Fachbereich Mathematik,
Universität Hamburg
WS 23/24

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Infinite graphs

Sheet 1

Besprechung am 23.10.2023

1. Find continuum sized families of countable graphs such that:

(i) all of them are pairwise non-isomorphic.

(ii) none of them is a subgraph of another.

(iii) for every two of them, one is a subgraph of the other but not the other way around.

(iv) all of them are pairwise non-isomorphic but every graph is a subgraph of every other graph in this family.

2. Can you make all graphs in the previous exercise locally finite trees?

3 (Written exercise). All locally finite, connected graphs are countable.

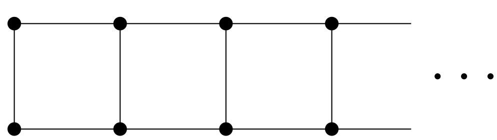
4. Show that there are only countably many non-isomorphic finite graphs.

5. Prove that every collection of disjoint homeomorphic copies of T's in the plane is countable.

The last exercise implies that infinite planar graphs have only countably many vertices of degree ≥ 3.

6. Show that the chromatic number of $G_1(\mathbb{R}^2)$ is at least 4 and at most 7.

7. Consider the one-way infinite ladder as follows:



Describe an enumeration $\{e_n : n \in \mathbb{N}\}$ of its edge set such that the algorithm 'deleting the next edge e_{n+1} as long as the graph stays connected' fails to produce a spanning tree.

8. Construct, for any given $k \in \mathbb{N}$, a planar k-connected graph. Can you construct one whose girth is also at least k? Can you construct an infinitely connected planar graph?

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Hints

1. (i) Use Sheet 0 Exercise 1.

(ii) Use Sheet 0 Exercise 1.

(iii) Use Sheet 0 Exercise 3 (ii).

(iv) –

2. –

3. Use Sheet 0 Exercise 2 (i).

4. Use Sheet 0 Exercise 2 (i) or (iii).

5. Use Sheet 0 Exercise 2 (ii) and the fact that $\mathbb{Q}$ is countable.

6. –

7. –

8. Construct the graph inductively, starting from a vertex or a cycle. To ensure that the final graph has high connectivity, join each new vertex by many edges to the infinite set of vertices yet to be defined.