Fachbereich Mathematik,
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WS 23/24

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Infinite graphs

Sheet 11

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1 (Written exercise). Every subgraph of a graph of $\mathcal{I}$-rank i has $\mathcal{I}$-rank $\leq i$.

2. If G is k-[edge-]connected and H is a k-[edge-]connected subgraph, then for every component C of $G-H$, the subgraph $H' := G[H \cup C]$ is k-[edge-]connected.

3. Let G be any k-[edge-]connected graph and let $U \subseteq V(G)$ be finite. Show that there are finitely many components $C_1, \dots, C_n$ of $G-U$ such that $U \cup C_1 \cup \dots \cup C_n$ induces a k-[edge-]connected subgraph of G.

4. Let G be any rayless k-[edge-]connected graph and let $U \subseteq V(G)$ be finite. Show that there is a finite k-[edge-]connected subgraph of G that contains U.

5. A graph is rayless and k-connected if and only if it admits a rayless tree-decomposition into finite, k-connected parts such that adjacent parts overlap in at least k vertices.

6. Let κ be an infinite cardinal. Show that the κ-regular rooted tree T_κ has an enumeration $V(T_\kappa) = \{t_i : i < \kappa\}$ such that if t_j is the parent of t_i, then $j < i$.

7. Apply The Robertson-Seymour-Thomas Theorem 7.3.6 to show for all infinite cardinals κ that $G \not\supseteq TK_\kappa$ implies $\mathrm{col}(G) \leq \kappa$, hence giving an alternative proof of Halin's infinite Hadwiger theorem.

8. If every connected graph contains an end-faithful tree, then every connected graph has an end-faithful spanning tree.

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Hints

1. –

2. –

3. –

4. Use induction on the rank of G and choose a reducing set U' of G containing U. Then apply Exercise 3 to the finite set U'. What can you say about the rank of the subgraph given by Exercise 3?

5. For the harder direction, show by rank induction the stronger statement that for an arbitrary finite set of vertices X, the graph G has a rayless tree-decomposition into k-connected finite parts such that the part corresponding to the root includes X.

6. –

7. –

8. Given a graph G, construct a graph G^+ by gluing onto every vertex $v \in V(G)$ a fresh ray R_v. Then relate end-faithful trees of G^+ to end-faithful spanning trees of G.