Exercise Sheet 4 for Infinite Graph Theory, WS 2019/20
(to be discussed on 11. November 2019)

1. Extend Nash-Williams' Theorem 2.4.3, which characterizes when the edge-set of a graph can be covered by k spanning trees, to infinite graphs, giving three proofs: one using the generalized infinity lemma, one using the compactness principle, and one applying Tychonov's theorem.

2. Show that a tree has a rank (as defined in the book in the second paragraph after the proof of Proposition 8.5.1) if and only if it is recursively prunable, and that it has rank α if and only if α is the maximum of its pruning labels.

3. Let G be a rayless graph, of rank α say, and let U be a finite set of vertices witnessing this, of minimal order. Show that U is unique.

4. (i) Construct a countable tree that has rank ω in the ranking of rayless graphs. Can you find one such tree that contains all the others?

   (ii)$^+$ Is there a tree of rank ω that is a subtree of every such tree?

5. A graph $G = (V, E)$ is called *bounded* if for every vertex labelling $\ell\colon V \to \mathbb{N}$ there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ that exceeds the labelling along any ray in G eventually. (Formally: for every ray $v_1 v_2 \dots$ in G there exists an n_0 such that $f(n) > \ell(v_n)$ for every $n > n_0$.) Prove the following assertions:

   (i) The ray is bounded.

   (ii) Every locally finite connected graph is bounded.

   (iii) A countable tree is bounded if and only if it contains no subdivision of the $\aleph_0$-regular tree $T_{\aleph_0}$.

Optional:

6.$^+$ Extend the packing-covering theorem (2.4.4) to infinite graphs.

7.$^+$ Let T be a tree with root r, and let $\leqslant$ denote the tree-order on $V(T)$ associated with T and r. Show that T contains no subdivision of the $\aleph_1$-regular tree $T_{\aleph_1}$ if and only if T has an ordinal labelling $t \mapsto o(t)$ such that $o(t) \geqslant o(t')$ whenever $t < t'$ and no more than countably many vertices of T have the same label.

Hinweise

1. –
2. Recall that the trees T_α defined in the recursive pruning algorithm are down-sets in T.
3. If U' is another such set, consider a vertex $u \in U \setminus U'$ and the rank of the component of $G - U'$ containing u.
4. How does subdividing the edges of a graph affect its rank?
5. For (i), note that a ray has countably many subrays. For the forward implication in (iii), prune the given tree recursively by chopping off subtrees you can bound already.

6.$^+$ First use Zorn's lemma to find a coarsest partition $\mathcal{P}$ of $V(G)$ such that every $G[U]$ for $U \in \mathcal{P}$ admits a packing of k spanning trees; and so that the packings of a coarser partition induce the packings of a finer partition. Then use the finite packing-covering theorem together with $Q1$ from this sheet to show that $G/\mathcal{P}$ can be covered by k trees.

7.$^+$ Does $T_{\aleph_1}$ have such a labelling? If $T \not\supseteq TT_{\aleph_1}$, construct a labelling of T inductively. Supposing a labelling exists: where in T will the vertices labelled zero lie? Where the vertices labelled 1?