Fachbereich Mathematik,
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WS 23/24

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Infinite graphs

Sheet 7

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1. Show that the graph G defined in the proof of the pressing down lemma is a tree.

2. Prove the following local version of Halin's result: Let $k \in \mathbb{N}$, and κ an uncountable regular cardinal. Suppose G contains a k-block U of size at least κ. Then G contains a subdivision of $K_{k,\kappa}$, with the κ-side included in U.

3. Show that the following assertions are equivalent for connected countable graphs G.

(i) G has a locally finite spanning tree.

(ii) For no finite separator $X \subseteq V(G)$ does $G - X$ have infinitely many components.

Deduce that every (countable) planar 3-connected graph has a locally finite spanning tree.

4. A family of sets $(A_i : i \in I)$ is a Δ-system if the pairwise intersection of its members is the same, i.e. there is a set S such that $A_i \cap A_j = S$ for all $i \neq j \in I$. Show that for all $n \in \mathbb{N}$, any infinite family of n-element sets contains an infinite Δ-system as a subfamily.

5. (Harder) Let G be a 2-connected graph. Then for every infinite set of vertices U, the graph G contains a one-way double ladder, a dominated ray, or a $K_{2,\aleph_0}$, each with infinitely many vertices in U.

6. Let κ be an infinite cardinal. An ultrafilter $\mathcal{U}$ on a set X is κ-uniform if every element $U \in \mathcal{U}$ has size at least κ. Show that on every set X of size at least κ there exists a κ-uniform ultrafilter.

7. (Written exercise) Let κ be any infinite cardinal, let $r \in \mathbb{N}$, and let G be a complete graph of size κ whose edges are coloured with r colours. Show that there is a monochromatic subdivided K_κ in G.

8. Let G by any countable, infinitely connected graph and let T be any countably infinite tree. Show that G has a spanning tree which is isomorphic to a subdivision of T.

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Hints

1. –

2. –

3. Normal spanning trees.

4. Induction on n.

5. Normal spanning trees. Note that we may assume that U and hence G are countably infinite by Proposition 2.6.1.

6. –

7. Combine the previous exercise with the strategy of Theorem 2.3.5.

8. Enumerate $V(T) = \{t_0, t_1, t_2, \dots\}$ such that $T_n := T[t_0, \dots, t_n]$ is connected for all $n \in \mathbb{N}$. Then recursively construct subdivisions of T_n in G extending each other and make sure that they eventually contain every vertex of G.