Fachbereich Mathematik,
Universität Hamburg
WS 23/24

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Infinite graphs

Sheet 8

Besprechung am 11.12.2023

1 (Written exercise). Show that the bipartite graph $K_{\omega,\omega}$ has colouring number $\aleph_0$, and that the bipartite graph K_{ω,ω_1} has colouring number $\aleph_1$.

2. Show that for any infinite cardinals κ and λ, we have

(1) $\mathrm{col}(K_\kappa) = \kappa$

(2) $\mathrm{col}(K_{\kappa,\kappa}) = \kappa$, and

(3) if $\kappa < \lambda$, then $\mathrm{col}(K_{\kappa,\lambda}) = \kappa^+$.

3. Show that every graph with a normal spanning tree has countable colouring number.

4. Directly prove that the rational distance graph on the plane $G_{\mathbb{Q}}(\mathbb{R}^2)$ has no copies of $K_{2,\aleph_1}$ and then deduce from a theorem from the lecture that it must have countable chromatic number.

5. Show that if $\mathrm{col}(G) > \mu$, then G contains a K_{n,μ^+} subgraph for every $n < \omega$.

6. Show that for every uncountably chromatic graph G there is $s \in \mathbb{N}$ and an edge e in $E(G)$ such that G contains cycles through e of all (even and odd) lengths greater than s.

7. Every graph of infinite chromatic number contains a $\bigsqcup_{n\in\mathbb{N}} TK_n$.

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Hints

1. –

2. –

3. –

4. –

5. Adapt the proof of Theorem 5.4.5 by constructing a continuous increasing transfinite sequence $(G_i : i < \sigma)$ of subgraphs all of size $< |G|$ such that if for some finite set of vertices $F \subset V(G_i)$ there *one* $v \in G \setminus G_i$ such that $N(v) \supseteq F$, then there are *at least* μ^+ such vertices in G.

6. Consider the subgraph G' of G from the proof of Theorem 5.2.7.

7. By Theorem 5.4.5, G contains a TK_n for all $n \in \mathbb{N}$.