LAVIOLETTE'S CUT-FAITHFUL DECOMPOSITIONS AND NASH-WILLIAMS' DECOMPOSITION THEOREM

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A family of graphs $\mathcal{H}$ forms a *decomposition*[1] of a graph G if the graphs in $\mathcal{H}$ are pairwise edge-disjoint subgraphs with $G = \bigcup \{H \colon H \in \mathcal{H}\}$. In this case, an element $H \in \mathcal{H}$ is also called a *fragment* of the decomposition $\mathcal{H}$. Following Thomassen [7], a subgraph $H \subset G$ is called *cut-faithful* if every finite bond of H is also a finite bond of G.

Theorem 1 (Laviolette's Theorem (2005) [2])**.** *Every graph has a decomposition into countable, cut-faithful graphs.*[2]

We will prove Laviolette's theorem in a minute. But let us first illustrate its power in reducing problems to their countable case by deducing an early classic from the theory of infinite graphs. A finite cut F in a graph G us called *even* or *odd* depending on whether $|F|$ is even or odd respectively. Recall that for *finite* connected graphs, we have the following equivalences:

Theorem 2 (Euler's Theorem)**.** *For a finite connected graph G, the following are equivalent:*

(i) G admits a closed Eulerian walk,
(ii) every vertex of G has even degree,
(iii) every (finite) cut of G is even, and
(iv) G has a decomposition into cycles.

The double ray shows that the equivalence $(ii) \Leftrightarrow (iii)$ no longer holds for infinite graphs. But a classic theorem of Nash-Williams says that $(iii) \Leftrightarrow (iv)$ continues to hold.

Theorem 3 (Nash-Williams' Theorem (1960) [3])**.** *A graph has a decomposition into cycles if and only if it contains no odd cut.*

Proof. Since every cycle meets every cut in an even number of edges, the necessity of our condition is clear. Conversely, by Laviolette's Theorem we may assume that our graph G is countable; let us enumerate its edges. We shall find the desired cycles in countably many steps. At each step $n \in \mathbb{N}$ we shall assume inductively that every cut in the remaining graph G_n is infinite or even; find some new cycle $C_n \subset G_n$. And delete its edges $G_{n+1} = G_n - E(C_n)$.

[1] The word *decomposition* traditionally refers to edge-disjoint subgraphs, the word *partition* to vertex-disjoint subgraphs.

[2] We remark that Laviolette's notion is a little stronger; first, it works for general cardinals, and second, Laviolette required that every countable cut of G is contained in a single graph $H \in \mathcal{H}$.

To find that cycle, consider the first remaining edge $e = xy$ in our enumeration. By our inductive assumption, e is not a bridge of G_n, so there is a $x-y$ path in G_n not using e. Together with e this path forms a cycle C_n. The deletion of C_n keeps every finite cut even and every infinite cut infinite, so our inductive assumptions continue to hold for G_{n+1}.

After countably many steps no edges remain, so $\{C_n : n \in \mathbb{N}\}$ is a decomposition. $\square$

We remark that Theorem 1 and Theorem 3 are closely related: We have seen that the first implies the latter, but in fact, Laviolette relied on Nash-Williams' theorem in the proof of his result, and hence did not provide an independent proof of Theorem 3. Direct proofs of Laviolette's (and hence Nash-Williams') theorem were found by Soukup [5] using the theory of elementary submodels, and a truly elementary (but tricky) proof was later found by Thomassen [7].

We are now ready to prove Theorem 1. We follow Soukup's proof idea from [5], but replace the elementary submodel argument by an explicit closure-operation, similar to the direct proof of Nash-Williams theorem in [1].

Proof. By induction on $|G|$. If G is countable, there is nothing to do, as the one-element decomposition $\mathcal{H} = \{G\}$ does the job.

So assume that $|G| = \kappa$ is uncountable. Define a sequence of cardinals $(\kappa_\alpha : \alpha < cf(\kappa))$ as follows. If $\kappa = \mu^+$, set $\kappa = 0$ and $\kappa_\alpha = \mu$ for all other α. Otherwise, let $\kappa_0 = 0$ and $(\kappa_\alpha : 0 < \alpha < cf(\kappa))$ be a strictly increasing continuous sequence of infinite cardinals with supremum κ.

For $v \neq w \in V(G)$ fix a maximal[3] system $\mathcal{P}(v,w)$ of edge-disjoint $v-w$ paths of size $\lambda_G(v,w)$ (the edge-connectivity of G between v and w). Fix an enumeration $\left\{P^i_{v,w} : i < \lambda_G(v,w)\right\}$.

Write G as a continuous, increasing union of induced subgraphs $G = \bigcup_{\alpha < cf(\kappa)} G_\alpha$ such that for all $\alpha < cf(\kappa)$ we have

(1) $|G_\alpha| = \kappa_\alpha$,
(2) if $v \neq w \in G_\alpha$ and $F \subset E(G_\alpha)$ is finite and there is a $v-w$ path in G avoiding F, then there is such a path in G_α,
(3) if $v \neq w \in G_\alpha$ and $F \subset E(G_{\alpha+1})$ is finite and there is a G_α-path[4] in G from v to w avoiding F, then there is such a G_α-path in $G_{\alpha+1}$, and
(4) for $v \neq w \in G_\alpha$, we have $P^i_{v,w} \subset G_\alpha$ for all $i < \kappa_\alpha$.

For the construction, pick an enumeration $V(G) = \{v_i : i < \kappa\}$, and find an increasing sequence of ordinals $\beta_\alpha < \kappa$ with supremum κ and $|\beta_\alpha| = \kappa_\alpha$. We will make sure that $\{v_i : i < \beta_\alpha\} \subset G_\alpha$. Define G_α by transfinite recursion on α. Let G_0 be the empty graph. For a limit λ, define $G_\lambda = \bigcup_{\alpha<\lambda} G_\alpha$. Since $\lambda < cf(\kappa)$, we have indeed $|G_\lambda| < |G|$. Since our sequence of cardinals is continuous, we have (1) for G_λ, and the other properties follow from the fact that $G_\alpha \subset G_\beta$ for all $\alpha < \beta < \lambda$. Lastly, suppose that G_α has already be defined according to the above requirements;

[3]Zorn's Lemma.

[4]For the purposes of this proof, an H-path is a path P in G which intersects H in start- and endvertex, and is edge-disjoint from H (but may have interior vertices in H).

our task is to define $G_{\alpha+1}$. We will do this in countably many steps, and define a chain of graphs $G_\alpha \subset G^0 \subset G^1 \subset \cdots$ and put $G_{\alpha+1} = \bigcup_{n\in\mathbb{N}} G^n$.

First, let $G^0 = G[G_\alpha \cup \{v_i \colon i < \beta_{\alpha+1}\}]$. Suppose inductively that G^n is already defined with $|G^n| = \kappa_{\alpha+1}$. Form G^{n+1} as follows:

- For every $v \neq w \in G^n$ and $F \subset E(G^n)$ as in (2) add in one such path to G^{n+1}.
- For every $v \neq w \in G_\alpha$ and $F \subset E(G^n)$ as in (3) add in one such G_α-path to G^{n+1}.
- For every $v \neq w \in G^n$ as in (4) add in all paths $P^i_{v,w}$ for $i < \kappa_{\alpha+1}$ to G^{n+1}.

Note that since there are only $\kappa_{\alpha+1}$ many such pairs $v \neq w \in G^n$ and only $\kappa_{\alpha+1}$ many finite edge sets $F \subset E(G^n)$, and for each case we are adding at most $\kappa_{\alpha+1}$ many new vertices. Hence, $|G^{n+1}| = \kappa_{\alpha+1}$. Finally, since $\kappa_{\alpha+1}$ is infinite, we have $|G_{\alpha+1}| = \aleph_0 \cdot \kappa_{\alpha+1} = \kappa_{\alpha+1}$ as desired. Lastly, it is easy to check that properties (2) – (4) are satisfied for $G_{\alpha+1}$: for (2) and (4) simply note that any pair $v \neq w \in G_{\alpha+1}$ is contained in some G^n and were therefore made happy in G^{n+1}. For (3), note that any finite $F \subset E(G_{\alpha+1})$ is contained in some $E(G^n)$, and therefore $v, w \in V(G_\alpha)$ were made happy with respect to F in G^{n+1}. This completes the construction.

First, let us see that property (2) implies that each G_α is a cut-faithful subgraph of G. Indeed, consider any finite bond $F \subset G_\alpha$. Then there is a connected component of C of G_α and a vertex partition (A, B) of C such that $F = E_{G_\alpha}(A, B)$. Let A' and B' be the components of $G - F$ containing A and B respectively. If $A' = B'$, there is $v \in A$ and $w \in B$ such that in G there is a $v - w$ path avoiding F, and hence by (2), there would be such a path in G_α, a contradiction. Hence, A' and B' are distinct components of G, and hence $F = E_G(A', B')$, i.e. F is also a bond of G. This shows that G_α is cut-faithful.

Now each G_α is a cut-faithful subgraph of G with $|G_\alpha| < |G|$ for all $\alpha < cf(\kappa)$ by (1), and so has decomposition into countable cut-faithful subgraphs by induction assumption. However, as the different G_α's are not disjoint, it is not clear how to use this.

To make the graphs edge-disjoint, consider the “onion rings” $H_\alpha = G_{\alpha+1} - E(G_\alpha)$ for $\alpha < cf(\kappa)$. Since $G_0 = \emptyset$ and the $\{G_\alpha \colon \alpha < cf(\kappa)\}$ formed a continuous chain, the collection $\mathcal{H} = \{H_\alpha \colon \alpha < cf(\kappa)\}$ forms a decomposition of G such that each $|H_\alpha| < |G|$ for all α by (1). We will show that each H_α is cut-faithful. By applying the induction assumption, we can further decompose G_0 and each H_α individually into countable cut-faithful subgraphs, which together form the desired decomposition for G.

Hence, it only remains to show that each $H_\alpha = G_{\alpha+1} - E(G_\alpha)$ is cut-faithful. This will follow from properties (3) and (4) as follows. Again, consider any finite bond $F \subset H_\alpha$, a connected component of C of H_α and a vertex partition (A, B) of C such that $F = E_{H_\alpha}(A, B)$. Let A' and B' be the components of $G_{\alpha+1}$ containing A and B respectively. If A' and B' are distinct, then F is a bond in $G_{\alpha+1}$ and hence in G by the above. Therefore, $A' = B'$ and hence for some $a \in A$ and $b \in B$ there is an $a - b$ path P in $G_{\alpha+1}$ avoiding F. The path P must use an edge from G_α, and hence at least two distinct vertices from G_α. Let v be the first and w be the last vertex of P in G_α. Note that $v, w \in C$ (in fact, $v \in A$ and $w \in B$ as witnessed by aPv and wPb), and hence there exists an $v-w$-path in C, which is then edge-disjoint from G_α. By property (4), it follows that $\lambda_G(v,w) > \kappa_\alpha$.Since $|E(G_\alpha) \cup F| \leqslant \kappa_\alpha$, one of the paths in $\mathcal{P}(v,w)$ is a G_α-path in G from v to w avoiding F, and hence by property (3), there would be one such G_α-path in $G_{\alpha+1}$, which would be a path in H_α from A to B avoiding F, a contradiction. □

Corollary 4. *Every 2-edge-connected graph has a collection $\mathcal{C}$ of cycles such that every edge of G is in at least one and at most countably many cycles in $\mathcal{C}$.*

Proof. Exercise. □

In [4], Nash-Williams also proved the following well-known orientation theorem.

Theorem 5 (Nash-Williams' Orientation Theorem 1959)**.** *Every $2k$-edge-connected finite graph has a strongly k-edge-connected orientation*

It is a well-known open problem whether Nash-Williams' orientation theorem also holds for infinite graphs (as a partial breakthrough, Thomassen has shown that edge-connectivity $8k$ suffices, [6]). However, using Laviolette's theorem one easily gets:

Corollary 6. *To establish Nash-Williams' orientation theorem for arbitrary graphs it suffices to prove the countable case.*

Proof. Exercise. □

In [3], Nash-Williams also proved the following statements, which can be proved using similar techniques as in our proof of Laviolette's theorem.

Theorem 7. *(1) A graph G can be decomposed into cycles and 2-way infinite tours*[5] *if and only if it has no vertex of odd degree. (2) G is decomposable into 2-way infinite tours if and only if it has no vertex of odd degree and no finite non-trivial component.*

Proof. In both cases, we may assume that G is connected. Since a cut-faithful subgraph of a connected graph is still connected, it looks like Laviolette's theorem allows us to reduce both assertions immediately to the countable case. However, note that even if G has no vertex of odd degree, this need not hold for a cut-faithful subgraph (example?).

Let us call a subgraph $H \subseteq G$ *degree-faithful* if for every vertex $v \in H$ with $0 < \deg_H(v) < \infty$ we have $\deg_H(v) = \deg_G(v)$. We upgrade the construction in Laviolette's theorem so that with G also every G_α and every Y_α has no odd-degree vertex. This can be achieved by fixing for every $v \in V(G)$ an enumeration of its neighbours $N_G(v) = \left\{x_v^i \colon i < \deg_G(v)\right\}$ and add properties

(5) For every $v \in G_\alpha$ we have $x_v^i \in G_\alpha$ for all $i < \kappa_\alpha$, and
(6) For every $v \in G_\alpha$ and every finite $F \subseteq E(G_{\alpha+1})$, if v has a neighbour in $G - G_\alpha - F$ then there is one such neighbour in $G_{\alpha+1} - G_\alpha - F$.

[5] A 2-way infinite tour is a walk $\{e_n = v_n v_{n+1} \colon n \in \mathbb{Z}\}$ without repeated edges; repeated vertices are allowed.

to our list of conditions in the above proof. With these conditions, one can show as above that all G_α and H_α are degree-faithful (in addition to being cut-faithful).

Hence, it suffices to prove both of Nash-Williams assertions for countable graphs G (as our decomposition fragments are now also degree-faithful, it follows easily that no fragment can be finite, or have a finite component), which we leave to the reader as an exercise. $\square$

References

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