Fachbereich Mathematik,
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WS 23/24

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Infinite graphs

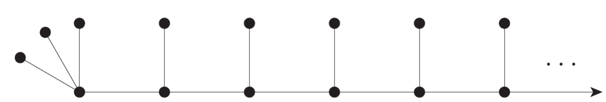
Sheet 5

Besprechung am 20.11.2023

1. Show that, in the ubiquity conjecture, the host graphs G considered can be assumed to be locally finite, too.

2. Let G be a graph such that for every natural number n there is a system of n pairwise disjoint double rays in G. Show that there is an infinite such system.

3. Show that the modified comb below is not ubiquitous with respect to the subgraph relation. Does it become ubiquitous if we delete its 3-star on the left?



4. Show that the following assertions about rays R, R' in a graph $G = (V, E)$ are equivalent:

(1) R, R' are parallel, i.e. are not separated by a finite $X \subseteq V$.

(2) For all finite $X \subseteq V$, both R and R' have a tail in the same component of $G - X$.

(3) There exist infinitely many disjoint R–R' paths in G.

(4) There exists a third ray K in G with $|K \cap R| = \infty = |K \cap R'|$.

5 (Written exercise). Suppose that $R_1, R_2, R_3, \ldots$ are countably many parallel rays in a graph G. Show that there exists a ray K in G such that $|K \cap R_i| = \infty$ for all $i \in \mathbb{N}$.

6. Let G be a graph, v a vertex and ε an end of G. Then the following assertions are equivalent:

(1) For some ε-ray R there is an infinite v–$(R - v)$ fan in G.

(2) For every ε-ray R there is an infinite v–$(R - v)$ fan in G.

(3) No finite subset of $V(G - v)$ separates v from some ray in ε.

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(4) No finite subset of $V(G - v)$ separates v from all rays in ε.

(5) $N(v) \cap C(X, \varepsilon) \neq \varnothing$ for all finite sets of vertices X.

7. Let ε be a dominated end in a graph G. Show that for every countable set U of vertices all dominating ε, there is an ε-ray R in G including U.

8. Show that a graph contains a $TK^{\aleph_0}$ if and only if some end (i.e., the end containing the rays from that $TK^{\aleph_0}$) is infinitely dominated.

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Hints

1. –

2. Modify the proof of Theorem 3.1.3. so that the paths in $\mathcal{P}_n$ get extended into both directions. It may help to take more than $n+1$ fresh double rays in each step.

3. To construct a graph that contains arbitrarily but not infinitely many copies of the modified comb T, start with infinitely many disjoint copies of T. Group these into disjoint sets $S_1, S_2, \dots$ so that S_n is a disjoint union of n copies of T. Then identify vertices from different sets S_n.

4. –

5. Let $(a_n : n \in \mathbb{N})$ be a sequence in $\mathbb{N}$ containing every number infinitely often. Construct K recursively, building in a vertex of R_{a_n} in step n.

6. –

7. –

8. –