Exercise Sheet 7 for Infinite Graph Theory, WS 2019/20

(to be discussed on 1. December 2019)

1.$^-$ If G is connected and none of its ends is dominated, then G has a normal spanning tree.

2.$^-$ Show that an ω_1-graph has no NST.

3. Prove the following special case of Fodor's Lemma: every decreasing function $f\colon \omega_1 \to \omega_1$ (one such that $f(\alpha) < \alpha$ for all $0 \neq \alpha \in \omega_1$) is constant on some uncountable subset of its domain. Then use this to show that every ω_1-graph contains a $TK^{\aleph_1}$.

4. (i)$^-$ Show that overloaded graphs have no NST.

   (ii) Show that AT-graphs have no NST.

An $(\aleph_0, \aleph_1)$-graph G with bipartition (A, B) is called *divisible* if it contains two disjoint $(\aleph_0, \aleph_1)$-subgraphs, i.e. there are partitions $A = A_1 \dot\cup A_2$ and $B = B_1 \dot\cup B_2$ such that (A_1, B_1) and (A_2, B_2) are both again $(\aleph_0, \aleph_1)$-graphs.

5. (i)$^-$ Show that a T_2 with $\aleph_1$ tops is divisible.

   (ii) Show that every $(\aleph_0, \aleph_1)$-minor of a T_2 with $\aleph_1$ tops is divisible.

The *colouring number* of a graph G is the smallest cardinal κ such that there is a well-ordering $V(G) = \{v_\alpha : \alpha < \mu\}$ in which every vertex v_α has strictly fewer than κ many neighbours amongst the earlier v_β with $\beta < \alpha$. In particular, a graph G has countable colouring number if there is a well-ordering $V(G) = \{v_\alpha : \alpha < \mu\}$ such that every vertex v_α has only finitely many neighbours amongst the earlier v_β with $\beta < \alpha$ (but may have infinitely many later neighbours). This generalizes the concept *Reihenzahl* from the colouring chapter in Diestel's book.

6.$^+$ Show that a graph has a normal spanning tree if and only if each minor has countable colouring number.

Optional:

7.$^+$ Construct, under the Continuum Hypothesis, an indivisible $(\aleph_0, \aleph_1)$-graph.

8.$^{++}$ Prove or disprove that every graph with an end containing uncountably many disjoint rays contains an $(\aleph_0, \aleph_1)$-graph as a minor.

Hinweise

1. Apply a suitable result from the lectures.
2.
3. To prove Fodor's Lemma, use that every countable subset of ω_1 has a supremum $< \omega_1$. Use the special case of Fodor's Lemma to show that every ω_1-graph has uncountably many vertices of uncountable degree. Then construct the desired $TK^{\aleph_1}$ inductively.
4. Levels and separators of the form $\lceil t \rceil$.
5. Which vertices have uncountably many tops above them? For (ii), study how the branch sets of an $(\aleph_0, \aleph_1)$-minor can lie in a binary tree with tops. Find a place where the latter splits that also splits the minor.

6.$^+$ Use the Diestel-Leader Theorem. To check that AT-graphs have uncountable colouring number, note that Fodor's lemma from Q2 remains true for regressive functions with domain all limit ordinals.

7.$^+$ A *free ultrafilter* on $\mathbb{N}$ is a collection $\mathcal{U}$ of infinite subsets of $\mathbb{N}$ which is closed under pairwise intersection such that for every bipartition $\mathbb{N} = A_1 \dot\cup A_2$, either $A_1 \in \mathcal{U}$ or $A_2 \in \mathcal{U}$ (they exist by Zorn's lemma). Using CH, enumerate $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\}$ and build an $(\aleph_0, \aleph_1)$-graph $(\mathbb{N}, B)$ with $B = \{b_\alpha : \alpha < \omega_1\}$ satisfying that $N(b_\alpha)$ is contained, up to finitely many vertices, in each U_β for $\beta < \alpha$ (note there are countably many such β).

8.$^{++}$ A related question is whether it must contain a subdivision of the cartesian product of an uncountable star with a ray? (This is a well-known problem due to Halin)