Exercise Sheet 5 for Topological Infinite Graph Theory, Summer 2020
(to be discussed on 25. May 2020)

1. Let $G = (V, E, \Omega)$ be a locally finite connected graph. Call a continuous map $\sigma: S^1 \to X$ a *topological Euler tour* of a connected standard subspace X of $|G|$ if it traverses every edge in $E(X)$ exactly once. (Formally: every inner point of an edge in $E(X)$ must be the image of exactly one point in S^1.) Show that X admits a topological Euler tour if and only if $E(X)$ meets every finite cut of G in an even number of edges.

2. An *open Euler tour* in an infinite connected (not necessarily locally finite) graph G is a 2-way infinite walk $\dots e_{-1} v_0 e_0 \dots$ that contains every edge of G exactly once. Show that G contains an open Euler tour if and only if G is countable, every vertex has even or infinite degree, and any finite cut $F = E(V_1, V_2)$ with both V_1 and V_2 infinite is odd.

3. Let $G = (V, E, \Omega)$ be a locally finite connected graph and suppose you have a sequence $C_n \subseteq G_n$ of compatible cycles in finite minors $G_n \preccurlyeq G$ arising from a cofinal increasing sequence S_n as usual. Is $\bigcup E(C_n)$ necessarily the edge set of a topological circle of $|G|$?

4. By Exercise 23 of Chapter 4, every finite 2-connected graph without a K^4 or $K_{2,3}$ minor contains a Hamilton cycle. Show that every locally finite such graph has a *Hamilton circle*, a circle in $|G|$ containing all the vertices (and ends) of G.
(Note that the Wild Circle graph 8.6.1 is such a graph).

Hinweise

1. Adapt the 2nd method for constructing arcs.
2. The last condition implies that deleting finitely many edges never leaves more than two infinite components (Handshaking Lemma). To prove sufficiency, construct an Euler tour inductively, incorporating at once any finite components arising in the remaining graph. Make a case distinction whether G has an odd cut or not.
3. End degrees.
4. (a) Choose special S_n such that the finite minors G_n arising from contracting components of $G - S_n$ are again 2-connected. (b) Use a compactness argument to find compatible Hamilton cycles $C_n \subseteq G_n$.