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WS 23/24

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Infinite graphs

Sheet 6

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1. Suppose A, B, C are subsets of a graph G such that each A and B are linked into C. Show that if $|A| < |B|$, then there is $b \in B \setminus A$ such that $A \cup \{b\}$ is linked into C.

2. Show that any graph G containing k edge disjoint rays for all $k \in \mathbb{N}$ also contains infinitely many edge disjoint rays.

3. Show directly that every subdivided countably infinite star of rays contains a subdivided ray of rays.

4. Find a function $f : \mathbb{N} \to \mathbb{N}$ such that for any collection $\mathcal{R}$ of $f(k)$ disjoint parallel rays in a graph G, there is a subdivision of the hexagonal $k \times \mathbb{N}$ half-grid in G such that all its vertical rays belong to $\mathcal{R}$.

5. Prove the following assertions about infinite graphs:

(1) Every infinite connected graph has a vertex of infinite degree or contains a ray.

(2) Every uncountable connected graph has a vertex of uncountable degree.

(3) Let κ be a regular uncountable cardinal. Every connected graph of size at least κ has a vertex of degree κ.

6. Starting with a single coin, you play a (transfinite) game with a simple automaton: at each step you insert a single coin to which the machine returns two new coins (but never one that at some point belonged to you).

(1) Show that if not careful, you might loose all your money in ω steps.

(2) Show that, with any strategy, the player will go bankrupt in countably many steps (i.e. you cannot continue the game for ω_1 steps).

7 (Written exercise). A finite set of vertices X in a connected graph G is critical if there are infinitely many components C of $G - X$ with $N(C) = X$. Show that every infinite connected graph contains a ray, or a critical vertex set.

8. Let U be an infinite set of vertices in a locally finite graph G.

(1) If G is 2-connected, then there is a ray in G containing infinitely many vertices from U.

(2) If G is 3-connected, then there is a double ray in G containing infinitely many vertices from U.

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Hints

1. When $k = |A| < |B|$, fix an $A - C$ path system $\mathcal{P} = \{P_a \colon a \in A\}$ witnessing that A is linked into B. If the assertion was wrong, then for every $b \in B$ there is separator S_b with $|S_b| \leq k$ separating $A \cup \{b\}$ from C.

For each path P_a let p_a^* be the last of the vertices in $\bigcup_{b \in B} S_b$ on P_a. Show that $S^* = \{p_a^* \colon a \in A\}$ separates B from C.

2. Wlog suppose that the graph G is locally finite and consider its line graph.

3. Use the leaf rays of the countable star of rays to define a ray-of-rays which connect up via the centre ray.

4. Use Proposition 3.5.2. and the fact that every large tree either contains a long path or a vertex of high degree.

5. Use König's infinity lemma for (1).

6. Pressing down lemma

7. Normal spanning trees (Exercise 2.1.14).

8. (1) Consider a rooted ray $r_0, r_1, r_2, \ldots$ in a normal spanning tree such that all $\lfloor r_n \rfloor$ meet U.

(2) Combine (1) and Corollary 3.3.5.