Fachbereich Mathematik,
Universität Hamburg
WS 23/24

PD Dr. Max Pitz
Thilo Krill

Infinite graphs

Sheet 0

Bearbeitung und Besprechung am 16.10.2023

1. Show that $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are countable, while $\mathbb{R}$ and $2^{\mathbb{N}}$ (the set of all infinite $0-1$ sequences, i.e. functions $f: \mathbb{N} \to \{0,1\}$) have size *continuum*; in particular, they are uncountable. By identifying subsets of $\mathbb{N}$ with their characteristic functions, it follows that $\mathcal{P}(\mathbb{N})$ has size continuum, too.

2. Prove the following facts about countable sets.

(i) A countable union of countable sets is again countable.

(ii) The pigeon hole principle: *If uncountably many objects are put into countably many boxes, then at least one box will contain uncountably many objects.*

(iii) A countable set has only countably many finite subsets.

3. Find continuum many subsets of $\mathbb{N}$ such that

(i) their pairwise intersections are all finite.

(ii) for any two of them, one is contained in the other.

4. How many disjoint circles are there in the plane? How many disjoint disks?

5. In the following, we want to find many disjoint, homeomorphic copies of a given letter in the plane.

(i) Prove that there is an uncountable collection of disjoint L's in the plane.

(ii) Prove that every collection of disjoint B's in the plane is countable.

6. Show that $G_1(\mathbb{Q}^2)$ is disconnected.

PD Dr. Max Pitz
Thilo Krill

Hints

1. Every number in $[0, 1)$ has a unique binary expansion $\frac{a_0}{2} + \frac{a_1}{2^2} + \frac{a_1}{2^3} + \dots$ with $a_i \in \{0, 1\}$ for all $i \in \mathbb{N}$.

2. -

3. It is easier to take a countable set different from $\mathbb{N}$. For example, the vertices of a binary tree. Or the rationals $\mathbb{Q}$.

4. Every disk in the plane contains an element of $\mathbb{Q}^2$.

5. Use Exercise 2 (ii) and the fact that every disk in the plane contains an element of $\mathbb{Q}^2$.

6. From a hypothetical $(0, 0) - (0, \frac{1}{2})$ path construct an odd cycle in $G_1(\mathbb{Q}^2)$.