Fachbereich Mathematik,
Universität Hamburg
WS 23/24

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Infinite graphs

Sheet 9

Besprechung am 18.12.2023

1. Show that every IG contains a lean IG as a subgraph.

2. Show that the assertion of Jung's Theorem 5.4.3 does not hold for the cardinal $\aleph_0$, i.e. find a graph containing $K_{\aleph_0}$ as a minor but not as a topological minor.

3. Let T be any order tree with comparability graph $G(T)$. Show that T is special if and only if $G(T)$ has countable chromatic number.[1]

4 (Written exercise). Construct for every infinite cardinal κ a graph of chromatic number κ^+ but without K_{κ^+} minor.

5. Prove, using the generalised infinite lemma or otherwise, that every uncountable tree T with all levels finite contains an uncountable branch.

6. Show that the following hold for any order tree $(T, \leq)$ and any T-graph G:

(1) For incomparable t, t' in T, the set $\lceil t \rceil \cap \lceil t' \rceil$ separates t from t' in G.
(2) Every connected subgraph of G has a unique T-minimal element.
(3) If $T' \subseteq T$ is down-closed, the components of $G - T'$ are spanned by the sets $\lfloor t \rfloor$ for t minimal in $T - T'$.

7. Let T be a normal tree in a graph G.

(1) Any two incomparable vertices t, s in T are separated in G by $\lceil t \rceil \cap \lceil s \rceil$.
(2) The boundary of T consists of all ends ε with $C(X, \varepsilon) \cap T \neq \varnothing$ for all finite $X \subseteq V(G)$.
(3) Every end of G in the boundary of T contains a unique normal ray of T.

8. Let T be a normal spanning tree of a graph G, and let ε be an end of G.

(1) The end ε has at most countable degree.
(2) Show that all vertices dominating ε lie on the unique normal ray representing ε. In particular, every end of a graph with a normal spanning tree is at most countably dominated.

[1] The colouring number might again be larger: It can be shown that for a special Aronszajn tree T, the graph $G(T)$ has uncountable colouring number.

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Hints

1. –

2. –

3. –

4. Consider the order tree T consisting of all injective functions $i \hookrightarrow \kappa$ for all ordinals $i < \kappa^+$, ordered by extension.

5. –

6. Take a look at the proof of Lemma 1.5.5 in Diestel's book.

7. Apply the Star-Comb Lemma inside T.

8. –