TOPOLOGICAL UBIQUITY OF COUNTABLE TREES

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Infinite Graph Theory, WS2019/20

§1. Introduction

Halin showed in [9] that all trees of maximum degree 3 are $\leqslant$-ubiquitous. Andreae improved this result to show that all *locally finite* trees are $\leqslant$-ubiquitous [1], and asked if his result could be extended to arbitrary trees [1, p. 214]. This was recently answered in the affirmative [3]:

Theorem 1. *Every tree is ubiquitous with respect to the topological minor relation.*

The purpose of these notes, which are essentially a trimmed-to-purpose version of [3], is to give a self-contained proof of Theorem 1 in the countable case.

§2. Preliminaries

We agree on the following notation.

- When H is a subdivision of G we write $G \leqslant^* H$. Then, $G \leqslant \Gamma$ means that there is a subgraph $H \subseteq \Gamma$ which is a subdivision of G, that is, $G \leqslant^* H$. If H is a subdivision of G and v a vertex of G, then we denote by $H(v)$ the corresponding vertex in H. More generally, given a subgraph $G' \subseteq G$, we denote by $H(G')$ the corresponding subdivision of G' in H.
- A *rooted graph* is a pair (G, v) where G is a graph and $v \in V(G)$ is a vertex of G which we call the *root*. Often, when it is clear from the context which vertex is the root of the graph, we will refer to a rooted graph (G, v) as simply G.
- Given a rooted tree (T, v), we define a partial order $\leq$, which we call the *tree-order*, on $V(T)$ by letting $x \leq y$ if the unique path between y and v in T passes through x. See [7, Section 1.5] for more background.
- For any edge $e \in E(T)$ we denote by e^- the endpoint closer to the root and by e^+ the endpoint further from the root.
- For any vertex t we denote by $N^+(t)$ the set of *children of* t in T, the neighbours s of t satisfying $t \leq s$.
- The subtree of T rooted at t is denoted by (T_t, t), that is, the induced subgraph of T on the set of vertices $\{s \in V(T) \colon t \leq s\}$. When the context is clear, we simply write T_t.

- We say that a rooted tree (S, w) is a *rooted subtree* of a rooted tree (T, v) if S is a subgraph of T such that the tree order on (S, w) agrees with the induced tree order from (T, v). In this case we write $(S, w) \subseteq_r (T, v)$.
- A rooted tree (S, w) is a *rooted topological minor* of a rooted tree (T, v) if there is a subgraph S' of T which is a subdivision of S such that for any $x \leq y \in V(S)$, $S'(x) \leq S'(y)$ in the tree-order on T. We call such an S' a *rooted subdivision of* S. In this case we write $(S, w) \leqslant_r (T, v)$, cf. [7, Section 12.2].

§3. Well-quasi-orders and ω-embeddability

Definition 2 (well-quasi-order). *A binary relation $\lhd$ on a set X is a* well-quasi-order *if it is reflexive and transitive, and for every sequence $x_1, x_2, \ldots \in X$ there is some $i < j$ such that $x_i \lhd x_j$.*

Lemma 3 (ω-embeddability). *If $\lhd$ is a well-quasi-order on a set X, then for every infinite sequence $(x_n)_{n \in \mathbb{N}}$ in X there is $N \in \mathbb{N}$ such that for every x_n with $n \geqslant N$ there are infinitely many later x_m with $x_n \lhd x_m$.*

Proof. Otherwise, if no N_i satisfies the assertion of the lemma, we inductively find a sequence $n_1 < N_1 < n_2 < N_2 < \cdots$ such that $x_{n_i} \ntriangleleft x_m$ for any $m \geqslant N_i$. But then $(x_{n_i})_{i \in \mathbb{N}}$ witnesses that $\lhd$ is not a well-quasi-order. □

We will use the following theorem of Nash-Williams on well-quasi-ordering of rooted trees, and its extension by Laver to labelled rooted trees.

Theorem 4 (Nash-Williams [11]). *The relation $\leqslant_r$ is a well-quasi order on the set of rooted trees.*

Theorem 5 (Laver [10]). *The relation $\leqslant_r$ is a well-quasi order on the set of rooted trees with finitely many labels, i.e. for every finite number $k \in \mathbb{N}$, whenever $(T_1, c_1), (T_2, c_2), \ldots$ is a sequence of rooted trees with k-colourings $c_i \colon T_i \to [k]$, there is some $i < j$ such that there exists a subdivision H of T_i with $H \subseteq_r T_j$ and $c_i(t) = c_j(H(t))$ for all $t \in T_i$.*

Together with Lemma 3 these results give us the following three corollaries:

Corollary 6. *Let (T, v) be a countable rooted tree, $t \in V(T)$ a vertex of infinite degree and $(t_i \in N^+(t) \colon i \in \mathbb{N})$ an enumeration of its countably many children. Then there exists $N_t \in \mathbb{N}$ such that for all $n \geqslant N_t$,*

$$\{t\} \cup \bigcup_{i > N_t} T_{t_i} \leqslant_r \{t\} \cup \bigcup_{i > n} T_{t_i}$$

(considered as trees rooted at t) fixing the root t.

Proof. Consider a labelling $c \colon T_t \to [2]$ mapping t to 1, and all remaining vertices of T_t to 2. By Theorem 5, the set $\mathcal{T} = \{\{t\} \cup \bigcup_{i > n} T_{t_i} \colon n \in \mathbb{N}\}$ is well-quasi-ordered by $\leqslant_r$ respecting the labelling, and so the claim follows by applying Lemma 3 to $\mathcal{T}$. □

Definition 7 (Self-similarity)**.** *A ray $R = r_1r_2r_3\dots$ in a rooted tree (T, v) which is upwards with respect to the tree order* displays self-similarity of T *if there are infinitely many n such that there exists a subdivision H of T_{r_1} with $H \subseteq_r T_{r_n}$ and $H(R) \subseteq R$.*

Corollary 8. *Let (T, v) be an infinite rooted tree and let $R = r_1r_2r_3\dots$ be a ray which is upwards with respect to the tree order. Then there is a $k \in \mathbb{N}$ such that r_kR displays self-similarity of T.*

Proof. Consider a labelling $c\colon T \to [2]$ mapping the vertices on the ray R to 1, and labelling all remaining vertices of T with 2. By Theorem 5, the set $\mathcal{T} = \{(T_{r_i}, c_i)\colon i \in \mathbb{N}\}$, where c_i is the natural restriction of c to T_{r_i}, is well-quasi-ordered by $\leqslant_r$ respecting the labellings. Now consider the N provided by Lemma 3. Then for every T_{r_k} with $k \geqslant N$, there are infinitely many $r_j \in r_kR$ such that $T_{r_k} \leqslant_r T_{r_j}$ respecting the labelling, i.e. mapping the ray to the ray, and hence r_kR displays the self similarity of T. □

§4. Linkages between rays

In this section we will establish a toolkit for constructing a disjoint system of paths from one family of disjoint rays to another.

Definition 9 (Tail of a ray)**.** *Given a ray R in a graph Γ and a finite set $X \subseteq V(\Gamma)$ the* tail of R after X, *denoted by $T(R, X)$, is the unique infinite component of R in $\Gamma - X$.*

Definition 10 (Linkage of families of rays)**.** *Let $\mathcal{R} = (R_i\colon i \in I)$ and $\mathcal{S} = (S_j\colon j \in J)$ be families of vertex disjoint rays, where the initial vertex of each R_i is denoted x_i. A family of paths $\mathcal{P} = (P_i\colon i \in I)$, is a* linkage *from $\mathcal{R}$ to $\mathcal{S}$ if there is an injective function $\sigma\colon I \to J$ such that*

- *each P_i joins a vertex $x_i' \in R_i$ to a vertex $y_{\sigma(i)} \in S_{\sigma(i)}$;*
- *the family $\mathcal{T} = (x_iR_ix_i'P_iy_{\sigma(i)}S_{\sigma(i)}\colon i \in I)$ is a collection of disjoint rays.*

We say that $\mathcal{T}$ is obtained by transitioning *from $\mathcal{R}$ to $\mathcal{S}$ along the linkage $\mathcal{P}$. Given a finite set of vertices $X \subseteq V(\Gamma)$, we say that $\mathcal{P}$ is* after X *if $x_i' \in T(R_i, X)$ and $x_i'P_iy_{\sigma(i)}S_{\sigma(i)}$ avoids X for all $i \in I$.*

Lemma 11 (Weak linking lemma)**.** *Let Γ be a graph and $\epsilon \in \Omega(\Gamma)$. Then for any families $\mathcal{R} = (R_i\colon i \in [n])$ and $\mathcal{S} = (S_j\colon j \in [n])$ of vertex disjoint rays in ϵ and any finite set X of vertices, there is a linkage from $\mathcal{R}$ to $\mathcal{S}$ after X.*

Proof. Let us write x_i for the initial vertex of each R_i and let x_i' be the initial vertex of the tail $T(R_i, X)$. Furthermore, let $X' = X \cup \bigcup_{i\in[n]} R_ix_i'$. For $i \in [n]$ we will construct inductively finite disjoint connected subgraphs $K_i \subseteq \Gamma$ for each $i \in [n]$ such that

- K_i meets $T(S_j, X')$ and $T(R_j, X')$ for every $j \in [n]$;
- K_i avoids X'.

Suppose that we have constructed $K_1, \dots, K_{m-1}$ for some $m \leq n$. Let us write $X_m = X' \cup \bigcup_{i<m} V(K_i)$. Since $R_1, \dots, R_n$ and $S_1, \dots, S_n$ lie in the same end ϵ, there exist paths $Q_{i,j}$ between

$T(R_i, X_m)$ and $T(S_j, X_m)$ avoiding X_m for all $i \neq j \in [n]$. Let $K_m = F \cup \bigcup_{i \neq j \in [n]} Q_{i,j}$, where F consists of an initial segment of each $T(R_i, X_m)$ sufficiently large to make K_m connected. Then it is clear that K_m is disjoint from all previous K_i and satisfies the claimed properties.

Let $K = \bigcup_{i=1}^{n} K_i$ and for each $j \in [n]$, let y_j be the initial vertex of $T(S_j, V(K))$. Note that by construction $T(S_j, V(K))$ avoids X for each j, since K_1 meets $T(S_j, X)$ and so $T(S_j, V(K)) \subseteq T(S_j, X)$.

We claim that there is no separator of size $< n$ between $\{x'_1, \ldots, x'_n\}$ and $\{y_1, \ldots, y_n\}$ in the subgraph $\Gamma' \subseteq \Gamma$ where $\Gamma' = K \cup \bigcup_{j=1}^{n} T(R_j, X') \cup T(S_j, X')$. Indeed, any set of $< n$ vertices must avoid at least one ray R_i, at least one graph K_m and one ray S_j. However, since K_m is connected and meets R_i and S_j, the separator does not separate x'_i from y_j.

Hence, by a version of Menger's theorem for infinite graphs [7, Proposition 8.4.1], there is a collection of n disjoint paths P_i from x'_i to $y_{\sigma(i)}$ in Γ'. Since Γ' is disjoint from X and meets each $R_i x'_i$ in x'_i only, it is clear that $\mathcal{P} = (P_i \colon i \in [n])$ is as desired. □

Lemma 12 (Strong linking lemma)**.** *Let Γ be a graph and $\epsilon \in \Omega(\Gamma)$. Let X be a finite set of vertices, $n \in \mathbb{N}$, and $\mathcal{R} = (R_i \colon i \in [n])$ a family of vertex disjoint rays in ϵ. Let x_i be the initial vertex of R_i and let x'_i the initial vertex of the tail $T(R_i, X)$.*

Then there is a finite number $N = N(\mathcal{R}, X)$ with the following property: For every collection $(H_j \colon j \in [N])$ of vertex disjoint connected subgraphs of Γ, all disjoint from X and each including a specified ray S_j in ϵ, there is a linkage $\mathcal{P} = (P_i \colon i \in [n])$ from $\mathcal{R}$ to $(S_j \colon j \in [N])$ which is after X and such that

$$\mathcal{T} = \big(x_i R_i x'_i P_i y_{\sigma(i)} S_{\sigma(i)} \colon i \in [n]\big)$$

avoids at least one H_i.

Proof. Let $X' = X \cup \bigcup_{i \in [n]} R_i x'_i$ and let $N_0 = |X'|$. We claim that the lemma holds with $N = N_0 + n^3 + 1$.

Indeed suppose that $(H_j \colon j \in [N])$ is a collection of vertex disjoint subgraphs as in the statement of the lemma. Since the H_j are vertex disjoint, we may assume without loss of generality that the family $(H_j \colon j \in [n^3 + 1])$ is disjoint from X'.

For each $i \in [n^2]$ we will build inductively finite, connected, vertex disjoint subgraphs $\hat{K}_i$ such that

- $\hat{K}_i$ meets $T(R_{i \pmod{n}}, X')$;
- $\hat{K}_i$ meets exactly n of the H_j, that is $|\{\, j \in [n^3 + a] \,:\, \hat{K}_i \cap H_j \neq \emptyset\}| = n$, and
- $\hat{K}_i$ avoids X'.

Suppose we have done so for all $i < m$. Let $X_m = X' \cup \bigcup_{i<m} V(\hat{K}_i)$. We will build inductively for $t = 0, \ldots, n$ increasing connected subgraphs $\hat{K}^t_m$ that meet $R_{i \pmod{n}}$, meet exactly t of the H_j, and avoid X_m.

We start with $\hat{K}^0_m = \emptyset$. For each $t = 0, \ldots n-1$, if $T(R_{m \pmod{n}}, X_m)$ meets some H_j not met by $\hat{K}^t_m$ then there is some initial vertex $z_t \in T(R_{m \pmod{n}}, X_m)$ where it does so and we set $\hat{K}_m^{t+1} := \hat{K}_m^t \cup T(R_{m \pmod n}, X_m)z_t$. Otherwise we may assume $T(R_{m \pmod n}, X_m)$ does not meet any such H_j. In this case, let $j \in [n^3+a]$ be such that $\hat{K}_m^t \cap H_j = \emptyset$. Since $R_{m \pmod n}$ and S_j belong to the same end ϵ, there is some path P between $T(R_{m \pmod n}, X_m)$ and $T(S_j, X_m)$ which avoids X_m. Since this path meets some H_k with $k \in [n^3+1]$ which $\hat{K}_m^t$ does not, there is some initial segment P' which meets exactly one such H_k. To form $\hat{K}_m^{t+1}$ we add this path to $\hat{K}_m^t$ together with an appropriately large initial segment of $T(R_{m \pmod n}, X_m)$ such that $\hat{K}_m^{t+1}$ is connected. Finally we let $\hat{K}_m = \hat{K}_m^n$.

Let $K = \bigcup_{i \in [n^2]} \hat{K}_i$. Since each $\hat{K}_i$ meets exactly n of the H_j, the set

$$J = \{j \in [n^3+1] \, : \, H_j \cap K \neq \emptyset\}$$

satisfies $|J| \leqslant n^3$. For each $j \in J$ let y_j be the initial vertex of $T(S_j, V(K))$.

We claim that there is no separator of size $< n$ between $\{x_1', \dots x_n'\}$ and $\{y_j \, : \, j \in J\}$ in the subgraph $\Gamma' \subseteq \Gamma$ where $\Gamma' = K \cup \bigcup_{j \in [n]} T(R_j, X') \cup \bigcup_{j \in J} H_j$. Suppose for a contradiction that there is such a separator S. Then S cannot meet every R_i, and hence avoids some R_q. Furthermore, there are n distinct $\hat{K}_i$ such that $i = q \pmod n$, all of which are disjoint. Hence there is some $\hat{K}_r$ with $r = q \pmod n$ disjoint from S. Finally, $|\{j \in J \, : \, \hat{K}_r \cap H_j \neq \emptyset\}| = n$ and so there is some H_s disjoint from S such that $\hat{K}_r \cap H_s \neq \emptyset$. Since $\hat{K}_r$ meets $T(R_q, X')$ and H_s, there is a path from x_q' to y_s in Γ', contradicting our assumption.

Hence, by a version of Menger's theorem for infinite graphs [7, Proposition 8.4.1], there is a family of disjoint paths $\mathcal{P} = (P_i \colon i \in [n])$ in Γ' from x_i' to $y_{\sigma(i)}$. Furthermore, since $|J| \leqslant n^3$ there is some subset $A \subseteq [n^3+a]$ of size a such that H_k is disjoint from K for each $k \in A$.

Therefore, since Γ' is disjoint from X' and meets each $R_i x_i'$ in x_i' only, the family $\mathcal{P}$ is a linkage from $\mathcal{R}$ to $(S_j)_{j \in [n^3+a]}$ which is after X such that

$$\mathcal{T} = \left(x_i R_i x_i' P_i y_{\sigma(i)} S_{\sigma(i)} \colon i \in [n]\right)$$

avoids H_i for $i \in [n^3+1] \setminus J$. □

§5. G-tribes and concentration of G-tribes towards an end

For showing that a given graph G is ubiquitous with respect to a fixed relation $\triangleleft$, we shall assume that $nG \triangleleft \Gamma$ for every $n \in \mathbb{N}$ and need to show that this implies that $\aleph_0 G \triangleleft \Gamma$. Since each subgraph witnessing that $nG \triangleleft \Gamma$ will be a collection of n disjoint subgraphs each being a witness for $G \triangleleft \Gamma$, it will be useful to introduce some notation for talking about these families of collections of n disjoint witnesses for each n.

To do this formally, recall that we write $G \leqslant^* H$ if H is a subdivision of G and $G \leqslant \Gamma$ if G is a topological minor of Γ.

Definition 13 (G-tribes)**.** *Let G and Γ be graphs.*

- *A G-*tribe *in Γ is a collection $\mathcal{F}$ of finite sets F (called* layer*) of disjoint subgraphs H of Γ such that $G \leqslant^* H$ for each* member of $\mathcal{F}$*, i.e. for each $H \in \bigcup \mathcal{F}$.*

- *A G-tribe $\mathcal{F}$ in Γ is called* thick*, if for each $n \in \mathbb{N}$ there is a* layer *$F \in \mathcal{F}$ with $|F| \geqslant n$; otherwise, it is called* thin.
- *A G-tribe $\mathcal{F}'$ in Γ is a G-*subtribe *of a G-tribe $\mathcal{F}$ in Γ, denoted by $\mathcal{F}' \lhd \mathcal{F}$, if there is an injection $\Psi \colon \mathcal{F}' \to \mathcal{F}$ such that for each $F' \in \mathcal{F}'$ there is an injection $\varphi_{F'} \colon F' \to \Psi(F')$ such that $V(H') \subseteq V(\varphi_{F'}(H'))$ for each $H' \in F'$. The G-subtribe $\mathcal{F}'$ is called* flat*, denoted by $\mathcal{F}' \subseteq \mathcal{F}$, if there is such an injection Ψ satisfying $F' \subseteq \Psi(F')$.*
- *A thick G-tribe $\mathcal{F}$ in Γ is* concentrated *at an end ϵ of Γ, if for every finite vertex set X of Γ, the G-tribe $\mathcal{F}_X = \{F_X \colon F \in \mathcal{F}\}$ consisting of the layers $F_X = \{H \in F : H \not\subseteq C(X, \epsilon)\} \subseteq F$ is a thin subtribe of $\mathcal{F}$.*

We first observe that removing a thin G-tribe from a thick G-tribe always leaves a thick G-tribe.

Lemma 14. *Let $\mathcal{F}$ be a thick G-tribe in Γ and let $\mathcal{F}'$ be a thin subtribe of $\mathcal{F}$, witnessed by $\Psi \colon \mathcal{F}' \to \mathcal{F}$ and $(\varphi_{F'} \colon F' \in \mathcal{F}')$. For $F \in \mathcal{F}$, if $F \in \Psi(\mathcal{F}')$, let $\Psi^{-1}(F) = \{F'_F\}$ and set $\hat{F} = \varphi_{F'_F}(F'_F)$. If $F \notin \Psi(\mathcal{F}')$, set $\hat{F} = \emptyset$. Then*

$$\mathcal{F}'' := \{F \setminus \hat{F} \colon F \in \mathcal{F}\}$$

is a thick flat G-subtribe of $\mathcal{F}$.

Proof. $\mathcal{F}''$ is obviously a flat subtribe of $\mathcal{F}$. As $\mathcal{F}'$ is thin, there is a $k \in \mathbb{N}$ such that $|F'| \leq k$ for every $F' \in \mathcal{F}'$. Thus $|\hat{F}| \leq k$ for all $F \in \mathcal{F}$. Let $n \in \mathbb{N}$. As $\mathcal{F}$ is thick, there is a layer $F \in \mathcal{F}$ satisfying $|F| \geq n + k$. Thus $|F \setminus \hat{F}| \geq n + k - k = n$. □

Given a thick G-tribe, the members of this tribe may have different properties, for example, some of them contain a ray belonging to a specific end ϵ of Γ whereas some of them do not. The next lemma allows us to restrict onto a thick subtribe, in which all members have the same properties, as long as we consider only finitely many properties. E.g. we find a subtribe in which either all members contain an ϵ-ray, or none of them contain such a ray.

Lemma 15 (Pigeon hole principle for thick G-tribes)**.** *Suppose for some $k \in \mathbb{N}$, we have a k-colouring $c \colon \bigcup \mathcal{F} \to [k]$ of the members of some thick G-tribe $\mathcal{F}$ in Γ. Then there is a monochromatic, thick, flat G-subtribe $\mathcal{F}'$ of $\mathcal{F}$.*

Proof. Since $\mathcal{F}$ is a thick G-tribe, there is a sequence $(n_i \colon i \in \mathbb{N})$ of natural numbers and a sequence $(F_i \in \mathcal{F} \colon i \in \mathbb{N})$ such that

$$n_1 \leqslant |F_1| < n_2 \leqslant |F_2| < n_3 \leqslant |F_3| < \cdots .$$

Now for each i, by pigeon hole principle, there is one colour $c_i \in [k]$ such that the subset $F'_i \subseteq F_i$ of elements of colour c_i has size at least n_i/k. Moreover, since $[k]$ is finite, there is one colour $c^* \in [k]$ and an infinite subset $I \subseteq \mathbb{N}$ such that $c_i = c^*$ for all $i \in I$. But this means that $\mathcal{F}' := \{F'_i \colon i \in I\}$ is a monochromatic, thick, flat G-subtribe. □

Lemma 16. *Suppose Γ contains a thick G-tribe $\mathcal{F}$ for some connected G. Then either $\aleph_0 G \lhd \Gamma$, or there is a thick flat subtribe $\mathcal{F}'$ of $\mathcal{F}$ and an end ϵ of Γ such that $\mathcal{F}'$ is concentrated at ϵ.*

Proof. For every finite vertex set $X \subseteq V(\Gamma)$, only a thin subtribe of $\mathcal{F}$ can meet X, so by Lemma 14 a thick flat subtribe $\mathcal{F}''$ is contained in the graph $\Gamma - X$. Since each member of $\mathcal{F}''$ is connected, any member H of $\mathcal{F}''$ is contained in a unique component of $\Gamma - X$. If for any X, infinitely many components of $\Gamma - X$ contain a subdivision of G, the union of all these copies is a subdivided copy of $\aleph_0 G$ in Γ. Thus, we may assume that for each X, only finitely many components contain elements from $\mathcal{F}''$, and hence, by colouring each H with a colour corresponding to the component of $\Gamma - X$ containing it, we may assume by the pigeon hole principle for G-tribes, Lemma 15, that at least one component of $\Gamma - X$ contains a thick flat subtribe of $\mathcal{F}$.

Let $C_0 = \Gamma$ and $\mathcal{F}_0 = \mathcal{F}$ and consider the following recursive process: If possible, we choose a finite vertex set X_n in C_n such that there are two components $C_{n+1} \neq D_{n+1}$ of $C_n - X_n$ where C_{n+1} contains a thick flat subtribe $\mathcal{F}_{n+1} \subseteq \mathcal{F}_n$ and D_{n+1} contains at least one subdivided copy H_{n+1} of G. Since by construction all H_n are pairwise disjoint, we either find infinitely many such H_n and thus an $\aleph_0 G \leqslant \Gamma$, or our process terminates at step N say. That is, we have a thick flat subtribe $\mathcal{F}_N$ contained in a subgraph C_N such that there is no finite vertex set X_N satisfying the above conditions.

Let $\mathcal{F}' := \mathcal{F}_N$. We claim that for every finite vertex set X of Γ, there is a unique component C_X of $\Gamma - X$ that contains a thick flat G-subtribe of $\mathcal{F}'$. Indeed, note that if for some finite $X \subseteq \Gamma$ there are two components C and C' of $\Gamma - X$ both containing thick flat G-subtribes of $\mathcal{F}'$, then since every G-copy in $\mathcal{F}'$ is contained in C_N, it must be the case that $C \cap C_N \neq \emptyset \neq C' \cap C_N$. But then $X_N = X \cap C_N \neq \emptyset$ is a witness that our process could not have terminated at step N.

Next, observe that whenever $X' \supseteq X$, then $C_{X'} \subseteq C_X$. By the *direction theorem* of Diestel and Kühn, [8], it follows that there is a unique end ϵ of Γ such that $C(X, \epsilon) = C_X$ for all finite $X \subseteq \Gamma$. It now follows easily from the uniqueness of $C_X = C(X, \epsilon)$ that $\mathcal{F}'$ is concentrated at this ϵ. □

We note that concentration towards an end ϵ is a robust property in the following sense:

Lemma 17. *Let G be a connected graph and Γ a graph containing a thick connected G-tribe $\mathcal{F}$ concentrated at an end ϵ of Γ. Then the following assertions hold:*

(1) For every finite set X, the component $C(X, \epsilon)$ contains a thick flat G-subtribe of $\mathcal{F}$.
(2) Every thick subtribe $\mathcal{F}'$ of $\mathcal{F}$ is concentrated at ϵ, too.

Proof. Let X be a finite vertex set. By definition, if the G-tribe $\mathcal{F}$ is concentrated at ϵ, then $\mathcal{F}$ is thick, and the subtribe $\mathcal{F}_X$ consisting of the sets $F_X = \{H \in F : H \not\subseteq C(X, \epsilon)\} \subseteq F$ for $F \in \mathcal{F}$ is a thin subtribe of $\mathcal{F}$, i.e. there exists $k \in \mathbb{N}$ such that $|F_X| \leqslant k$ for all $F_X \in \mathcal{F}_X$.

For (1), observe that the G-tribe $\mathcal{F}' = \{F \setminus F_X : F \in \mathcal{F}\}$ is a thick flat subtribe of $\mathcal{F}$ by Lemma 14, and all its members are contained in $C(X, \epsilon)$ by construction.

For (2), observe that if $\mathcal{F}'$ is a subtribe of $\mathcal{F}$, then for every $F' \in \mathcal{F}'$ there is an injection $\varphi_{F'} \colon F' \to F$ for some $F \in \mathcal{F}$. Therefore, $|\varphi_{F'}^{-1}(F_X)| \leqslant k$ for $F_X \subseteq F$ as defined above, and so only a thin subtribe of $\mathcal{F}'$ is not contained in $C(X, \epsilon)$. □

§6. Countable subtrees

In this section we prove the countable version of Theorem 1. Let T be a countable tree. By Lemma 16, we may assume without loss of generality that there are an end ϵ of Γ and a thick T-tribe $\mathcal{F}$ concentrated at ϵ.

Without loss of generality, we may assume that ϵ is undominated in Γ. Indeed, an end of Γ is dominated by infinitely many distinct vertices if and only if Γ contains a subdivision of $K_{\aleph_0}$ [7, Exercise 19, Chapter 8], in which case proving ubiquity becomes trivial:

Lemma 18. *For any countable graph G, we have $\aleph_0 \cdot G \subseteq K_{\aleph_0}$.*

Proof. By partitioning the vertex set of $K_{\aleph_0}$ into countably many infinite parts, we see that $\aleph_0 \cdot K_{\aleph_0} \subseteq K_{\aleph_0}$. Also, clearly $G \subseteq K_{\aleph_0}$. Hence, we have $\aleph_0 \cdot G \subseteq \aleph_0 \cdot K_{\aleph_0} \subseteq K_{\aleph_0}$. □

Therefore, ϵ is only finitely dominated, but then, if X denotes the vertices dominating ϵ, we may simply work in the connected graph $C(X, \epsilon) \subset \Gamma$, in which now ϵ is undominated and which by concentration still contains a thick T-tribe concentrated at ϵ.

6.1. Preprocessing.

We begin by picking a root v for T. Let $V_\infty(T)$ be the set of vertices of infinite degree in T.

Definition 19. *Given T as above, define a locally finite subtree $T^* \subseteq T$ by*

$$T^* := T \setminus \bigcup_{t \in V_\infty(T)} \{T_{t_i} \colon t_i \in N^+(t), i > N_t\},$$

where N_t is as in Corollary 6.

Definition 20. *An edge e of T^* is an* extension edge *if there is a ray in T^* starting at e^+ which displays self-similarity of T.*[1] *For each extension edge e we fix one such a ray R_e. Write $Ext(T^*) \subseteq E(T^*)$ for the set of extension edges.*

Consider the forest $T^* - Ext(T^*)$ obtained from T^* by removing all extension edges. Since every ray in T^* must contain an extension edge by Corollary 8, each component of $T^* - Ext(T^*)$ is a locally finite rayless tree and so is finite. We enumerate the components of $T^* - Ext(T^*)$ as $T_0^*, T_1^*, \dots$ in such a way that for every $n \geqslant 0$, the set

$$T_n := T\left[\bigcup_{i \leqslant n} V(T_i^*)\right]$$

[1] Recall that all such rays by definition go upwards with respect to the tree order. Also note that it should display self-similarity of all of T, not just of T^*.

is a finite subtree of T^* containing the root r. Let us write $\partial(T_n) = E(T_n, T^* \setminus T_n)$, and note that $\partial(T_n) \subseteq Ext(T^*)$. We make the following definitions:

- For a given T-tribe $\mathcal{F}$ and ray R of T, we say that R *converges to* ϵ *according to* $\mathcal{F}$ if for all members H of $\mathcal{F}$ the ray $H(R)$ is in ϵ. We say that R is *cut from* ϵ *according to* $\mathcal{F}$ if for all members H of $\mathcal{F}$ the ray $H(R)$ is not in ϵ. Finally we say that $\mathcal{F}$ *determines whether* R *converges to* ϵ if either R converges to ϵ according to $\mathcal{F}$ or R is cut from ϵ according to $\mathcal{F}$.
- Given $n \in \mathbb{N}$, we say a thick T-tribe $\mathcal{F}$ *agrees about* $\partial(T_n)$ if for each extension edge $e \in \partial(T_n)$, it determines whether R_e converges to ϵ.
- Since $\partial(T_n)$ is a finite set of edges for all n, it follows from Lemma 15 that given some $n \in \mathbb{N}$, any thick T-tribe has a flat thick T-subtribe $\mathcal{F}$ such that $\mathcal{F}$ agrees about $\partial(T_n)$. Under these circumstances we set

$$\partial_\epsilon(T_n) := \{e \in \partial(T_n) \colon R_e \text{ converges to } \epsilon \text{ according to } \mathcal{F}\},$$
$$\partial_{\neg\epsilon}(T_n) := \{e \in \partial(T_n) \colon R_e \text{ is cut from } \epsilon \text{ according to } \mathcal{F}\}.$$

- Also, under these circumstances, let us write $T_n^{\neg\epsilon}$ for the component of the forest $T - \partial_\epsilon(T_n)$ containing the root of T. Note that $T_n \subseteq T_n^{\neg\epsilon}$.

The following lemma contains a large part of the work needed for our inductive construction.

Lemma 21 (T-tribe refinement lemma)**.** *Suppose we have a thick T-tribe $\mathcal{F}_n$ concentrated at ϵ which agrees about $\partial(T_n)$ for some $n \in \mathbb{N}$. Let f denote the unique edge from T_n to $T_{n+1} \setminus T_n$. Then there is a thick T-tribe $\mathcal{F}_{n+1}$ concentrated at ϵ with the following properties:*

(i) $\mathcal{F}_{n+1}$ *agrees about* $\partial(T_{n+1})$.
(ii) $\mathcal{F}_{n+1} \cup \mathcal{F}_n$ *agree about* $\partial(T_n) \setminus \{f\}$.
(iii) $T_{n+1}^{\neg\epsilon} \supseteq T_n^{\neg\epsilon}$.
(iv) For all $H \in \mathcal{F}_{n+1}$ *there is a finite* $X \subseteq \Gamma$ *such that* $H(T_{n+1}^{\neg\epsilon}) \cap C_\Gamma(X, \epsilon) = \emptyset$.

Moreover, if $f \in \partial_\epsilon(T_n)$, and $R_f = v_0 v_1 v_2 \ldots \subseteq T^*$ (with $v_0 = f^+$) denotes the ray displaying self-similarity of T at f, then we may additionally assume:

(v) For every $H \in \mathcal{F}_{n+1}$ and every $k \in \mathbb{N}$, there is $H' \in \mathcal{F}_{n+1}$ with
  - $H' \subseteq_r H$
  - $H'(T_n) = H(T_n)$,
  - $H'(T_{v_0}) \subseteq_r H(T_{v_k})$, *and*
  - $H'(R_f) \subseteq H(R_f)$.

Proof. Concerning (v), if $f \in \partial_\epsilon(T_n)$ recall that according to Definition 20, the ray R_f satisfies that for all $k \in \mathbb{N}$ we have $T_{v_0} \leqslant_r T_{v_k}$ such that R_f gets embedded into itself. In particular, there is a subtree $\hat{T}_1$ of T_{v_1} which is a rooted subdivision of T_{v_0} with $\hat{T}_1(R_f) \subseteq R_f$, considering $\hat{T}_1$ as a rooted tree given by the tree order in T_{v_1}. If we define recursively for each $k \in \mathbb{N}$ $\hat{T}_k = \hat{T}_{k-1}(\hat{T}_1)$ then it is clear that $(\hat{T}_k \colon k \in \mathbb{N})$ is a family of rooted subdivisions of T_{v_0} such that for each $k \in \mathbb{N}$

- $\hat{T}_k \subseteq T_{v_k}$;
- $\hat{T}_k \supseteq \hat{T}_{k+1}$;
- $\hat{T}_k(R_f) \subseteq R_f$.

Hence, for every subdivision H of T with $H \in \bigcup \mathcal{F}_n$ and every $k \in \mathbb{N}$, the subgraph $H(\hat{T}_k)$ is also a rooted subdivision of T_{v_0}. Let us construct a subdivision $H^{(k)}$ of T by letting $H^{(k)}$ be the minimal subtree of H containing $H(T \setminus T_{v_0}) \cup H(\hat{T}_k)$, where $H^{(k)}(T \setminus T_{v_0}) = H(T \setminus T_{v_0})$ and $H^{(k)}(T_{v_0}) = H(\hat{T}_k)$. Note that

$$H^{(k)}(T_{v_0}) = H(\hat{T}_k) \subseteq_r H^{(k-1)}(T_{v_0}) = H(\hat{T}_{k-1}) \subseteq_r \ldots \subseteq_r H(T_{v_k}).$$

In particular, for every subdivision $H \in \bigcup \mathcal{F}_n$ of T and every $k \in \mathbb{N}$, there is a subdivision $H^{(k)} \subseteq H$ of T such that $H^{(k)}(T_n^{\neg\epsilon}) = H(T_n^{\neg\epsilon})$, $H^{(k)}(T_{v_0}) \subseteq_r H(T_{v_k})$, and $H^{(k)}(R_f) \subseteq H(R_f)$. By the pigeon hole principle, there is an infinite index set $K_H = \{k_1^H, k_2^H, \ldots\} \subseteq \mathbb{N}$ such that $\{\{H^{(k)}\} \colon k \in K_H\}$ agrees about $\partial(T_{n+1})$. Consider the thick subtribe $\mathcal{F}'_n = \{F'_i \colon F \in \mathcal{F}_n, i \in \mathbb{N}\}$ of $\mathcal{F}_n$ with

$$(\dagger) \qquad F'_i := \{H^{(k_i^H)} \colon H \in F\}.$$

Observe that $\mathcal{F}'_n \cup \mathcal{F}_n$ still agrees about $\partial(T_n)$. (If $f \in \partial_{\neg\epsilon}(T_n)$, then skip this part and simply let $\mathcal{F}'_n := \mathcal{F}_n$.)

Concerning (iii), observe that for every $H \in \bigcup \mathcal{F}'_n$, since the rays $H(R_e)$ for $e \in \partial_{\neg\epsilon}(T_n)$ do not tend to ϵ, there is a finite vertex set X_H such that $H(R_e) \cap X_H = \emptyset$ for all $e \in \partial_{\neg\epsilon}(T_n)$. Furthermore, since X_H is finite, for each such extension edge e there exists $x_e \in R_e$ such that

$$H(T_{x_e}) \cap C(X_H, \epsilon) = \emptyset.$$

By definition of extension edges, cf. Definition 20, for each $e \in \partial_{\neg\epsilon}(T_n)$ there is a rooted embedding of T_{e^+} into $H(T_{x_e})$. Hence, there is a subdivision $\tilde{H}$ of T with $\tilde{H} \leqslant H$ and $\tilde{H}(T_n) = H(T_n)$ such that $\tilde{H}(T_{e^+}) \subseteq H(T_{x_e})$ for each $e \in \partial_{\neg\epsilon}(T_n)$.

Note that if $e \in \partial_{\neg\epsilon}(T_n)$ and g is an extension edge with $e \leqslant g \in \partial(T_{n+1}) \setminus \partial(T_n)$, then $\tilde{H}(R_g) \subseteq \tilde{H}(T_{e^+}) \subseteq H(Tx_e)$, and so

$$(\ddagger) \qquad \tilde{H}(R_g) \text{ doesn't tend to } \epsilon.$$

Define $\tilde{\mathcal{F}}_n$ to be the thick T-subtribe of $\mathcal{F}'_n$ consisting of the $\tilde{H}$ for every H in $\bigcup \mathcal{F}'_n$. Now use Lemma 15 to chose a maximal thick flat subtribe $\mathcal{F}^*_n$ of $\tilde{\mathcal{F}}_n$ which agrees about $\partial(T_{n+1})$, so it satisfies (i) and (ii). By ($\ddagger$), the tribe $\mathcal{F}^*_n$ satisfies (iii), and by maximality and ($\dagger$), it satisfies (v).

In our last step, we now arrange for (iv) while preserving all other properties. For each $H \in \bigcup \mathcal{F}^*_n$, since $H(T_{n+1})$ is finite and ϵ undominated, we may find a finite separator Y_H such that

$$H(T_{n+1}) \cap (Y_H \cup C(Y_H, \epsilon)) = \emptyset.$$

Since Y_H is finite, for every vertex $t \in V(T_{n+1}) \cap V_\infty(T)$, say with $N^+(t) = (t_i)_{i \in \mathbb{N}}$, there exists $n_t \in \mathbb{N}$ such that $C(Y_H, \epsilon) \cap H(T_{t_j}) = \emptyset$ for all $j \geqslant n_t$. Using Corollary 6, for every such t there is a rooted embedding

$$\{t\} \cup \bigcup_{j>N_t} T_{t_j} \leqslant_r \{t\} \cup \bigcup_{j>n_t} T_{t_j}.$$

fixing the root t. Hence there is a subdivision H' of T with $H' \leqslant H$ such that $H'(T^*) = H(T^*)$ and for every for every vertex $t \in V(T_{n+1}) \cap V_\infty(T)$

$$H'\left[\{t\} \cup \bigcup_{j>N_t} T_{t_j}\right] \cap (Y_H \cup C(Y_H, \epsilon)) = \emptyset.$$

Moreover, note that by construction of $\tilde{F}_n$, every such H' automatically satisfies that

$$H(T_{e^+}) \cap C(X_H \cup Y_H, \epsilon) = \emptyset$$

for all $e \in \partial_{\neg\epsilon}(T_{n+1})$. Let $\mathcal{F}_{n+1}$ consist of the set of H' as defined above for all $H \in \mathcal{F}_n^*$. Then $X_H \cup Y_H$ is a finite separator witnessing that $\mathcal{F}_{n+1}$ satisfies (iv). □

6.2. **The construction.** So let T be a countable tree. Recall that we may assume that there are an undominated end ϵ of Γ and a thick T-tribe $\mathcal{F}$ concentrated at ϵ.

Definition 22 (Bounder, extender)**.** *Suppose that some thick T-tribe $\mathcal{F}$ which is concentrated at ϵ agrees about $\partial(T_n)$ for some given $n \in \mathbb{N}$, and $Q_1^n, Q_2^n, \ldots, Q_n^n$ are disjoint subdivisions of $T_n^{\neg\epsilon}$ (note, $T_n^{\neg\epsilon}$ depends on $\mathcal{F}$).*

- *A* bounder *for the $(Q_i^n : i \in [n])$ is a finite set X of vertices in Γ separating all the Q_i from ϵ, i.e. such that*

$$C(X, \epsilon) \cap \bigcup_{i=1}^{n} Q_i^n = \emptyset.$$

- *An* extender *for the $(Q_i^n : i \in [n])$ is a family $\mathcal{E}_n = (E_{e,i}^n : e \in \partial_\epsilon(T_n), i \in [n])$ of rays in Γ tending to ϵ which are disjoint from each other and also from each Q_i^n except at their initial vertices, and where the start vertex of $E_{e,i}^n$ is $Q_i^n(e^-)$.*

To prove Theorem 1 for T, we now assume inductively that for some $n \in \mathbb{N}$, with $r := \lfloor n/2 \rfloor$ and $s := \lceil n/2 \rceil$ we have:

(1) A thick T-tribe $\mathcal{F}_r$ in Γ concentrated at ϵ which agrees about $\partial(T_r)$, with a boundary $\partial_\epsilon(T_r)$ such that $T_{r-1}^{\neg\epsilon} \subseteq T_r^{\neg\epsilon}$.

(2) a family $(Q_i^n : i \in [s])$ of s pairwise disjoint subdivisions of $T_r^{\neg\epsilon}$ in Γ with $Q_i^n(T_{r-1}^{\neg\epsilon}) = Q_i^{n-1}$ for all $i \leqslant s-1$,

(3) a bounder X_n for the $(Q_i^n : i \in [s])$, and

(4) an extender $\mathcal{E}_n = (E_{e,i}^n : e \in \partial_\epsilon(T_r^{\neg\epsilon}), i \in [s])$ for the $(Q_i^n : i \in [s])$.

The base case $n = 0$ it easy, as we simply may choose $\mathcal{F}_0 \leqslant_r \mathcal{F}$ to be any thick T-subtribe in Γ which agrees about $\partial(T_0)$, and let all other objects be empty.

So, let us assume that our construction has proceeded to step $n \geqslant 0$. Our next task splits into two parts: First, if $n = 2k-1$ is odd, we extend the already existing k subdivisions $(Q_i^n : i \in [k])$

of $T_{k-1}^{\neg\epsilon}$ to subdivisions $(Q_i^{n+1}\colon i \in [k])$ of $T_k^{\neg\epsilon}$. And secondly, if $n = 2k$ is even, we construct a further disjoint copy Q_{k+1}^{n+1} of $T_k^{\neg\epsilon}$.

Construction part 1: $n = 2k-1$ **is odd.** By assumption, $\mathcal{F}_{k-1}$ agrees about $\partial(T_{k-1})$. Let f denote the unique edge from T_{k-1} to $T_k \setminus T_{k-1}$. We first apply Lemma 21 to $\mathcal{F}_{k-1}$ in order to find a thick T-tribe $\mathcal{F}_k$ concentrated at ϵ satisfying properties (i)–(v). In particular, $\mathcal{F}_k$ agrees about $\partial(T_k)$ and $T_{k-1}^{\neg\epsilon} \subseteq T_k^{\neg\epsilon}$

We first note that if $f \notin \partial_\epsilon(T_{k-1})$, then $T_{k-1}^{\neg\epsilon} = T_k^{\neg\epsilon}$, and we can simply take $Q_i^{n+1} := Q_i^n$ for all $i \in [k]$, $\mathcal{E}_{n+1} := \mathcal{E}_n$ and $X_{n+1} := X_n$.

Otherwise, we have $f \in \partial_\epsilon(T_{k-1})$. By Lemma 17(2) $\mathcal{F}_k$ is concentrated at ϵ, and so we may pick a collection $\{H_1, \ldots, H_N\}$ of disjoint subdivisions of T from some $F \in \mathcal{F}_k$, all of which are contained in $C(X_n, \epsilon)$, where $N = |\mathcal{E}_n|$. By Lemma 11 there is some linkage $\mathcal{P} \subseteq C(X_n, \epsilon)$ from

$$\mathcal{E}_n \text{ to } (H_j(R_f)\colon j \in [N]),$$

which is after X_n. Let us suppose that the linkage $\mathcal{P}$ joins a vertex $x_{e,i} \in E_{e,i}^n$ to $y_{\sigma(e,i)} \in H_{\sigma(e,i)}(R_f)$ via a path $P_{e,i} \in \mathcal{P}$. Let $z_{\sigma(e,i)}$ be a vertex in R_f such that $y_{\sigma(e,i)} \leqslant H_{\sigma(e,i)}(z_{\sigma(e,i)})$ in the tree order on $H_{\sigma(e,i)}(T)$.

By property (v) of $\mathcal{F}_k$ in Lemma 21, we may assume without loss of generality that for each H_j there is a another member $H'_j \subseteq H_j$ of $\mathcal{F}_k$ such that $H'_j(T_{f^+}) \subseteq_r H_j(T_{z_j})$. Let $\hat{P}_j \subseteq H'_j$ denote the path from $H_j(y_j)$ to $H'_j(f^+)$.

Now for each $i \in [k]$, define

$$Q_i^{n+1} = Q_i^n \cup E_{f,i}^n x_{f,i} P_{f,i} y_{\sigma(f,i)} \hat{P}_{\sigma(f,i)} \cup H'_{\sigma(f,i)}(T_k^{\neg\epsilon} \setminus T_{k-1}^{\neg\epsilon}).$$

By construction, each Q_i^{n+1} is a subdivision of $T_k^{\neg\epsilon}$.

By Lemma 21(iv) we may find a finite set $X_{n+1} \subseteq \Gamma$ with $X_n \subseteq X_{n+1}$ such that

$$C(X_{n+1}, \epsilon) \cap \Big(\bigcup_{i \in [k]} Q_i^{n+1} \Big) = \emptyset.$$

This set X_{n+1} will be our bounder.

Define an extender $\mathcal{E}_{n+1} = (E_{e,i}^{n+1}\colon e \in \partial_\epsilon(T_k), i \in [k])$ for the Q_i^{n+1} as follows:

- For $e \in \partial_\epsilon(T_{k-1}) \setminus \{f\}$, let $E_{e,i}^{n+1} := E_{e,i}^n x_{e,i} P_{e,i} y_{\sigma(e,i)} H_{\sigma(e,i)}(R_f)$.
- For $e \in \partial_\epsilon(T_k) \setminus \partial(T_{k-1})$, let $E_{e,i}^{n+1} := H'_{\sigma(e,i)}(R_e)$.

Since each $H_{\sigma(e,i)}, H'_{\sigma(e,i)} \in \bigcup \mathcal{F}_k$, and $\mathcal{F}_k$ determines that R_f converges to ϵ, these rays belong indeed to the end ϵ. Furthermore, since $H'_{\sigma(e,i)} \subseteq H_{\sigma(e,i)}$ and $\{H_1, \ldots, H_N\}$ are disjoint, it follows that the rays are disjoint.

Construction part 2: $n = 2k$ **is even.** If $\partial_\epsilon(T_k) = \emptyset$, then $T_k^{\neg\epsilon} = S$, and so picking any element Q_{k+1}^{n+1} from $\mathcal{F}_k$ with $Q_{k+1}^{n+1} \subseteq C(X_n, \epsilon)$ gives us a further copy of S disjoint from all the previous ones. Using Lemma 21(iv), there is a suitable bounder $X_{n+1} \supseteq X_n$ for Q_{k+1}^{n+1}, and we are done. Otherwise, pick $e_0 \in \partial_\epsilon(T_k)$ arbitrary.

Since $\mathcal{F}_k$ is concentrated at ϵ, we may pick a collection $\{H_1, \dots, H_N\}$ of disjoint subdivisions of T from $\mathcal{F}_k$ all contained in $C(X_n, \epsilon)$, where N is large enough so that we may apply Lemma 12 to find a linkage $\mathcal{P} \subseteq C(X_n, \epsilon)$ from

$$\mathcal{E}_n \text{ to } (H_i(R_{e_0})\colon i \in [N]),$$

after X_n, avoiding say H_1. Let us suppose the linkage $\mathcal{P}$ joins a vertex $x_{e,i} \in E^n_{e,i}$ to $y_{\sigma(e,i)} \in H_{\sigma(e,i)}(R_{e_0})$ via a path $P_{e,i} \in \mathcal{P}$. Define

$$Q^{n+1}_{k+1} = H_1(T_k^{\neg\epsilon}).$$

Note that Q^{n+1}_{k+1} is a T-suitable subdivision of $T_k^{\neg\epsilon}$.

By Lemma 21(iv) there is a finite set $X_{n+1} \subseteq \Gamma$ with $X_n \subseteq X_{n+1}$ such that $C(X_{n+1}, \epsilon) \cap Q^{n+1}_{k+1} = \emptyset$. This set X_{n+1} will be our new bounder.

Define the extender $\mathcal{E}_{n+1} = (E^{n+1}_{e,i}\colon e \in \partial_\epsilon(T_{k+1}), i \in [k+1])$ of ϵ-rays as follows:

- For $i \in [k]$, let $E^{n+1}_{e,i} := E^n_{e,i} x_{e,i} P_{e,i} y_{\sigma(e,i)} H_{\sigma(e,i)}(R_{e_0})$.
- For $i = k+1$, let $E^{n+1}_{e,k+1} := H_1(R_e)$ for all $e \in \partial_\epsilon(T_{k+1})$.

Once the construction is complete, let us define $H_i := \bigcup_{n \geqslant 2i-1} Q^n_i$. Since $\bigcup_{n\in\mathbb{N}} T_n^{\neg\epsilon} = T$, and due to the extension property (2), the collection $(H_i)_{i\in\mathbb{N}}$ is a topological minor of $\aleph_0 T$ in Γ, and the proof is complete. □

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