Correction: Theorem 8.5.3

The following argument, marked blue in the screenshot below, needs one extra step to work:

Theorem 8.5.3. *Every countable rayless graph G has an unfriendly partition.*

Proof. To help with our formal notation, we shall think of a partition of a set V as a map $\pi: V \to \{0,1\}$. We apply induction on the rank of G. When this is zero then G is finite, and an unfriendly partition can be obtained by maximizing the number of edges across the partition. Suppose now that G has rank $\alpha > 0$, and assume the theorem as true for graphs of smaller rank.

Let U be a finite set of vertices in G such that each of the components $C_0, C_1, \dots$ of $G - U$ has rank $< \alpha$. Partition U into the set U_0 of vertices that have finite degree in G, the set U_1 of vertices that have infinitely many neighbours in some C_n, and the set U_2 of vertices that have infinite degree but only finitely many neighbours in each C_n.

For every $n \in \mathbb{N}$ let $G_n := G[U \cup V(C_0) \cup \ldots \cup V(C_n)]$. This is a graph of some rank $\alpha_n < \alpha$, so by induction it has an unfriendly partition π_n. Each of these π_n induces a partition of U. Let π_U be a partition of U induced by π_n for infinitely many n, say for $n_0 < n_1 < \ldots$. Choose n_0 large enough that G_{n_0} contains all the neighbours of vertices in U_0, and the other n_i large enough that every vertex in U_2 has more neighbours in $G_{n_i} - G_{n_{i-1}}$ than in $G_{n_{i-1}}$, for all $i > 0$. Let π be the partition of G defined by letting $\pi(v) := \pi_{n_i}(v)$ for all $v \in G_{n_i} - G_{n_{i-1}}$ and all i, where $G_{n_{-1}} := \emptyset$. Note that $\pi|_U = \pi_{n_0}|_U = \pi_U$.

Let us show that π is unfriendly. We have to check that every vertex is *happy with* π, i.e., that it has at least as many neighbours in the opposite class under π as in its own.[8] To see that a vertex $v \in G - U$ is happy with π, let i be minimal such that $v \in G_{n_i}$ and recall that v was happy with π_{n_i}. As both v and its neighbours in G lie in $U \cup V(G_{n_i} - G_{n_{i-1}})$, and π agrees with π_{n_i} on this set, v is happy also with π. Vertices in U_0 are happy with π, because they were happy with π_{n_0}, and π agrees with π_{n_0} on U_0 and all its neighbours. Vertices in U_1 are also happy. Indeed, every $u \in U_1$ has infinitely many neighbours in some C_n, and hence in some $G_{n_i} - G_{n_{i-1}}$. Then u has infinitely many opposite neighbours in $G_{n_i} - G_{n_{i-1}}$ under π_{n_i}. Since π_{n_i} agrees with π on both U and $G_{n_i} - G_{n_{i-1}}$, our vertex u has infinitely many opposite neighbours also under π. Vertices in U_2, finally, are happy with every π_{n_i}. By our choice of n_i, at least one of their opposite neighbours under π_{n_i} must lie in $G_{n_i} - G_{n_{i-1}}$. Since π_{n_i} agrees with π on both U_2 and $G_{n_i} - G_{n_{i-1}}$, this gives every $u \in U_2$ at least one opposite neighbour under π in every $G_{n_i} - G_{n_{i-1}}$. Hence u has infinitely many opposite neighbours under π, which clearly makes it happy. □

We need to be a little more careful, and replace the blue lines by the following argument:

> Indeed, every $u \in U_1$ has infinitely many neighbours in some C_n, and hence in some $G_{n_i} - G_{n_{i-1}}$. Let n_i be minimal such that $u \in U_1$ has infinitely many neighbours in $G_{n_i} - G_{n_{i-1}}$. Since π_{n_i} was unfriendly, u has infinitely many opposite neighbours in G_{n_i} under π_{n_i}. However, by minimality of n_i, our vertex u has only finitely many neighbours in $G_{n_{i-1}}$ altogether, and hence u must still have infinitely many opposite neighbours in $G_{n_i} - G_{n_{i-1}}$ under π_{n_i}.