FRAÏSÉ LIMITS

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Definition 1. *A* language *L is a countable family of symbols R_i, $i \in I$, which represent relations, and symbols f_j, $j \in J$, which represent functions. To each symbol we associate a particular arity, i.e. a positive integer $n(i)$ for relation symbols, or a nonnegative integer $m(j)$ for function symbols.*

Definition 2. *A* structure *for a language L is a set A together with a family of relations $R_i^A \subseteq A^{n(i)}$, one for each $i \in I$, and a family of functions $f_j^A : A^{m(j)} \to A$, one for each $j \in J$. The relations R_i^A and functions f_j^A are called* interpretations *of the relation and function symbols in L. If L has no function symbols, then we call A a relational structure.*

Examples. (1) If $L = \emptyset$, then the structures for L are just sets.

(2) If $L = \{<\}$, where $<$ is a relation symbol of arity 2, then linearly ordered sets are examples of structures in L. So are partially ordered sets.

(3) Suppose L again contains just one relation symbol E of arity 2. A graph $G = (V, E)$ is a structure in L on the set V where E is an irreflexive, symmetric relation on V specifying the edges. This can easily be extended to multigraphs.

(4) Let $L = \{D_q : q \in \mathbb{Q} \cap (0, \infty)\}$ consist of binary relation symbols. A rational metric space (X, d) is a structure in L, where we understand that $D_q^X(x, y)$ holds if and only if $d(x, y) = q$ for all $x, y \in X$.

(5) A Boolean algebra is a structure for two binary function symbols $\vee$ and $\wedge$, a unary function symbol $\neg$, and constant symbols 0 and 1.

Definition 3. *An injective (bijective) map $f : A \to B$ between two structures A and B for the same language L is called an* embedding (isomorphism) *if for every relation symbol $R_i \in L$, we have*

$$(a_1, ..., a_{n(i)}) \in R_i^A \leftrightarrow (f(a_1), ..., f(a_{n(i)})) \in R_i^B$$

and similar for all functions symbols.

Definition 4. *A structure F is called* locally finite *if every finite subset of F is contained in some finite substructure $D \subseteq F$.*

Remark. All relational structures are locally finite, as every finite subset induces a substructure. Groups are examples, where for given finite sets, need to consider the generated subgroup.

Definition 5. *A structure F is called* homogenous *if every isomorphism between finite substructures $A \subseteq F$ and $B \subseteq F$ extends to an automorphism of F.*

Definition 6. *If A is a structure, then the* age *of A, denoted $Age(A)$, is the class of all finite structures which are isomorphic to a substructure of A.*

Theorem 7. *Let F be a countable locally finite structure. The following are equivalent:*

(1) F is homogeneous.

(2) F has the finite extension property*, i.e., whenever $A, B \in Age(F)$, and $A \subseteq B$, every embedding of A into F extends to an embedding of B into F.*

Proof. For (1) $\Rightarrow$ (2), consider any embedding $f\colon A \hookrightarrow F$. Since $B \in Age(F)$, there is also an embedding $h\colon B \hookrightarrow F$, and there is another copy $h[A]$ of A in $h[B] \subset F$. Since F is homogeneous, the partial isomorphism $f \circ h^{-1}$ between $h[A]$ and $f[A]$ extends to an automorphism α of F. But then is clear that $\alpha \circ h$ is the desired extendion of f.

For (2) $\Rightarrow$ (1), consider any partial isomorphism f_0 between two finite substructures A_0 and A'_0 of F. Since F is countable and locally finite, it is clear that we can write $\bigcup_{n\in\mathbb{N}} A_n = F = \bigcup_{n\in\mathbb{N}} A'_n$ each as the union of a chain of finite substructures starting with A_0 and A'_0 respectively. We will define the isomorphism f in countably many steps using a back-and-forth argument. Suppose a partial isomorphism $f_n\colon B_n \to C_n$ is already defined. At even steps: Let k be minimal such that $B_n \subset A_k$ and use the finite extension property to extend f_n to A_{k+1}. At odd steps: Let k be minimal such that $C_n \subset A'_k$ and use the finite extension property to extend f_n^{-1} to A'_{k+1}. □

Theorem 8. *Every two countable locally finite homogenous structures in the same language, having the same age, are isomorphic.*

Proof. Consider two such structure N and N' with $Age(N) = Age(N')$. Since both are countable and locally finite, we can write $N = \bigcup_{n\in\mathbb{N}} A_n$ and $N' = \bigcup_{n\in\mathbb{N}} A'_n$ each as the union of a chain of finite substructures.

We will define the isomorphism f in countably many steps using a back-and-forth argument. Suppose a partial isomorphism $f_n\colon B_n \to C_n$ is already defined.

At even steps: Let k be minimal such that $B_n \subset A_k$. Since $B_n \subset A_{k+1} \in Age(N')$, the embedding f_n extends to a partial isomorphism f_{n+1} with domain A_{k+1} by the finite extension property of N' (Theorem 7).

At odd steps: Let k be minimal such that $C_n \subset A'_k$ and extend f_n to a partial isomorphism f_{n+1} with codomain A'_{k+1} using a symmetric argument. □

Lemma 9. *If N is a countable, locally finite homogeneous L-structure, then $\mathcal{C} = Age(N)$ satisfies the following properties:*

(1) $\mathcal{C}$ is closed under isomorphisms;
(2) $\mathcal{C}$ is closed under taking substructures;
(3) $\mathcal{C}$ contains structures of arbitrarily high finite cardinality;
(4) $\mathcal{C}$ satisfies the joint embedding property*, i.e., whenever $A, B \in \mathcal{C}$, then there is $D \in C$ containing both (isom. copies of) A and B as substructures; and*
(5) $\mathcal{C}$ satisfies the amalgamation property*, i.e. whenever $f_1 : A \to B_1$ and $f_2 : A \to B_2$ are embeddings of structures in $\mathcal{C}$, then there is $D \in \mathcal{C}$ and embeddings $g_1 : B_1 \to D$ and $g_2 : B_2 \to D$ such that $g_1 \circ f_1 = g_2 \circ f_2$.*

Proof. We check the amalgamation property (the others are easy).

Assume $A, B_1, B_2 \subset N$. Since N is homogeneous, the maps

$$f_i\colon A_i \to f(A_i) \subset B_i$$

extend to automorphisms $\hat{f}_i$ of N. Since N is locally finite, there is a finite substructure D of N containing $\hat{f}_1^{-1}(B_1) \cup \hat{f}_2^{-1}(B_2)$, and D with $g_i := \hat{f}_i^{-1} \restriction B_i$ is as desired. □

Definition 10. *Any class $\mathcal{C}$ of finite L-structures satisfying (1) - (5) is called an* amalgamation class *or* Fraissé class*. It is called* essentially countable *if it contains at most countably many isomorphism types.*

Theorem 11 (Fraïssé 1954)**.** *Every essentially countable amalgamation class $\mathcal{C}$ of finite L-structures is the age of a unique countable, locally finite homogeneous L-structure, which is called the Fraïssé-limit of $\mathcal{C}$.*

Examples. The set of finite (K^r-free) graphs is an essentially countable amalgamation class with Fraissé-limit R (R^r). Similarly, the set of finite linear orders is an essentially countable amalgamation class with Fraissé-limit $(Q, <)$.

Proof. Enumerate representatives of isomorphism types in $\mathcal{C}$ as $A_0, A_1, A_2, \ldots$.

Step I: Construct a chain $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ of structures $M_n \in \mathcal{C}$ such that for all $n \in \mathbb{N}$ we have

(1) M_n has a substructure isomorphic to A_n, and
(2) whenever $i, j \leqslant n$ and $\alpha\colon A_i \to A_j$ and $f\colon A_i \to M_n$ are embeddings, there is an embedding $g\colon A_j \to M_{n+1}$ such that $f = g \circ \alpha$.

Start by taking $M_0 = A_0$. Suppose that some finite chain $M_0 \subseteq \cdots \subseteq M_n$ has been constructed as required. Enumerate the finitely many, say r many, pairs of embeddings $(\alpha_k \colon A_{i_k} \to A_{j_k},\ f_k \colon A_{i_k} \to M_n)_{k=1}^r$ with $i_k, j_k \leqslant n$. Using the joint embedding property, find B_0 containing M_n and A_{n+1}. Using the amalgamation property, we build inductively a chain $M_n \subseteq B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots \subseteq B_r =: M_{n+1}$ where for $1 \leqslant k \leqslant r$, we have that B_k is the amalgam of B_{k-1} and A_{j_k} over A_{i_k} via the embeddings α_k and f_k. Then M_{n+1} is as desired.[1]

Step II: Put $M = \bigcup_{n \in \mathbb{N}} M_n$. It remains to check that

- M is countable and locally finite.

Follows since countable union of finite sets is countable. Moreover, and finite set is contained in some finite substructure M_n.

- $Age(M) = \mathcal{C}$.

'$\subseteq$' Let $A \subset M$ be a finite substructure. Then $A \subset M_n$ for some M_n, and since $M_n \in \mathcal{C}$ has the hereditary property, we have $A \in \mathcal{C}$. '$\supseteq$' Any $A \in \mathcal{C}$ is isomorphic to some $A \cong A_n$, and $A_n \subset M_n$ by construction, so $A \in Age(M)$ by definition.

- M has the 'finite extension property' (then M is homogeneous by Theorem 7, and uniqueness follows from Theorem 8).

If $A \subseteq B \in Age(M) = \mathcal{C}$ and $f \colon A \hookrightarrow M$ is an embedding, need to show that f extends to embedding $g \colon B \hookrightarrow M$. But $A \cong A_i$ and $B \cong A_j$ and $f \colon A \hookrightarrow M_n$ for some large enough M. The identity map $id \colon A \hookrightarrow B$ translates to some embedding $\alpha \colon A_i \to A_j$ and $f \colon A_i \hookrightarrow M_n$, so by construction, there is an extension $g \colon A_j \to M_{n+1}$. $\square$

[1]To see that without loss of generality, the M_n form a chain, we may always pretend that $M_n, B_i \subset \mathbb{N}$ as a set. When we get a new amalgam $B_i \hookrightarrow B_{i+1}$, then simply relabel the new elements of B_{i+1} by unused integers.