Exercise Sheet 0 for Topological Infinite Graph Theory, Summer 2020
(to be discussed on 20. April 2020)

The purpose of this exercise sheet is a quick revision of some fundamental topological facts.

1. A collection $\mathcal{B}$ of subsets of a set X is a *basis* for a topology on X if (1) for every $x \in X$ there is $B \in \mathcal{B}$ with $x \in B$, and (2) whenever $x \in B_1 \cap B_2$ for $B_i \in \mathcal{B}$ there is B_3 with $x \in B_3 \subseteq B_1 \cap B_2$. The unions of elements in $\mathcal{B}$ are the open sets of the topology.
   - Verify that in a metric space (X, d), the collection $\mathcal{B}$ of ε-balls $B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$ for $\varepsilon > 0$ and $x \in X$ is a basis.
   - What about the closed ε-balls $\overline{B}_\varepsilon(x) = \{y \in X : d(x, y) \leq \varepsilon\}$?

2. A topological space X is *Hausdorff* if distinct points of X lie in disjoint open sets. The space X is *normal* if for any disjoint closed sets C, D there are disjoint open sets U, V with $C \subseteq U$ and $D \subseteq V$.
   - Show that every metric space is Hausdorff.
   - Show that every metric space is normal.

3. A space X is *compact* if every open cover of X has a finite subcover.
   - Show that a closed subset $Y \subseteq X$ of a compact space X is compact in the subspace topology.
   - Show that if X is Hausdorff and $Y \subseteq X$ is compact, then Y is closed in X.
   - Let $\mathcal{B}$ be a basis for X. Show that X is compact if and only if every open cover consisting of elements from $\mathcal{B}$ has a finite subcover.
   - A family $\mathcal{C}$ of subsets of X has the *finite intersection property* if $C_1 \cap \cdots \cap C_n \neq \emptyset$ for all $n \in \mathbb{N}$ and $C_i \in \mathcal{C}$. Show that a space is compact if and only if every family $\mathcal{C}$ of closed sets with the f.i.p. satisfies $\bigcap \mathcal{C} \neq \emptyset$.

4. A map $f : X \to Y$ is *continuous* if preimages of open sets are open. It is a *homeomorphism* if it is bijective and both f and f^{-1} are continuous.
   - Show that a map $f : X \to Y$ is continuous if and only if preimages of closed sets are closed.
   - Show that if X is compact and $f : X \to Y$ is a continuous surjection, then Y is compact, too.
   - Show that a continuous bijection $f : X \to Y$ between a compact space X and a Hausdorff space Y must be a homeomorphism.

5. Given a Cartesian product $X = \prod_{s \in S} X_s$ of topological spaces X_s, the *product topology* on X is given by the basis $\mathcal{B}$ where elements are of the form $B = \prod_{s \in S} B_s$ with $B_s \subseteq X_s$ open and $B_s \subsetneq X_s$ for only finitely many $s \in S$.
   - Verify that $\mathcal{B}$ is indeed a basis (see Ex 1.)
   - Prove carefully that the Cantor middle third set is homeomorphic to $\{0, 1\}^{\mathbb{N}}$ (the space of all 0-1-sequences) with the product topology.

Hinweise

1.
2. For normality, use the distance function to find for a closed set $C \subset X$ a continuous function $f_C\colon X \to \mathbb{R}$ such that $C = f^{-1}(0)$. Then consider the function $f_C/(f_C + f_D)\colon X \to \mathbb{R}$.
3.
4. For the last item, combine earlier results from Q4 and Q3.
5. The previous exercise helps.