Exercise Sheet 8 for Infinite Graph Theory, WS 2019/20
(to be discussed on 8. December 2019)

1. Prove that if a given end of a graph contains k disjoint rays for every $k \in \mathbb{N}$ then it contains infinitely many disjoint rays.

2. Prove that if a graph contains k disjoint double rays for every $k \in \mathbb{N}$ then it contains infinitely many disjoint double rays.

3.$^+$ Prove Theorem 8.2.5 (ii).

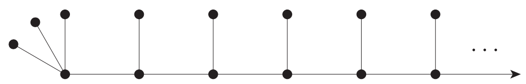
4. Show that, in the ubiquity conjecture, the host graphs G considered can be assumed to be locally finite too.

5. Prove that every locally finite tree T is minor-ubiquitous by proving and combining the following ingredients:
   (i) The grid satisfies $\aleph_0(\mathbb{N} \times \mathbb{N}) \preccurlyeq (\mathbb{N} \times \mathbb{N})$.
   (ii) Every tree is a minor of the grid $\mathbb{N} \times \mathbb{N}$.
   (iii)$^+$ Let $nT \preccurlyeq G$ for all $n \in \mathbb{N}$. Find either $\aleph_0 T \preccurlyeq G$ or a thick end of G.

6. Imitate the proof of Theorem 8.2.6 to find a function $f\colon \mathbb{N} \to \mathbb{N}$ such that whenever an end ω of a graph G contains $f(k)$ disjoint rays, G contains a subdivision of the $k \times \mathbb{N}$ hexagonal grid whose rays all belong to ω.

Optional:

7. Show that the modified comb below is not ubiquitous with respect to the subgraph relation. Does it become ubiquitous if we delete its 3-star on the left?



Hinweise

1. Imitate the proof of Theorem 8.2.5, choosing all the rays used from the given end. Do the rays constructed also belong to that end? If not, how can this be achieved?
2. Imitate the proof of Theorem 8.2.5. Work with rays rather than double rays whenever possible.
3. $^{+}$ Adapt the vertex case so as to construct infinitely many edge-disjoint walks that do not repeat edges and repeat vertices only finitely often. Construct these walks in segments, making each segment vertex-disjoint from all the earlier segments except the last.
4. The task is to find in any graph G that contains arbitrarily many disjoint IH a locally finite subgraph with the same property. In a first step, find a countable such subgraph G', and enumerate its vertices. Then use the enumeration to find a locally finite such subgraph $G'' \subseteq G'$ by ensuring that each vertex of G' is used by only finitely many IH.
5. $^{+}$ For (iii), you could for example use the infinity lemma together with the direction theorem to find an end ω such that $nT \preccurlyeq C(X, \omega)$ for all finite separators X and all $n \in \mathbb{N}$.
6. Unlike in the proof of Theorem 8.2.6, you can use suitable tails of *all* the rays in the (large but finite) set $\mathcal{R}_0$ as rays R_n. The part of the proof that starts with assumption $(**)$ can thus be replaced by a much simpler algorithm that finds R_n and an infinite set of disjoint R_n–$R_{p(n)}$ paths. To determine how many rays are needed, start with a suitable finite analogue to the infinity lemma: any large enough rooted tree either has a vertex with at least k successors or contains a path of length k.
7. To construct a graph that contains arbitrarily but not infinitely many copies of the modified comb T, start with infinitely many disjoint copies of T. Group these into disjoint sets $S_1, S_2, \ldots$ so that S_n is a disjoint union of n copies of T. Then identify vertices from different sets S_n, so as to spoil infinite 'diagonal' sets of disjoint copies of T.