Fachbereich Mathematik,
Universität Hamburg
WS 23/24

PD Dr. Max Pitz
Thilo Krill

Infinite graphs

Sheet 2

Besprechung am 30.10.2023

1. The infinite van-der-Waerden theorem states that if one colours $\mathbb{N}$ with finitely many colours, then one colour class contains arbitrarily long arithmetic progressions. Deduce from this the corresponding finite theorem: For every number of colours $k \in \mathbb{N}$ and any length $r \in \mathbb{N}$ there is $N \in \mathbb{N}$ such that if one colours $\{0, 1, \ldots, N\}$ with k colours, then one colour class contains an arithmetic progression of length r.

2. Prove the following version of König's theorem for locally finite graphs: Every bipartite graph G has a matching M and a vertex cover U of the edges of G such that U consists of one vertex from each edge in M.

3. Suppose two players, Alice and Bob, play a game on $K_{\mathbb{N}}$ where in each step, Alice colours one (still uncoloured) edge with colour blue, then Bob colours one (still uncoloured) edge with colour red, and then it's Alice's turn again. Show that Alice has a strategy to build a blue infinite complete subgraph.

4. In the proof of the Canton-Schröder Bernstein Theorem 1.5.1, it was sufficient to show that H contains no finite paths of even length. Determine exactly for which odd numbers m there might be a path in H of length m.

5. Show that every infinite linear order has an infinite, strictly increasing sequence, or an infinite, strictly decreasing sequence.

6. Show that every partial order $(X, \leq)$ embeds into $(\mathcal{P}(X), \subseteq)$.

7 (Written exercise). Show that every 2-edge connected graph G has a strongly connected orientation $\vec{G}$, i.e. a way to orient each edge xy of G either from $\vec{xy}$ or from $\vec{yx}$ such that for every two vertices, there is a directed path from the first to the second (and vice versa).

8. Show that every rayless connected graph has a (rayless) normal spanning tree.

PD Dr. Max Pitz
Thilo Krill

Hints

1. Follow the proof of Theorem 1.4.4.

2. Apply the infinity lemma to a suitably weakened statement about finite subgraphs.

3. Show that Alice can build an infinite subtree of the (rooted) binary tree such that each vertex sends blue edges to all vertices below it towards the root.

4. –

5. Use the infinite Ramsey Theorem 1.3.1.

6. –

7. Apply Zorn's Lemma 2.1.8 to the poset of oriented subgraphs of G which are strongly connected, ordered by the (directed) subgraph relation.

8. First, look again at the proof of Theorem 1.2.2 that every countable graph has a normal spanning tree. Then apply Zorn's Lemma to a suitable poset.