Fachbereich Mathematik,
Universität Hamburg
WS 23/24

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# Infinite graphs

## Sheet 3

Besprechung am 06.11.2023

**1.** a) Prepare a drawing of the well-order $X = \bigoplus_{n\in\mathbb{N}}(\bigoplus_{n\in\mathbb{N}} \mathbb{N})$.
b) By Cantor's theorem, every countable linear order embeds into the rationals. Describe formally a subset of the rationals isomorphic to $X$.

**2.** Show that if $X$ is a well-ordered set and $Y \subseteq X$, then $Y \leq X$. What about the stronger assertion that if $Y \subsetneq X$, then $Y < X$?

**3.** i) Show that an ordinal embeds into $(\mathbb{R}, \leq)$ if and only if it is countable.
ii) Show that there is a 2-colouring of the edges of $K_{\mathbb{R}}$ without a monochromatic uncountable clique.

**4.** Show that the Spanning Tree Theorem implies the Axiom of Choice.

**5 (Written exercise).** Present the second proof of Theorem 2.1.9 in the successor / limit step version of transfinite induction.

**6.** Rephrase the second proof of the de Bruijn-Erdős Theorem 2.1.15 (the one via transfinite induction) so that it uses Zorn's lemma.

**7.** Spell out the details of the proof of the Ultrafilter Lemma 2.3.1.

**8.** Show that every countable, infinitely connected graph $G$ contains infinitely many edge-disjoint spanning trees.

*Optional:*

**9.** Show, using Zorn's Lemma or otherwise, that every infinitely edge-connected graph contains infinitely many edge-disjoint spanning trees.

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# *Hints*

**1.** –

**2.** –

**3.** (i) –

(ii) Compare the usual order on $\mathbb{R}$ with a well-order on $\mathbb{R}$, and encode this information in a 2-colouring.

**4.** Given a family $\{A_i : i \in I\}$ of disjoint sets, we need to find a formula that defines a choice function. Can you do this using a spanning tree $T$ for the graph $G$ with vertex set $\{x\} \cup \bigcup_{i \in I}(A_i \cup \{y_i, z_i\})$ and edge set $xy_i$ for all $i$ and $y_i a$, $az_i$ for all $a \in A_i$ and all $i$?

**5.** Let $G = (V, E)$ be a connected graph, with enumeration $V = (v_j : j < i)$ for some ordinal $i$. For $j < i$ define spanning trees $T_j$ of $G_j = G[\{v_m : m < j\}]$.

**6.** Consider the set $X$ of all pairs $(f, W_f)$, where $W_f \subseteq V(G)$ and $f$ is a proper $k$-colouring of $G[W_f]$ such that for all finite $U \subseteq V(G)$, the colouring $f$ can be extended to a colouring of $G[W_f \cup U]$.

**7.** –

**8.** List $\mathbb{N} \times V(G) = \{(a_n, v_n) \colon n \in \mathbb{N}\}$ and make sure that in the $n$th step of the construction, the $a_n$'th spanning tree contains the vertex $v_n$.

**9.** Consider the poset of families $(T_1, T_2, \ldots)$ of edge-disjoint subtrees of $G$ with $V(T_i) = V(T_j)$, ordered by the coordinate-wise subgraph relation.