Exercise Sheet 3 for Infinite Graph Theory, WS 2019/20
(to be discussed on 4. November 2019)

1. Give two proofs, one using Zorn's lemma and one using ordinals, of the fact that a graph is 2-edge-connected if and only if it has a strongly connected orientation, one in which every vertex can be reached from every other vertex by a directed path.

2. Extend Theorem 2.4.4 to infinite graphs, giving three proofs: one using the generalized infinity lemma, one using the compactness principle, and one applying Tychonov's theorem.

3. Prove the compactness principle and the generalized infinity lemma from Tychonoff's theorem.

4. In the lecture on compactness, the unfriendly partition conjecture was proved for locally finite graphs, using the infinity lemma.
   (i) Give an alternative proof using the compactness principle.
   (ii) The proof using the infinity lemma required a modification of the statement. Is this still necessary? Which step in the proof using the compactness principle reflects the requirement in the infinity lemma that every admissible partial solution must induce an admissible solution on a smaller substructure? Where is the local finiteness used?

5. Prove the following version of König's theorem for locally finite graphs: Every bipartite graph G has a matching M and a vertex cover U of the edges of G such that U consists of one vertex from each edge in M.

Optional:

6. (i) Prove the unfriendly partition conjecture for countable graphs with all degrees infinite.

   (ii) Can you adapt the proof to cover also those countable graphs that have finitely many vertices of finite degree?

7. Gallai's partition theorem of Exercise 39^+, Ch 1, clearly fails for infinite graphs. (Why?) However, rephrase the theorem in terms of degrees, and extend the new version to locally finite graphs. (You do not have to prove the finite theorem.)

Hinweise

1.
2. You may (and should) use Theorem 2.4.4 for finite graphs.
3.
4. For (ii), try to prove the statement without a modification.
5. Apply the infinity lemma or compactness principle to a suitably weakened statement about finite subgraphs.
6. For (i), consider the vertices in one infinite sequence that lists each vertex infinitely often. Make a vertex a little happier every time it is considered. For (ii), make one type of vertices happy first, then the other.
7. Apply the Infinity Lemma. Find a statement about a vertex partition of $G_n = G[v_1, \ldots, v_n]$ that implies the corresponding statement for the induced partition of G_{n-1}, and whose truth for the partitions of the G_n induced by a given partition of G implies that this partition of G is as desired.