Exercise Sheet 2 for Topological Infinite Graph Theory, Summer 2020
(to be discussed on 4. May 2020)

A *circle* in $|G|$ is a subspace homeomorphic to the unit circle S^1, the subspace of $\mathbb{R}^2$ consisting of all points at distance 1 from the origin. A *(topological) circuit* in $|G|$ is the set of all edges of G contained in some circle.

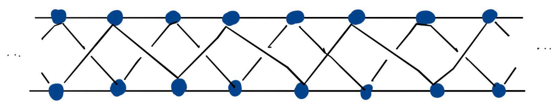
1. Does every infinite locally finite 2-connected graph contain an infinite circuit? Does it contain an infinite bond?

2. Consider a locally finite graph.
   (i) Show that every infinite circuit meets some infinite bond in exactly one edge.
   (ii) Show that every infinite bond meets some infinite circuit in exactly one edge.

Let G be a locally finite connected graph. A *topological Euler tour* is a continuous (but not necessarily injective) map $f\colon S^1 \to |G|$ such that every inner point of an edge of G is the image of exactly one point of S^1. (Thus, every edge is traversed exactly once, and in a straight manner.)

3.$^-$ Show that every topological Euler tour $f\colon S^1 \to |G|$ is surjective.

4.$^-$ Prove that the following graph has a topological Euler tour.



5. Show that if there exists a topological Euler tour $f\colon S^1 \to |G|$, then every finite cut of G is even.

A topological space X is *locally connected* if for every $x \in X$ and every neighbourhood U of x there is an open connected neighbourhood $U' \subseteq U$ of x. A *continuum* is a compact, connectedmetric space. By a theorem of general topology, every locally connected continuum is arc-connected.

6.$^+$ Show that, for G connected and locally finite, every connected closed subspace X of $|G|$ is locally connected. Using the theorem cited above, deduce that every such X is arc-connected.

Hinweise

1. Construct two rays that belong to the same end and start at the same vertex but are otherwise disjoint. This can be done by considering a normal ray and using the fact that none of its vertices is a cutvertex.
2. What the components of the subgraph induced by a circuit? You will need the jumping arc lemma for both parts.
3. That inner points on edges are in the image is part of the definition of a topological Euler tour. What about vertices and ends?
4. Guess a map f and then verify it's continuous.

5.$^-$ Jumping arc lemma.

6.$^+$ Given a closed connected subspace X and an end $\omega \in X$ and a neighbourhood $U = \hat{C}_\epsilon(S, \omega)$, show that $X \cap U$ has at most $|E(S, \omega)|$ many components, which must then be open.