Exercise Sheet 6 for Topological Infinite Graph Theory, Summer 2020
(to be discussed on 8. Juni 2020)

1.$^-$ Where in the proof of Theorem 8.8.2 do we use that G is connected?

2. Prove the fundamental theorem for inverse systems (Lemma 8.8.1)

3. Let G be a connected graph. Let $\mathcal{X}$ be the collection of all finite subsets of $E(G)$, and observe that $\mathcal{X}$ is directed by $\subseteq$. For $F \in \mathcal{X}$ consider the finite multi-graph G_F with vertices the connected components of $G - F$, and edge set F (some of these will be loops in G_F). Naturally, if $F \subseteq F'$, then $G_F \preccurlyeq G_{F'}$, giving compatible bonding maps $f_{F',F}\colon G_{F'} \to G_F$. Hence, $(G_F\colon F \in \mathcal{X})$ forms an inverse system.
Give two proofs of the result that if G is locally finite, then $|G| \cong \lim_{\leftarrow}(G_F\colon F \in \mathcal{X})$:
   (1) A direct proof along the lines of Theorem 8.8.2.
   (2) By combining Theorem 8.8.2 with Lemma 8.8.3.

4. Let G be the countably infinite star. Consider the space $\lim_{\leftarrow}(G_F\colon F \in \mathcal{X})$ from the previous question. Find a subspace of the plane that is homeomorphic to $\lim_{\leftarrow}(G_F\colon F \in \mathcal{X})$.

Bonus (this has nothing to do with inverse limits):

5.$^+$ Let (G, V, Ω) be a connected, not necessarily locally finite graph. Show that the endspace Ω is normal.

Hinweise

1.
2.
3.
4. You could use Lemma 8.8.3 to better see what's going on. The inverse limit is compact.
5.+ Consider two disjoint closed subsets $A, B \subseteq \Omega(G)$, and suppose for a contradiction that they cannot be separated by open sets.
   (1) Show that for any finite set of vertices S, there is at least one component C of $G - S$ such that $A \cap \hat{C}$ and $B \cap \hat{C}$ cannot be separated by open sets.
   (2) Let $a_0 \in A$. Since $a_0 \notin B$ and B is closed, there is a finite S_0 which separates a_0 from B. Use observation (1) to construct a sequence $a_n \in A$ $(n \in \mathbb{N})$ of ends in A that converge to an end $b \in B$ (recall Sheet 1 Q7).