Exercise Sheet 6 for Infinite Graph Theory, WS 2019/20
(to be discussed on 25. November 2019)

1. Show that any dispersed vertex set in a connected graph G can be covered by a tree which is normal in G.

2. Show that any connected minor of a graph with an NST also has an NST.

3. Use normal spanning trees to show that the following assertions are equivalent for connected countable graphs G.
   (i) G has a locally finite spanning tree.
   (ii) For no finite separator $X \subseteq V(G)$ does $G - X$ have infinitely many components.

   Deduce that every (countable) planar 3-connected graph has a locally finite spanning tree.

4. (i) Show that the vertices of any infinite connected locally finite graph can be enumerated in such a way that every vertex is adjacent to some later vertex.

   (ii) Characterize the class of all graphs which has such a labelling as in (i), countable but not necessarily locally finite, by their separation properties.

5. Prove the following infinite version of the Erdős-Pósa theorem: an infinite graph G either contains infinitely many disjoint cycles or it has a finite set Z of vertices such that $G - Z$ is a forest.

6. Use normal spanning trees to show that a countable connected graph has either countably many or continuum many ($= |2^{\mathbb{N}}|$ many) ends.

Hinweise

1.
2. Use Jung's theorem. (Can you find a direct proof?)$^{++}$
3. Find an equivalent assertion involving normal spanning trees.
4. For simplicity, replace the graph with a spanning tree in it, T say. Which vertices have to appear earlier in the enumeration than others? For (ii), consider a normal spanning tree, or use the previous exercise.
5. Fundamental cycles in normal spanning trees.
6. To prove the assertion for trees, either follow the proof of Proposition 8.5.1 or, more interestingly, assume the tree has uncountably many ends and directly construct a subdivided T_2 inside it.