Exercise Sheet 9 for Infinite Graph Theory, WS 2019/20
(to be discussed on 16. December 2019)

1.$^-$ Let A, B be two vertex sets in a locally finite connected graph G. Can there be an infinite sequence $\mathcal{P}_1, \mathcal{P}_2, \ldots$ of disjoint A–B paths such that each $\mathcal{P}_{n+1}$ arises from $\mathcal{P}_n$ by applying an alternating walk, and such that some edge $e \in G$ lies in $E[\mathcal{P}_n]$ for infinitely many n but not in $E[\mathcal{P}_n]$ for infinitely many other n?

2. Prove the following strengthening of Lemma 3.3.2: If for a system $\mathcal{P}$ of disjoint $A-B$-paths there is an alternating walk W ending in $B \setminus V[\mathcal{P}]$, then there also exists such an alternating walk W' such that the symmetric difference $E[\mathcal{P}] \triangle E(W')$ is *precisely* the edge-set of a system $\mathcal{P}'$ of disjoint $A-B$-paths with $|\mathcal{P}'| = |\mathcal{P}| + 1$.

3. Let G be a locally finite graph. Let us say that a finite set S of vertices *separates* two ends ω and ω' if $C(S,\omega) \neq C(S,\omega')$. Use Proposition 8.4.1 to show that if ω can be separated from ω' by $k \in \mathbb{N}$ but no fewer vertices, then G contains k disjoint double rays each with one tail in ω and one in ω'. Is the same true for all graphs that are not locally finite?

4. Prove the following more structural version of Exercise 2 on Sheet 8. Let ω be an end of a graph G. Show that either G contains a $TK^{\aleph_0}$ with all its rays in ω, or there are disjoint finite sets $S_0, S_1, \ldots$ such that, if C_i is the component of $G - (S_0 \cup S_i)$ that contains a tail of every ray in ω, we have for all $i < j$ that $C_i \supseteq C_j$ and $G[S_i \cup C_i]$ contains $|S_i|$ disjoint S_i–S_{i+1} paths for all $i \geqslant 1$.

5.$^+$ Is there a planar $\aleph_0$-regular graph all whose ends have infinite vertex-degree?

Hinweise

1.$^-$ Try to construct G together with the $\mathcal{P}_n$.

2. Make sure that $E[\mathcal{P}] \triangle E(W')$ contains no cycles.

3. Look at next exercise and its hint. For locally finite G the sets S'_i are very easy to find, and no normal spanning tree is needed.

4.$^+$ Use a normal spanning tree to find provisional sets $S'_1, S'_2, \dots$ of arbitrary finite cardinality that have the separation properties required of the S_i. Then use these to find the S_i.

5.$^+$ Use the previous exercise.