Fachbereich Mathematik,
Universität Hamburg
WS 23/24

PD Dr. Max Pitz
Thilo Krill

Infinite graphs

Sheet 4

Besprechung am 13.11.2023

1. Let $(X, \leq)$ be a well-ordered set and suppose that $f : X \to X$ is an embedding (not necessarily onto an initial segment). Then f is non-decreasing (i.e. $x \leq f(x)$ for all $x \in X$).

2. An ultrafilter $\mathcal{U}$ on a set X is free if and only if all sets in $\mathcal{U}$ are infinite.

3. a) Read and understand the proof of Theorem 1.4.5 in the countable case.
b) **(Written exercise)** Use the Generalized Infinity Lemma to extend Theorem 1.4.5 from countable to graphs of arbitrary size.

4. Prove the Ultrafilter Lemma 2.3.1 from the Generalized Infinity Lemma 2.3.7

5. For every colouring of the edges of a $K_{\mathbb{N},\mathbb{N}}$ with $r \in \mathbb{N}$ many colours, the vertex set can be partitioned into $2r - 1$ monochromatic paths, where 'path' means either a finite path or a ray. Furthermore, this number of paths is optimal.

6. Show that there are exactly $2^{\aleph_0}$ many pairwise non-isomorphic, countable graphs.

7. Show that there exists a 2-colouring of $\mathbb{R}$ such that for every continuous map $f : \mathbb{R} \to \mathbb{R}$ other than the identity there exists $x \in \mathbb{R}$ such that x and $f(x)$ have different colours.

8. Show that every infinite partial order $(P, \leq)$ without infinite antichains contains a chain of size $|P|$; and that every infinite partial order $(P, \leq)$ without infinite chains contains an antichain of size $|P|$.

Fachbereich Mathematik,
Universität Hamburg
WS 23/24

PD Dr. Max Pitz
Thilo Krill

Hints

1. –

2. –

3. –

4. Suppose $\mathcal{A}$ is a collection of subsets of a set X with the finite intersection property. Consider the collection Π of all finite partitions of X, directed by $\pi \leq \pi'$ if π' refines π. Then for each π consider the finite set V_π consisting of those $U \in \pi$ such that $\mathcal{A} \cup \{U\}$ has the finite intersection property.

5. Write A and B for the partition classes of $K_{\mathbb{N},\mathbb{N}}$. Can you do it with $2r$ monochromatic paths by employing free ultrafilters $\mathcal{U}_A$ on A and $\mathcal{U}_B$ on B? Is it possible to merge one pair of paths of the same colour?

6. From a previous exercise, we already know that there are at least continuum many such graphs.

7. Show first that any continuous such function is determined by its values on $\mathbb{Q}$ and hence that there are $2^{\aleph_0}$ many continuous functions $\mathbb{R} \to \mathbb{R}$.

Then enumerate the set of all such continuous functions (other than the identity) as $\{f_i : i < 2^{\aleph_0}\}$ and colour recursively for every i a fresh point $x_i \in \mathbb{R}$ with $f_i(x_i) \neq y_i$ by distinct colours.

8. Dushnik-Erdős-Miller.