Exercise Sheet 5 for Infinite Graph Theory, WS 2019/20
(to be discussed on 18. November 2019)

1. The following assertions about rays R, R' in $G = (V, E)$ are equivalent:
   (1) R, R' are not separated by finite $S \subset V$.
   (2) For every finite $S \subset V$, both R and R' have a tail in the same component of $G - S$.
   (3) There exist infinitely many disjoint $R - R'$ paths in G.
   (4) There exists a third ray Q in G with $|Q \cap R| = \infty = |Q \cap R'|$.

2. Show that a locally finite spanning tree of a graph G contains a ray from every end of G.

3. A vertex $v \in G$ is said to *dominate* an end ω of G if any of the following four assertions holds; show that they are all equivalent.
   (i) For some ray $R \in \omega$ there is an infinite v–$(R - v)$ fan in G.
   (ii) For every ray $R \in \omega$ there is an infinite v–$(R - v)$ fan in G.
   (iii) No finite subset of $V(G - v)$ separates v from a ray in ω.
   (iv) $v \in S^*_\omega := \bigcap_{S \in \mathcal{X}} \widehat{C}(S, \omega)$ (where $\widehat{C} = V(C) \cup N(C)$, and $\mathcal{X}$ the collection of finite subsets of $V(G)$).

4. Show that a graph G contains a $TK^{\aleph_0}$ if and only if some end of G is dominated by infinitely many vertices.

An end is *thick* if it contains infinitely many disjoint rays.

5. Construct a countable graph with uncountably many thick ends. Can you find a locally finite such graph?

6. Show that a locally finite connected vertex-transitive graph has exactly 0, 1, 2 or infinitely many ends.

Optional:

7.$^+$ Show that the automorphisms of a graph $G = (V, E)$ act naturally on its ends, i.e., that every automorphism $\sigma\colon V \to V$ can be extended to a map $\sigma\colon \Omega(G) \to \Omega(G)$ such that $\sigma(R) \in \sigma(\omega)$ whenever R is a ray in an end ω. Prove that, if G is connected, every automorphism σ of G maps a finite set of vertices to itself or fixes an end. If σ fixes no finite set of vertices, can it fix more than one end? More than two?

Hinweise

1.
2.
3.
4. For the backwards direction, construct the $TK^{\aleph_0}$ inductively.
5. Start with the binary tree T_2, and make its ends thick while keeping the graph countable.
6. Suppose a locally finite connected graph G has three distinct ends. Let S be a finite set of vertices separating these pairwise. Take an automorphism that maps S 'far away' into a component of $G - S$. Can you show that the image of S separates this component in such a way that G must have more than three ends?

7.$^+$ Pick a vertex v. Is its orbit $U = \{v, \sigma(v), \sigma(\sigma(v)), \ldots\}$ finite or infinite? To determine the position of U within G, let P be a path from v to $\sigma(v)$ and consider the infinite union $P \cup \sigma(P) \cup \sigma(\sigma(P)) \cup \ldots$. Does this, somehow, define an end? And what about the sequence $v, \sigma^{-1}(v), \sigma^{-2}(v), \ldots$?