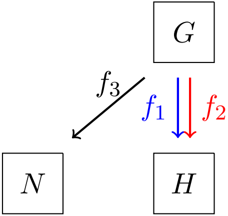

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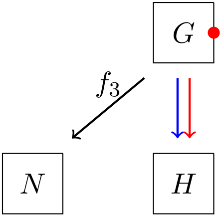

Promise leaves: ① Viewed from G and H: attach same graph behind red promise leaves respectively. Then iso's $f_1, f_2 \colon G \to H$ extend.

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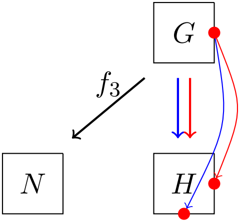

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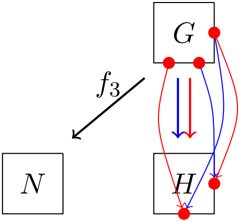

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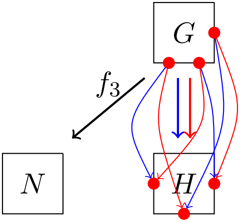

Promise leaves: Either orbit closes a loop after finitely many iterations...

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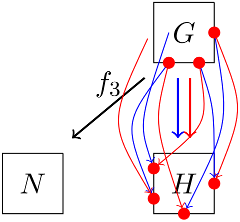

Promise leaves: ... or the orbit forms an infinite double ray.

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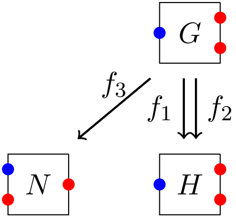

Promise: Suppose have two distinct orbits of promise leaves coloured blue and red.

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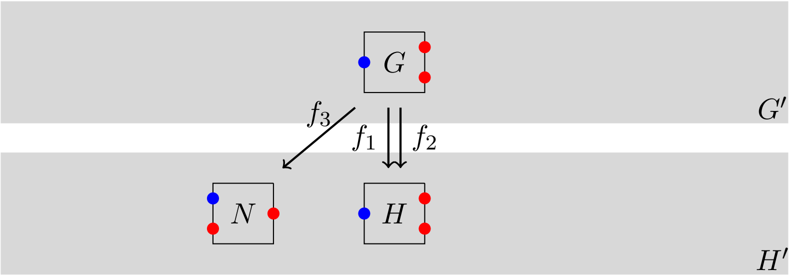

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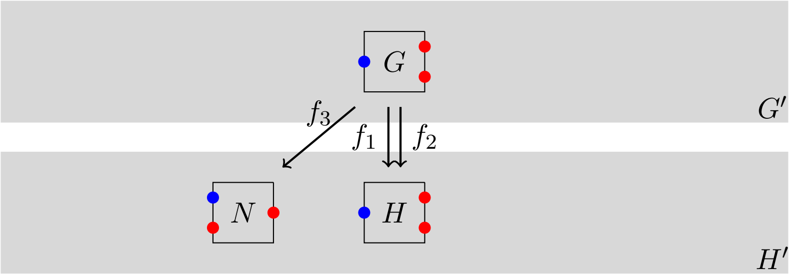


Promise: 2 Viewed from G and N: attach same graph behind red and blue leaves respectively. Then new iso $f_3 \colon G \to N$ extends.

Ⓐ Amalgamating isomorphisms via promise structures

A versatile proof technique to control isomorphisms of locally finite connected graphs

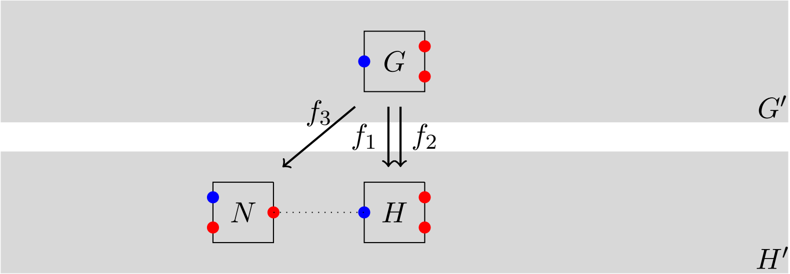
A general set-up. Given:

- Disjoint graphs G, H, N,
- isomorphisms $f_1, \dots, f_n \colon G \to H$,
- (new) isomorphism $f_{n+1} \colon G \to N$.

Want:

- $G' \supset G$ and $H' \supset N \dot\cup H$ s.t.
- all maps lift to isomorphisms $f_i' \colon G' \to H'$ for $i \leq n+1$.

Case $n = 2$.



To make H' connected, add an edge between N and H.

Amalgamating isomorphisms via promise structures

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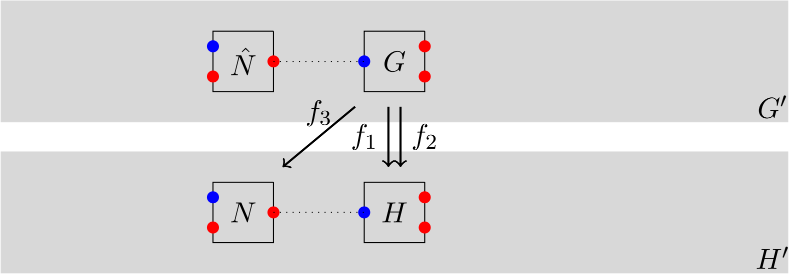
Case $n = 2$.



To make f_1, f_2 happy: Add copy $\hat{N}$ of N behind blue leaf of G.

Amalgamating isomorphisms via promise structures

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- Disjoint graphs G, H, N,
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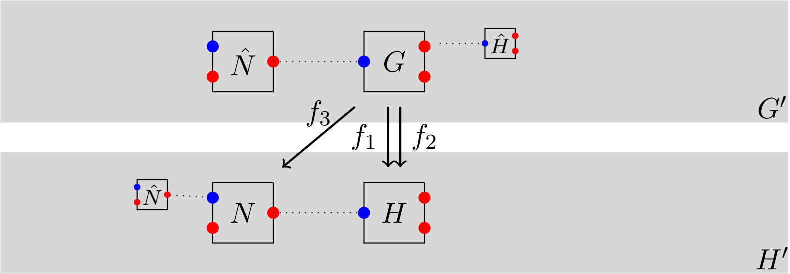
Want:

- $G' \supset G$ and $H' \supset N \dot\cup H$ s.t.
- all maps lift to isomorphisms $f_i' \colon G' \to H'$ for $i \leq n+1$.

Case $n = 2$.



To make f_3 happy: Add copy $\hat{N}$ of N behind blue leaf of N, and copy $\hat{H}$ of H behind correct red leaf of G.

A Amalgamating isomorphisms via promise structures

A versatile proof technique to control isomorphisms of locally finite connected graphs

A general set-up. Given:

- Disjoint graphs G, H, N,
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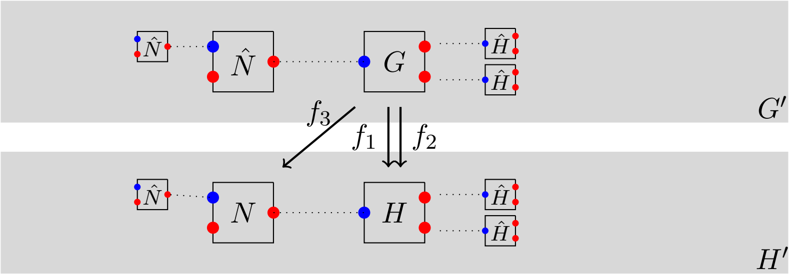
- (new) isomorphism $f_{n+1} \colon G \to N$.

Want:

- $G' \supset G$ and $H' \supset N \dot\cup H$ s.t.
- all maps lift to isomorphisms $f_i' \colon G' \to H'$ for $i \leq n+1$.

Case $n = 2$.



To make f_1, f_2 happy: Add copy $\hat{H}$ of H behind red leaf of H, and another copy $\hat{N}$ of N upstairs.

Amalgamating isomorphisms via promise structures

A versatile proof technique to control isomorphisms of locally finite connected graphs

A general set-up. Given:

- Disjoint graphs G, H, N,
- isomorphisms $f_1, \dots, f_n \colon G \to H$,
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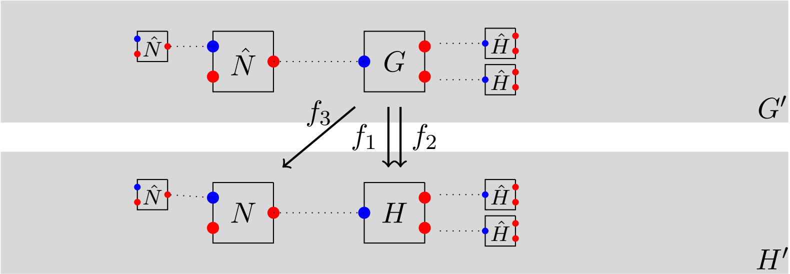
Case $n = 2$.



To make f_3 happy: ...

Amalgamating isomorphisms via promise structures

A versatile proof technique to control isomorphisms of locally finite connected graphs

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- Disjoint graphs G, H, N,
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Want:

- $G' \supset G$ and $H' \supset N \dot\cup H$ s.t.
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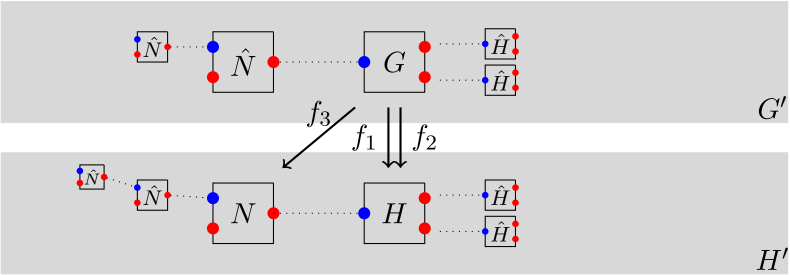
Case $n = 2$.



To make f_3 happy: ...

Amalgamating isomorphisms via promise structures

A versatile proof technique to control isomorphisms of locally finite connected graphs

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Want:

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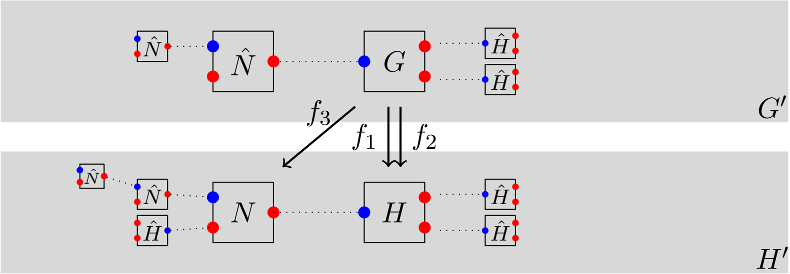
Case $n = 2$.



To make f_3 happy: ...

Amalgamating isomorphisms via promise structures

A versatile proof technique to control isomorphisms of locally finite connected graphs

A general set-up. Given:

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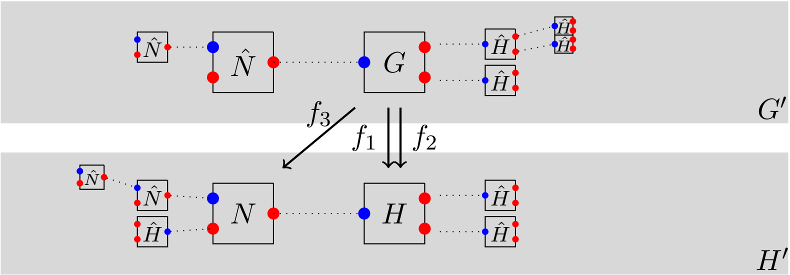
- (new) isomorphism $f_{n+1} \colon G \to N$.

Want:

- $G' \supset G$ and $H' \supset N \dot\cup H$ s.t.
- all maps lift to isomorphisms $f_i' \colon G' \to H'$ for $i \leq n+1$.

Case $n = 2$.



Now f_3 is happy. Continue, by adding in turn copies of H behind red promise leaves, and new copies of N behind blue promise leaves.

Amalgamating isomorphisms via promise structures

A versatile proof technique to control isomorphisms of locally finite connected graphs

A general set-up. Given:

- Disjoint graphs G, H, N,
- isomorphisms $f_1, \dots, f_n \colon G \to H$,
- (new) isomorphism $f_{n+1} \colon G \to N$.

Want:

- $G' \supset G$ and $H' \supset N \dot\cup H$ s.t.
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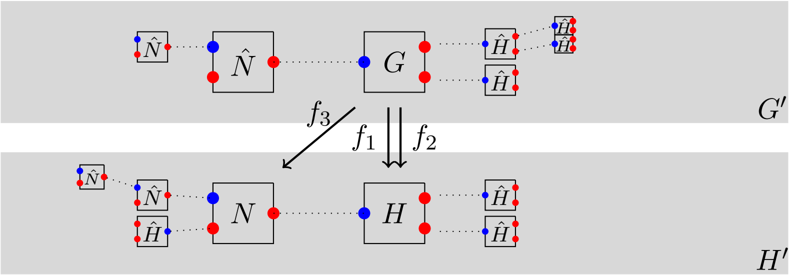
Case $n = 2$.



At the end of time, all of f_1, f_2 and f_3 are simultaneously happy.

A Amalgamating isomorphisms via promise structures

A versatile proof technique to control isomorphisms of locally finite connected graphs

A general set-up. Given:

- Disjoint graphs G, H, N,
- isomorphisms $f_1, \dots, f_n \colon G \to H$,
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Want:

- $G' \supset G$ and $H' \supset N \dot\cup H$ s.t.
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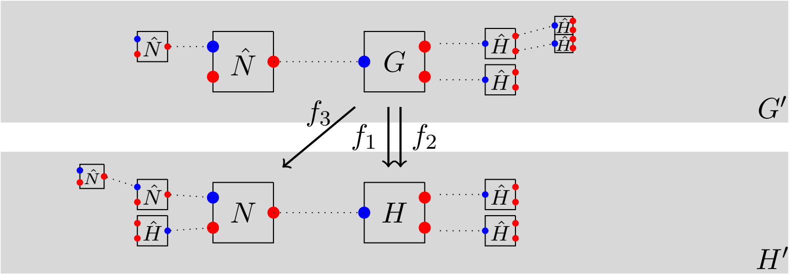
Case $n = 2$.



Note: Obtain global structure of k-regular tree (where $k \in \mathbb{N} \cup \infty$ the number of promise leaves) and hence uncountably many ends.

A Amalgamating isomorphisms via promise structures

A versatile proof technique to control isomorphisms of locally finite connected graphs

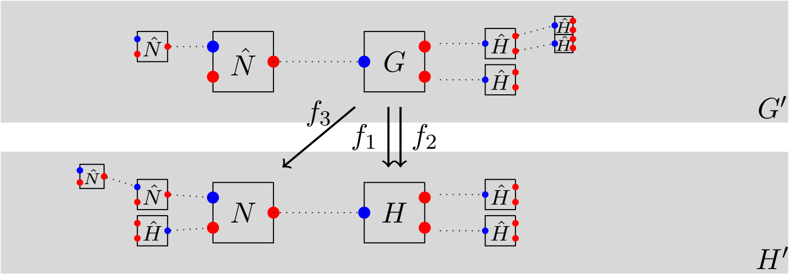
A general set-up. Given:

- Disjoint graphs G, H, N,
- isomorphisms $f_1, \dots, f_n \colon G \to H$,
- (new) isomorphism $f_{n+1} \colon G \to N$.

Want:

- $G' \supset G$ and $H' \supset N \dot\cup H$ s.t.
- all maps lift to isomorphisms $f_i' \colon G' \to H'$ for $i \leq n+1$.

Case $n = 2$.

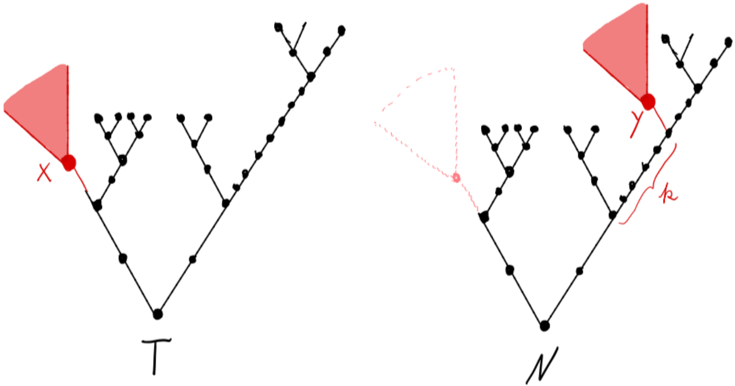


Note as well: If G, H, N were (locally finite) trees to start with, then so will be G' and H'.

Ⓑ Shifting single vertices

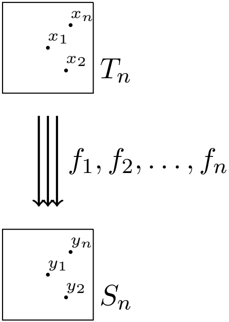
Constructing non-isomorphic trees sharing a common card

Shifting Lemma: Given a 'nice' tree T and $x \in T$, may construct tree $N \ncong T$ and $y \in N$ such that cards satisfy $T - x \cong N - y$.



A non-reconstructible tree of maximum degree 3.

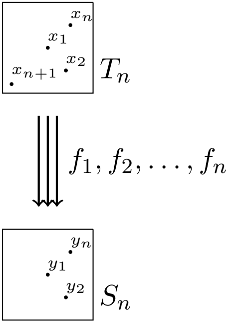
A back-and-forth construction using the amalgamation theorem.



At step n, have constructed trees $T_n \ncong S_n$ with n common cards, witnessed by isomorphisms $f_i \colon T_n - x_i \to S_n - y_i$.

A non-reconstructible tree of maximum degree 3.

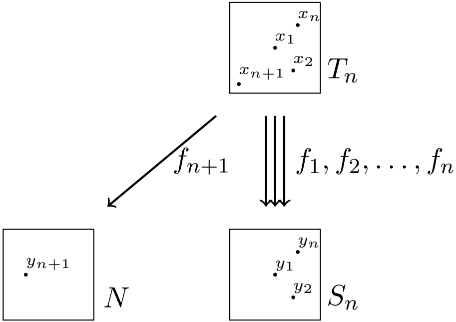
A back-and-forth construction using the amalgamation theorem.



Consider $x_{n+1} \in T_n$, for which we want to find a corresponding card.

A non-reconstructible tree of maximum degree 3.

A back-and-forth construction using the amalgamation theorem.



Shifting Lemma: Construct new tree N and $y_{n+1} \in N$ so that

$$T_n - x_{n+1} \cong N - y_{n+1}.$$

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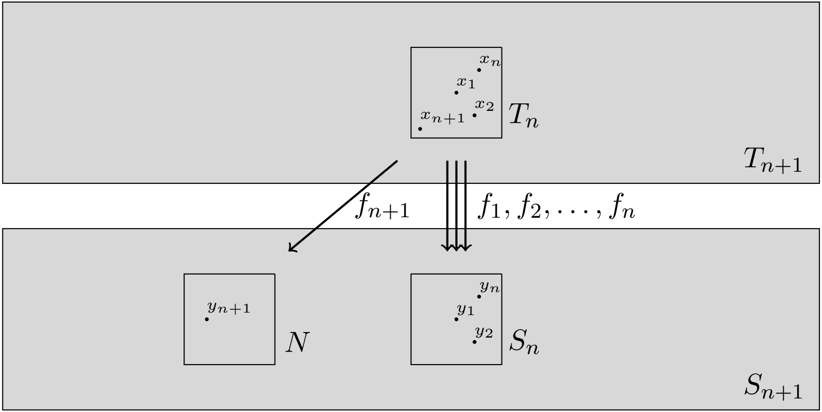

Amalgamate. Obtain trees $T_{n+1} \ncong S_{n+1}$ with $n+1$ common cards, witnessed by isomorphisms $f'_i \colon T_{n+1} - x_i \to S_{n+1} - y_i$.

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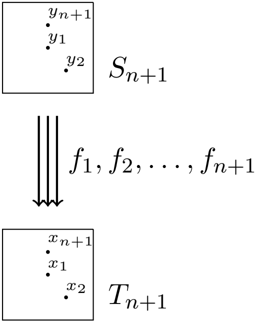

At step $n+1$, have constructed trees $T_{n+1} \ncong S_{n+1}$ with $n+1$ common cards, witnessed by isomorphisms $f_i \colon S_{n+1} - y_i \to T_{n+1} - x_i$.

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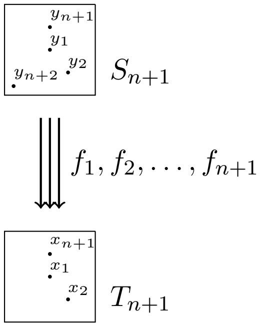
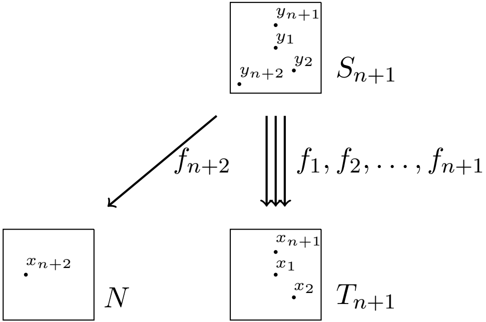


Consider $y_{n+2} \in S_{n+1}$, for which we want to find a corresponding card.

A non-reconstructible tree of maximum degree 3.

A back-and-forth construction using the amalgamation theorem.

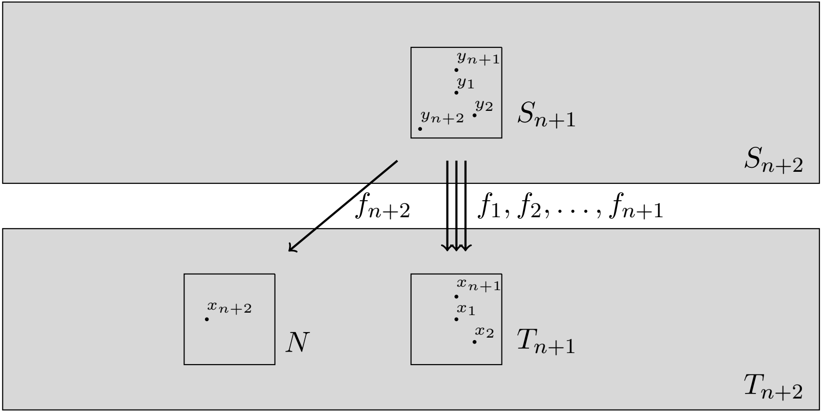


Shifting Lemma: Construct new tree N and $x_{n+2} \in N$, such that

$$f_{n+2} \colon S_{n+1} - y_{n+2} \cong N - x_{n+2}.$$

A non-reconstructible tree of maximum degree 3.

A back-and-forth construction using the amalgamation theorem.



Amalgamate. Obtain trees $S_{n+2} \ncong T_{n+2}$ with $n+2$ common cards, witnessed by isomorphisms $f_i' \colon S_{n+2} - y_i \to T_{n+2} - x_i$.

A non-reconstructible tree of maximum degree 3.

A back-and-forth construction using the amalgamation theorem.

Get a sequence of trees and points

$$\begin{array}{ccccccc} T_0 & \subset & T_1 & \subset & T_2 & \subset \dots \\ \Psi & & \Psi & & \Psi & \\ x_0 & & x_1 & & x_2 & \dots \end{array} \quad \text{and} \quad \begin{array}{ccccccc} S_0 & \subset & S_1 & \subset & S_2 & \subset \dots \\ \Psi & & \Psi & & \Psi & \\ y_0 & & y_1 & & y_2 & \dots \end{array}$$

such that $T = \bigcup T_n$ and $S = \bigcup S_n$ satisfiy $T - x_i \cong S - y_i$.

A non-reconstructible tree of maximum degree 3.

A back-and-forth construction using the amalgamation theorem.

Get a sequence of trees and points

$$\begin{array}{ccccccc} T_0 & \subset & T_1 & \subset & T_2 & \subset & \dots \\ \Cup & & \Cup & & \Cup & & \\ x_0 & & x_1 & & x_2 & & \dots \end{array} \quad \text{and} \quad \begin{array}{ccccccc} S_0 & \subset & S_1 & \subset & S_2 & \subset & \dots \\ \Cup & & \Cup & & \Cup & & \\ y_0 & & y_1 & & y_2 & & \dots \end{array}$$

such that $T = \bigcup T_n$ and $S = \bigcup S_n$ satisfiy $T - x_i \cong S - y_i$.

Question: Do their decks agree? Need to arrange $V(T) = \{x_i \colon i \in \mathbb{N}\}$ and $V(S) = \{y_i \colon i \in \mathbb{N}\}$!

A non-reconstructible tree of maximum degree 3.

A back-and-forth construction using the amalgamation theorem.

Get a sequence of trees and points

$$\begin{array}{ccccccc} T_0 & \subset & T_1 & \subset & T_2 & \subset & \dots \\ \Cup & & \Cup & & \Cup & & \\ x_0 & & x_1 & & x_2 & & \dots \end{array} \quad \text{and} \quad \begin{array}{ccccccc} S_0 & \subset & S_1 & \subset & S_2 & \subset & \dots \\ \Cup & & \Cup & & \Cup & & \\ y_0 & & y_1 & & y_2 & & \dots \end{array}$$

such that $T = \bigcup T_n$ and $S = \bigcup S_n$ satisfiy $T - x_i \cong S - y_i$.

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. . . .

$V(T) = \mathbb{N}$

A non-reconstructible tree of maximum degree 3.

A back-and-forth construction using the amalgamation theorem.

Get a sequence of trees and points

$$\begin{array}{ccccccc} T_0 & \subset & T_1 & \subset & T_2 & \subset & \dots \\ \Uparrow & & \Uparrow & & \Uparrow & & \\ x_0 & & x_1 & & x_2 & & \dots \end{array} \quad \text{and} \quad \begin{array}{ccccccc} S_0 & \subset & S_1 & \subset & S_2 & \subset & \dots \\ \Uparrow & & \Uparrow & & \Uparrow & & \\ y_0 & & y_1 & & y_2 & & \dots \end{array}$$

such that $T = \bigcup T_n$ and $S = \bigcup S_n$ satisfiy $T - x_i \cong S - y_i$.

Question: Do their decks agree? Need to arrange $V(T) = \{x_i \colon i \in \mathbb{N}\}$ and $V(S) = \{y_i \colon i \in \mathbb{N}\}$!

• · •

$V(T) = \mathbb{N}$, $V(T_1) = 2\mathbb{N} - 1$

A non-reconstructible tree of maximum degree 3.

A back-and-forth construction using the amalgamation theorem.

Get a sequence of trees and points

$$\begin{array}{ccccccc} T_0 & \subset & T_1 & \subset & T_2 & \subset \dots \\ \Cup & & \Cup & & \Cup & \\ x_0 & & x_1 & & x_2 & \dots \end{array} \quad \text{and} \quad \begin{array}{ccccccc} S_0 & \subset & S_1 & \subset & S_2 & \subset \dots \\ \Cup & & \Cup & & \Cup & \\ y_0 & & y_1 & & y_2 & \dots \end{array}$$

such that $T = \bigcup T_n$ and $S = \bigcup S_n$ satisfiy $T - x_i \cong S - y_i$.

Question: Do their decks agree? Need to arrange $V(T) = \{x_i : i \in \mathbb{N}\}$ and $V(S) = \{y_i : i \in \mathbb{N}\}$!

$V(T) = \mathbb{N}$, $V(T_1) = 2\mathbb{N} - 1$, $V(T_2 \setminus T_1) = 4\mathbb{N} - 2$

A non-reconstructible tree of maximum degree 3.

A back-and-forth construction using the amalgamation theorem.

Get a sequence of trees and points

$$\begin{array}{ccccccc} T_0 & \subset & T_1 & \subset & T_2 & \subset & \dots \\ \Cup & & \Cup & & \Cup & & \\ x_0 & & x_1 & & x_2 & & \dots \end{array} \quad \text{and} \quad \begin{array}{ccccccc} S_0 & \subset & S_1 & \subset & S_2 & \subset & \dots \\ \Cup & & \Cup & & \Cup & & \\ y_0 & & y_1 & & y_2 & & \dots \end{array}$$

such that $T = \bigcup T_n$ and $S = \bigcup S_n$ satisfiy $T - x_i \cong S - y_i$.

Question: Do their decks agree? Need to arrange $V(T) = \{x_i \colon i \in \mathbb{N}\}$ and $V(S) = \{y_i \colon i \in \mathbb{N}\}$!

$V(T) = \mathbb{N}$, $V(T_1) = 2\mathbb{N} - 1$, $V(T_2 \setminus T_1) = 4\mathbb{N} - 2$, $V(T_3 \setminus T_2) = 8\mathbb{N} - 4$, ...

A non-reconstructible tree of maximum degree 3.

A back-and-forth construction using the amalgamation theorem.

Get a sequence of trees and points

$$\begin{array}{ccccccc} T_0 & \subset & T_1 & \subset & T_2 & \subset & \dots \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \\ x_0 & & x_1 & & x_2 & & \dots \end{array} \quad \text{and} \quad \begin{array}{ccccccc} S_0 & \subset & S_1 & \subset & S_2 & \subset & \dots \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \\ y_0 & & y_1 & & y_2 & & \dots \end{array}$$

such that $T = \bigcup T_n$ and $S = \bigcup S_n$ satisfiy $T - x_i \cong S - y_i$.

Question: Do their decks agree? Need to arrange $V(T) = \{x_i : i \in \mathbb{N}\}$ and $V(S) = \{y_i : i \in \mathbb{N}\}$!

$V(T) = \mathbb{N}$, $V(T_1) = 2\mathbb{N} - 1$, $V(T_2 \setminus T_1) = 4\mathbb{N} - 2$, $V(T_3 \setminus T_2) = 8\mathbb{N} - 4$, ...

Then T and S are non-isomorphic reconstructions of each other.

The second counterexample

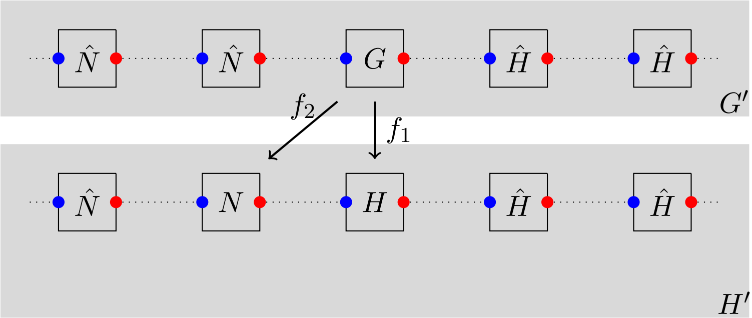
Constructing a non-reconstructible locally finite one-ended graph: Modify ingredient A.

A general set-up. Given:

- Disjoint graphs G, H, N,
- isomorphisms $f_1, \dots, f_n \colon G \to H$,
- (new) isomorphism $f_{n+1} \colon G \to N$.

Want:

- $G' \supset G$ and $H' \supset N \dot\cup H$ s.t.
- all maps lift to isomorphisms $f'_i \colon G' \to H'$ for $i \leq n+1$.



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Want: One-ended graphs

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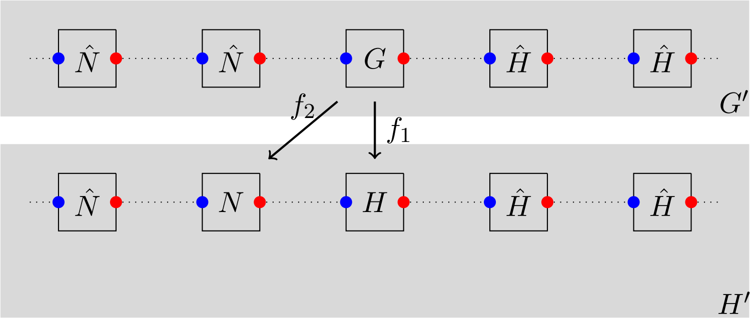

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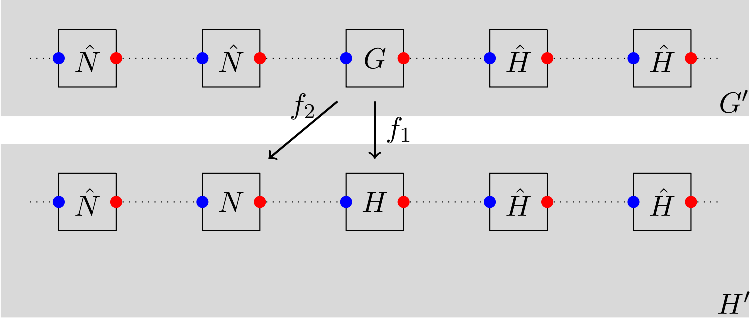
Want: One-ended graphs

- $G' \supset G$ and $H' \supset N \dot\cup H$ s.t.
- all maps lift to isomorphisms $f'_i \colon G' \to H'$ for $i \leq n+1$.



To make the graphs one-ended, for finite G, H, N, glue on a big half-grid $\mathbb{Z}\square\mathbb{N}$. Maps f_1, f_2 still lift to maps f'_1 and f'_2.

The second counterexample

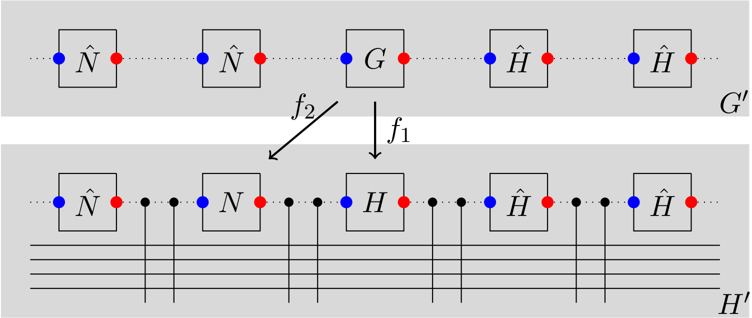
Constructing a non-reconstructible locally finite one-ended graph: Modify ingredient A.

A general set-up. Given:

- Disjoint graphs G, H, N,
- isomorphisms $f_1, \dots, f_n \colon G \to H$,
- (new) isomorphism $f_{n+1} \colon G \to N$.

Want: One-ended graphs

- $G' \supset G$ and $H' \supset N \dot\cup H$ s.t.
- all maps lift to isomorphisms $f_i' \colon G' \to H'$ for $i \leq n+1$.



To make the graphs one-ended, for finite G, H, N, glue on a big half-grid $\mathbb{Z}\Box\mathbb{N}$. Maps f_1, f_2 still lift to maps f_1' and f_2'.

The second counterexample

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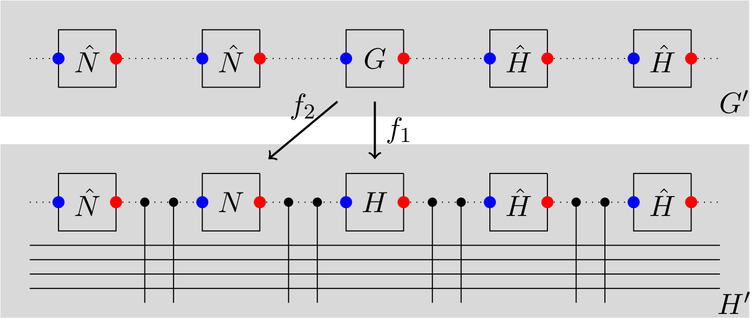

For $n \geq 2$, under mild assumptions on promise leaves, glueing on a tree-grid $T \square \mathbb{N}$ for some suitable locally finite tree T works.

Open questions for reconstruction of infinite graphs

When restricting the end-degree, our counterexample techniques no longer work.

# ends	Locally finite trees	Locally finite graphs
1	✓ Thomassen '78	✗ BEHLP '18
2	✓ Bondy/Hemminger '74	✓ NW '91
$3, 4, \ldots$	✓ Bondy/Hemminger '74	✓ NW '87
$\lvert\mathbb{N}\rvert$	✓ Andreae '81	✗ BEHLP '18
$\lvert\mathbb{R}\rvert$	✗ BEHLP '17	(✗ BEHLP '17)

Question A (Nash-Williams): Is every one-ended locally finite connected graph with finite end-degree reconstructible?

Question B: Is every countably-ended connected graph with of finite tree-width reconstructible?