Fachbereich Mathematik,
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WS 23/24

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Infinite graphs

Sheet 12

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1. Find a direct proof (2) ⇔ (3) in Theorem 7.2.1, i.e. let κ be an uncountable regular cardinal and show that any graph G contains a T_κ minor if and only if G contains a T_κ subdivision.

2. Find a graph that contains an end-faithful spanning tree but no normal spanning tree.

3. Let H be a locally finite spanning subgraph of some (not necessarily locally finite) graph G. Show that $\iota : \mathcal{E}(H) \to \mathcal{E}(G)$ is always onto.

4. Let X be any set of vertices in a graph G.

(i) Every end of G is in the boundary of X or in the boundary of some component of $G - X$.

(ii) Every end of G that is in the boundary of two distinct components of $G - X$ is also in the boundary of X.

(iii) Moreover, if $X = V(T)$ for a normal tree T in G, then for every component C there is at most one end in $\partial C \cap \partial T$, and this end is given by the ray $\lceil N(C) \rceil_T$.

5. Let W be a finite set of vertices in a graph G, and $U = N(W)$ its neighbourhood. Then the boundary ∂U consists of precisely those ends of G that are dominated by a vertex in W.

6 (Written exercise). Let G be an infinitely connected graph. Show that G has a rayless spanning tree if and only if it has and end-faithful spanning tree.

7. Let G be any graph and let $U \subseteq V(G)$ be normally spanned. Then there is a rayless tree S that includes U if and only if all the ends of G in the closure of U are dominated in G.

8. Show that the κ-star is subgraph-ubiquitous for any infinite cardinal κ.

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Hints

1. Use Exercises 4.3.6 and 4.3.7.

2. –

3. –

4. For (iii), apply Exercise 6.1.5.

5. –

6. Adapt the strategy of Lemma 8.1.5.

7. Let T be a normal tree containing U cofinally. Build a spanning tree S by incorporating, in step $n \in \mathbb{N}$, the vertices on the nth level of T into S, following the strategy of the proof of Theorem 8.3.1.

8. –