Exercise Sheet 11 for Topological Infinite Graph Theory, Summer 2020
(to be discussed on 13. July 2020)

Let $G = (V, E, \Omega)$ be a locally finite connected graph, $\mathcal{C}$ its topological cycle space, $\mathcal{B}$ its cut space, $\hat{\mathcal{C}}$ its set of circuits and $\hat{\mathcal{B}}$ its set of bonds.

1.$^-$ If a finite set D of edges meets every finite cut of G evenly, must it also meet every infinite cut evenly?

2.$^+$ Write $\hat{\mathcal{C}}$ for the set of circuits in G, and $\hat{\mathcal{B}}$ for the set of bonds.

   (i) Show that every element of $\hat{\mathcal{C}}^\perp$ is a disjoint union of finite bonds, and that every element of $\hat{\mathcal{B}}^\perp$ is a disjoint union of finite circuits.

   (ii) Construct 2-connected graphs with $\mathcal{C}^\perp \subsetneqq \hat{\mathcal{C}}^\perp$ or $\mathcal{B}^\perp \subsetneqq \hat{\mathcal{B}}^\perp$.

3.$^+$ Let (X, V, E) be a metrizable graph-like continuum, $\mathcal{C} = \mathcal{C}(X)$ its topological cycle space, and $\mathcal{B} = \mathcal{B}_{top}(X)$ its topological cut space. Can you generalize Theorem 2.6 (iii) to show that for 2-connected X, we have $\mathcal{C}^\perp = \mathcal{B}$?

4. Find an example of G whose ordinary (meaning maximal number of vertex disjoint rays) end degrees and whose vertex degrees are all even but with $E(G) \notin \mathcal{C}$.

5. Show that the number of odd vertices and ends in some $|G|$ is even or infinite.

Bonus:

6.$^+$ Let $G = (V, E)$ be a locally finite connected graph. The kth power G^k is the graph with vertex set V, and $vw \in E(G^k)$ if their distance in G is at most k. Show that G^3 has a topological Hamilton circle.

7.$^+$ Prove Bruhn & Stein's conjecture on weakly even degrees for $D = E(G)$ in the case where G has at most countably many ends.

Hinweise

1.$^-$ Review all you know about the cycle space.

2.$^+$ (i) For the first assertion, show first that every set $F \in \hat{\mathcal{C}}^\perp$ is a cut. Then delete a maximal set of disjoint finite bonds from F to obtain a cut F', and show that $F' = \emptyset$. The second, dual, part of (i) is similar except for the last step, where a spanning tree will come in handy.

(ii) Any set $F \in \hat{\mathcal{C}}^\perp \setminus \mathcal{C}^\perp$ will be an infinite cut that meets every circuit evenly. It is not hard to construct G and F so that F meets no circuit infinitely. A trick in the construction, which works for 2-connected but not for 3-connected graphs (for which it is unknown whether $\hat{\mathcal{C}}^\perp = \mathcal{C}^\perp$), will prevent it from meeting a circuit oddly. The construction of a 2-connected graph with $\mathcal{C}^{*\perp} \subsetneqq \hat{\mathcal{B}}^\perp$ can use the dual trick.

3. First find a proof for Theorem 2.6(iii) that uses TSTs instead of NSTs.

4.

5.

6.$^+$ For finite graphs, this is easily proved by induction, see Exercise 10.14. First, argue why it suffices to consider the case $G = T$ a tree. Also, show that $\Omega(T)$ is homeomorphic to $\Omega(T^3)$. Finally, pick a root $r \in T$ and let $S_n = B_n(r)$ be the ball of radius n around the root, and consider the usual minors $G_n = T_n$ with respect to S_n. Can you find compatible Hamilton cycles for the minors $(T_n)^3$?

7.$^+$ Find disjoint neighbourhoods $U_1, \dots, U_n$ with $|\partial U_i|$ even covering the endspace and contract every U_i.