Fachbereich Mathematik,
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WS 23/24

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Infinite graphs

Sheet 10

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1. Let U be a dispersed set of vertices in a graph G. If for each component C of $G - U$ we select a finite set of vertices $U_C \subseteq C$, then $U \cup \bigcup_C U_C$ is again dispersed.

2 (Written exercise). Suppose G is a graph with NST. Show that every connected minor of G also has an NST.

3. Show that a connected graph G has a tree-decomposition into finite parts if and only if G has an NST.

4. Give a direct proof of Diestel's criterion in Theorem 6.2.10 by employing a suitable closure argument.

5. Use NSTs to show that a countable connected graph has either countably many or continuum many ends.

6. Show that an uncountable clique with all edges subdivided once cannot be represented as a T-graph and deduce that the class of T-graphs is not closed under minors or subgraphs.

7 (Holiday Special – optional). A group of infinitely many prisoners are offered the following game. On New Year's Eve, each of the prisoners will receive a label on their forehead showing some real number. Every prisoner can see the other inmates' labels but not their own. They know each other and can tell each other apart. The moment they hear the New Year Canon, which by tradition is fired once to announce an amnesty at the beginning of the new year, they each have to shout a real number. If all but finitely many prisoners shout the number on their own forehead, then all prisoners are freed.

The prisoners are allowed to get together beforehand to agree a strategy. Can you advise them? You may assume the axiom of choice. You may also assume that real numbers can be written on foreheads and shouted out in an instance and that an infinite amount of information can be exchanged when agreeing to a strategy.

8 (Open problem – optional). Characterise the class of graphs that can be represented as a T-graph for a suitable order tree T (depending on the graph). Maybe easier: Is the class of T-graphs closed under induced subgraphs?

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Hints

1. –

2. Use Jung's Theorem 6.2.6.

3. For the forward implication, use Jung's Theorem. For the backward implication, use the normal spanning tree and define a suitable part for every node of it. (Some parts may be subsets of other parts.)

4. By induction on $|G| = \sigma > \aleph_0$.

(a) Using Lemma 2.6.6, construct a continuous increasing ordinal-indexed sequence $(G_i : i < \sigma)$ of induced infinite subgraphs all of size less than $|G|$ with $G = \bigcup_{i<\sigma} G_i$ such that

   (i) the end vertices of any G_i-path in G have infinite connectivity in G_i, and

   (ii) the end vertices of any G_i-path in G have uncountable connectivity in G.

(b) Then construct an increasing sequence of normal spanning trees T_i of G_i all with the same root by transfinite recursion on i. Use (ii) to argue that $N(C)$ is finite for every component C of $G - T_i$, and use (i) together with Lemma 6.1.2 (1) to argue that $N(C)$ lies on a chain of T_i.

5. By Lemma 6.1.3, it suffices to prove the exercise for trees. Assuming that a tree has uncountably many ends, construct a subdivision of T_2 in it.

6. Show that an uncountable clique G with all edges subdivided once does not have an NST. Hence, if G is a T-graph for some tree T, then T must contain a limit point.

7. –